separately:

$$I=\left(\frac{4\pi a}{k}\right)^2\int_0^{\frac{1}{2}\pi}d\chi\,\sin2\chi\int_0^\infty\frac{\rho d\rho}{(1+\rho^2\sin^2\chi)(1+\rho^2\cos^2\chi)}\times\int_{\Delta\phi}^{\pi}\frac{d\phi}{1-\sin2\chi\,\cos\phi}$$

$$=-2\left(\frac{4\pi a}{k}\right)^2\int_0^{\frac{1}{2}\pi}d\chi\,\frac{\tan2\chi}{|\cos2\chi|}\times\ln\tan\chi\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{(1+\sin2\chi)}{|\cos2\chi|}\frac{\Delta\phi}{2}\right)\right\}.$$

Since $\Delta\phi\ll1$ we have replaced $\tan(\frac{1}{2}\Delta\phi)$ by $\frac{1}{2}\Delta\phi$ in the last integrand.

Introducing the new variable $t=\tan\chi$ and noting that

$$\frac{\tan2\chi}{\cos2\chi}\ln\tan\chi d\chi=\frac{2t\ln t}{(1-t^2)^2}dt=-\frac{1}{2}\frac{d}{dt}\left\{\frac{t^2\ln t^2}{1-t^2}+\ln|1-t^2|\right\}dt,$$

a partial integration gives

$$I=-\left(\frac{4\pi a}{k}\right)^2\int_0^\infty dt\left\{\frac{t^2\ln t^2}{1-t^2}+\ln|1-t^2|\right\}\times\frac{\Delta\phi}{(1+t)^2(\frac{1}{2}\Delta\phi)^2+(1-t)^2}.$$

Since $\Delta\phi\ll1$ the contribution to the integral comes only from a small region around $t=1$. Thus we obtain, with $1-t=y\Delta\phi$ and neglecting terms of relative order $\Delta\phi$,

$$I=-\left(\frac{4\pi a}{k}\right)^2\int_{-\infty}^\infty dy\,\{-1+\ln(2\Delta\phi)+\ln|y|\}\frac{1}{1+y^2}$$
$$=(4\pi a/k)^2\pi\{-\ln(2\Delta\phi)+1\},$$

which is the result used in Eq. (26) in the text. The same method may be used to obtain the integral I_e given in Eq. (27).

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Coordinate Covariance and the Particle Spectrum*

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An attempt is made to find an analog, for the quantum mechanics of non-Euclidean space-time, to the classification of representations of the Lorentz group. The difficulty of obtaining any such classification in terms of curvilinear coordinates is pointed out, and the use of a higher-dimensional set of pseudo-Euclidean coordinates is chosen as an alternative mode of attack. A class of representations then follows easily. On the basis of an intuitive approximation it is found that spectra of elementary particles, with conservation of quantities of the nature of isotopic spin, seem to arise from these representations.

I. INTRODUCTION

CONSIDERATIONS of covariance have often been of value to the development of a theory[1,2]; perhaps this will also be the case for the generalization of quantum mechanics to allow for the curvature of space. It is, of course, true that such generalizations are always expressed in covariant language. However, there has been no attempt to enumerate the modes of covariance allowed to a physical quantity—that is, the set of representations of the group of coordinate transformations—such as has been done for the case of Minkowski space, where this group is the Lorentz group.[3] The purpose of the present article is to consider an approach which may shed light on this problem, and at the same time does suggest a possible origin for the observed multiplet structure of the elementary particle spectrum.

The following considerations are based on the familiar assumption:

> Any physical system (in particular, an elementary particle) can be representated by a wave function belonging to some irreducible representation of the group of all coordinate transformations.

This is a combination of the quantum-mechanical assumption that systems may be represented by wave functions, with the requirement that a system describable in one coordinate system be describable also in any other coordinate system[4]; the "Schrödinger picture" is

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[1] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1950).
[2] E. P. Wigner, Nuovo cimento **3**, 517 (1956).
[3] E. P. Wigner, Ann. Math. **40**, 149 (1939); Iu. M. Shirokov, JETP **6**, 919, 929 (1958).

[4] The requirement that the group of wave function transformations be a "representation" of the group of coordinate transformations means essentially that if wave function transformations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ correspond to coordinate transformations T_1, T_2,

being assumed here. The reader is referred to reference 2 for a fuller discussion of these considerations, for the case of flat space-time and the Lorentz group. In our case, of course, space-time is not flat, and the coordinate systems must be chosen accordingly. It is usually tacitly assumed that the appropriate coordinate systems are those consisting of four curvilinear coordinates. However, this choice implies a transformation group with an infinite number of parameters[5] (as compared to the ten of the Lorentz group)—a quite unwieldly entity. A less usual but much more amenable choice (namely, pseudo-Euclidean "hypercoordinates") will be considered in Sec. III of this paper; and in Sec. IV the results so obtained will be compared with conventional theory by means of an intuitive approximation. But first we shall briefly review the situation for curvilinear coordinates, in order to emphasize the difficulties inherent in that approach.

II. REPRESENTATIONS WITH CURVILINEAR COORDINATES

To avoid subtle considerations in the following discussion, it will be assumed that wave functions are always multi-component functions of the coordinates transforming according to $\psi(x)\to A_x\psi(f^{-1}x)$ under the coordinate transformation $x\to fx$, where A_x is a finite-dimensional matrix. Then $\psi(a)\to A_a\psi(a)$ under the set $\{f_a\}$ of all transformations leaving a fixed ($f_a a=a$); thus the set $\{A_a\}$ must be a representation of the set $\{f_a\}$.

The classical representations known as tensor fields (i.e. scalar fields, vector fields, second-rank-tensor fields, and so forth) could be discovered in the following fashion. The Jacobian matrices $\{\|\partial f_a/\partial x\|_a\}$ form a representation (the "vector" representation) of the set $\{f_a\}$, since $hx=fgx$ implies that $\|\partial x''/\partial x\| =\|\partial x''/\partial x'\|\ \|\partial x'/\partial x\|$ where $x''\equiv hx$, $x'\equiv gx$, and $x''=fx'$. We obtain the nth rank (contravariant) tensor representation by taking the direct product of this representation with itself n times:

$$F'^{\alpha\beta\cdots\gamma}(x')=(\partial x'^\alpha/\partial x^\mu)(\partial x'^\beta/\partial x^\nu)\cdots \times(\partial x'^\gamma/\partial x^\rho)F^{\mu\nu\cdots\rho}(f^{-1}x'), \quad (1)$$

where F carries n indices. Examples of such tensor fields are provided by the photon field, A^μ or $F^{\mu\nu}$.

Such fields cannot describe the electron since they always correspond to integral spin (where this last phrase denotes merely a particular variety of behavior under spatial rotations). In the physics of *flat* space-time, the matrices $\|\partial f/\partial x\|\equiv\|a_{\mu\nu}\|$ are required to be independent of x and to satisfy the relation $a_{\mu\nu}a_{\mu\rho}=\delta_{\nu\rho}$; such matrices possess a spin one-half representation $\{U\}$ (with the Pauli spin matrices as generators), so that wave functions transforming in the fashion $\psi(x)\to U\psi(f^{-1}x)$ can describe electrons. However, in the case of curved space-time, there is no restriction on the matrices $\|\partial f_a/\partial x\|_a$, other than that they be real and nonsingular; and the group of such matrices has *no* double-valued finite-dimensional representations.[6,7] Thus in curved space-time the matrices $\{A_a\}$ for the electron *cannot* be chosen to form a representation of the matrices $\|\partial f_a/\partial x\|_a$. But this does not rule out the existence of suitable representations of the set $\{f_a\}$ (in fact, such representations can be related to the wave functions used in conventional curved-space quantum theory). These representations owe their existence to the fact that with each point of space-time (coordinates x^μ) we may associate the tangent flat space (coordinates y^μ), so oriented, for example, that the tangent to the x^1 coordinate line lies along the y^1 axis, the tangent to the x^2 coordinate line lies in the y^1-y^2 plane, etc. Any transformation of the curvilinear coordinates then induces, at each point of space-time, a rotation of the tangent flat space. If the matrices $\{A_a\}$ are chosen to form a representation of the set of these (Lorentz) rotations of the tangent space at the point a, then they will also form a representation of the set $\{f_a\}$[8]—and, in fact, of the whole set $\{f\}$. Thus the transformations $\psi(x)\to A_x\psi(f^{-1}x)$ will indeed form a representation of the set $\{f\}$. In particular, we can choose the matrices A_x to belong to the spin one-half representation of these "local Lorentz rotations," so that ψ can represent the electron.

In this way one finds representations of two sorts: tensor wave functions, presumably applicable to the photon; and spinor wave functions, quite possibly appropriate to the electron. The question naturally arises of whether there may be other representations of physical interest. In fact, the above consideration of spinor fields might encourage such a thought, for it is conceivable that one could specify the orientation of higher-dimensional "tangent" flat spaces at each point of space-time. This would mean that wave functions could belong to representations of rotations in spaces of dimension greater than four, and we would have the very interesting situation of multiplets of spin one-half particles.

Considering the fairly complex known multiplet structure of elementary particles and the possibility of a connection with coordinate covariance, there can be no doubt that the determination of all representations of

and T_3, respectively, for which $T_1T_2=T_3$, then $\mathcal{T}_1\mathcal{T}_2=\mathcal{T}_3$. A representation is "irreducible" if the group of transformations leaves no vector subspace invariant; this condition must evidently be satisfied for the set of vectors associated with *one* physical system viewed from different coordinate systems.

[5] For this choice, the new coordinates are arbitrary functions of the old; the coefficients of a power series expansion constitute such an infinite set of parameters.

[6] This group splits into two commuting subgroups, the group of uniform dilatations at the point a and the group of transformations having determinant $+1$. The latter is locally, except for conditions of reality, the group $O^+(6)$ of proper rotations in six-dimensional Euclidean space.[7] Although $O^+(6)$ has both single- and double-valued finite-dimensional representations, $\|\partial f_a/\partial x\|_a$ itself corresponds to a double-valued representation.

[7] E. B. Dynkin, Amer. Math. Soc. Transl. Series 2, **6**, 319 ff (1957).

[8] But not of the set $\{\|\partial f_a/\partial x\|_a\}$, as each of those matrices can correspond to a continuous infinity of the matrices A_a.

general (curvilinear) coordinate transformations would be of interest. But, on the other hand, the difficulty of solution of the much simpler problem of finding representations of the Lorentz group suggests that this may be an insoluble problem. In this situation it seems justifiable to abandon a customary assumption, in order to find a solvable problem which may be related to physics. This assumption is that the coordinate systems most appropriate for the description of physics are those with the minimum possible number (four) of coordinates; when it is dropped, one is naturally led to the approach of the next section: the use of pseudo-Euclidean "hypercoordinates."

III. REPRESENTATIONS WITH PSEUDO-EUCLIDEAN COORDINATES

Attention has recently been called to the mathematical theorem which states that any four-dimensional Riemannian space can be locally embedded in a pseudo-Euclidean space of dimension ten or less.[9] To put it another way, we can introduce a (redundant) set of coordinates z^i into space-time such that

$$ds^2=\sum_{i=1}^{n}\epsilon_i(dz^i)^2, \tag{2}$$

(where $\epsilon_i=\pm1$ and $n\leq10$), at least in some finite region surrounding any given point. The possibility that all of space-time cannot be so embedded, or the consequences if it cannot, will not be considered here.[10] The legitimacy of choosing this particular set of coordinate systems out of the collection of all possible sets of curvilinear coordinates rests on the same ground as does the choice of Minkowski coordinates in flat space; namely, such a set is (in principle) physically distinguishable, through comparison of the physically significant quantity ds^2 with its value as given by (2).[11]

As is obvious, the different possible coordinate systems that can be introduced satisfying (2) correspond precisely to rotations and translations in the pseudo-Euclidean space E_n. Thus wave functions will be multicomponent functions in E_n, transforming as $\psi(x)\rightarrow A\psi(t^{-1}x)$ where the (constant) matrices $\{A\}$ belong to some representation of the group $O^+(n)$ of proper rotations in n-dimensional Euclidean space (temporarily neglecting complications due to the indefinite metric). As for three-dimensional rotations all elementary particles seem to correspond to representations of low dimension (scalar, spinor, and vector), so we shall assume that also for $O^+(n)$ only the lowest-dimensional representations need be considered. For the groups considered here ($n\leq10$), the two representations of lowest dimension (next to the scalar) are again the "spinor" and "vector" representations. These are mathematically distinguished as the "elementary" representations (for $n\geq5$), and the spinor is further distinguished by the fact that any representation arises from repeated direct products of the spinor with itself. The vectors of the vector representation (transforming like the coordinates) have, of course, n components. The dimensions of the spinor representations can be expressed as 2^m; if n is odd, there is one spinor representation and $m=\frac{1}{2}(n-1)$; if n is even, there are two, and $m=\frac{1}{2}(n-2)$.[12]

As for $n=3$ or 4, so in general it is convenient to define generators of rotations, or "angular momentum" matrices, $M_{\mu\nu}$. The spinor representations result from setting

$$M_{\mu\nu}=\frac{1}{4i}[\Gamma_\mu,\Gamma_\nu], \tag{3}$$

where

$$\{\Gamma_\mu,\Gamma_\nu\}=2\delta_{\mu\nu}.$$

The Γ matrices can, in turn, be easily constructed from Pauli matrices, according to some scheme such as

$$\begin{aligned}\Gamma_1&=\sigma_1\otimes I\otimes I\otimes\cdots,\\ \Gamma_2&=\sigma_2\otimes I\otimes I\otimes\cdots,\\ \Gamma_3&=\sigma_3\otimes\sigma_1\otimes I\otimes\cdots,\\ \Gamma_4&=\sigma_3\otimes\sigma_2\otimes I\otimes\cdots,\ \text{etc.}\end{aligned} \tag{4}$$

For n odd, the resulting representation of $O^+(n)$ is irreducible; for n even, it is the direct sum of the two spinor representations. Finally, it can easily be proved from the commutation relations that the quantity $\psi^\dagger\Gamma_\mu\psi$ transforms according to the vector representation.

So far in the discussion of wave functions, we have made no mention of the space-time surface R_4. However, there must be some feature of the theory corresponding to the fact that physical observations are always restricted to R_4. The simplest assumption is that the wave functions themselves are so restricted; and we shall suppose this restriction to be "smooth." Thus, wave functions will be assumed to be functions defined on E_n which vanish at any appreciable distance from R_4, but possess all necessary derivatives, including those in directions perpendicular to R_4. We might describe this by saying that particles move in a sort of "potential trough" centered about the surface R_4 (note also the last paragraph of Sec. IV). Since the shape of R_4 is determined by the distribution of matter throughout the universe, such a "cosmic potential" is evidently to be interpreted as a form of interaction with matter at large distances.

Without investigating the nature of such a "cosmic

[9] C. Fronsdal, Nuovo cimento **13**, 988 (1959); L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926), p. 187 ff.

[10] However, it may be noted that the Schwarzschild solution can be entirely embedded in a six-dimensional pseudo-Euclidean space.[9]

[11] It should be noted that the above-mentioned difficulties in finding representations of curvilinear-coordinate transformations do not depend on the nonflatness of space-time; they would arise for flat space-time if we did not restrict consideration to Minkowski coordinate systems, by the condition that $ds^2=dr^2-dt^2$.

[12] The reader will find in reference 7 a general, though not elementary, account of the representations of these groups.

potential," nor even whether this picture has any sense, let us use it as a model to suggest an approximation by which the present discussion can be brought into comparison with current physical knowledge. According to the above picture, laboratory experiments take place in a sort of external field due to the presence of distant matter. We wish to find those coordinate transformations under which laboratory experiments demonstrate covariance; these will be those which leave this external field invariant. According to our "potential trough" picture, they must at least preserve R_4. However, we do not expect R_4 to possess any exact symmetries; but this does not preclude approximate symmetries, some of which may be obeyed to a high degree of accuracy by laboratory experiments. The simplest approximation is to assume that R_4 can be replaced by its tangent hyperplane T_4 and that laboratory experiments will show covariance under those transformations which preserve T_4. Although this rather violent approximation may seem almost equivalent to the usual assumption of a four-dimensional pseudo-Euclidean space from the outset,[13] the geometry of R_4 still enters the picture through its determination of the dimension n of our pseudo-Euclidean embedding space. This approximation will be considered in the next section.

IV. THE HYPERPLANE APPROXIMATION

In this section we shall suppose (as has been argued in the preceding section) that the physics of elementary particles shows symmetry under those transformations of coordinates (in E_n) which preserve the hyperplane T_4 tangent to the space-time hypersurface R_4. This group of transformations is the product of the group of rotations and translations within T_4 with the group of rotations leaving T_4 point-wise fixed. The first of these is evidently to be (locally) identified with Lorentz transformations, and the second, with isospace rotations (i.e., those related to such internal symmetries as conservation of isotopic spin and strangeness). It is convenient for the following to introduce pseudo-Euclidean coordinates x^μ ($\mu=1$ to 4) within T_4, and $y^\mu=x^{\mu+4}$ ($\mu=1$ to $n-4$) within the space orthogonal to it; then Lorentz transformations act on the coordinates x^μ, and isospace rotations on the coordinates y^μ.

An interesting feature of the elementary representations is the manner in which they split up under the reduced symmetry group of this hyperplane approximation. According to the prescriptions (3) and (4), it is possible to choose the spinor representation so that the first four Γ matrices are of the form $\Gamma_\mu=\gamma_\mu\otimes I\otimes I\otimes\cdots$, where the γ_μ are Dirac matrices; in this case the spinors are reducible with respect to Lorentz rotations, of course, but irreducible with respect to isospace rotations (as can be seen by comparing the dimensions given above). The vector representation, on the other hand, can be chosen so that the first four components transform as a space-time vector/isospace scalar, and the last $n-4$ components transform as an isospace vector/space-time scalar.

These remarks will be illustrated by a simple example, the case of $n=7$.[14] For this we shall choose

$$\Gamma_\mu=\gamma_\mu\otimes I \quad \text{for} \quad \mu=1 \text{ to } 4;$$
and
$$\Gamma_\mu=\gamma_5\otimes\sigma_{\mu-4} \quad \text{for} \quad \mu=5 \text{ to } 7. \tag{5}$$

The generators $M_{\mu\nu}$ of Lorentz rotations of the spinor components are then the usual ones (but for a factor of I), while the generators of isospace rotations are given by $\mathbf{N}=I\otimes\boldsymbol{\sigma}$. Thus we have an isotopic doublet of Dirac particles, similar to the nucleon. The simplest interaction term that can be written down involving spinor and vector wave functions is then

$$g\psi^\dagger\Gamma_\mu\psi\phi_\mu\sim g_1\psi^\dagger\gamma_\mu\psi B_\mu+g_2\psi^\dagger\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\pi} \tag{6}$$

(if one is allowed suggestive notation), where ψ is the spinor wave function, and B_μ and $\boldsymbol{\pi}$ constitute the first four and last three components, respectively, of the vector wave function ϕ_μ. Full seven-dimensional invariance results if $g_1=g_2$, and invariance under the Lorentz and isospace groups results in any case.

Larger values of n will, of course, result in more complicated multiplet structures. It might be worth noting that $O^+(10)$ contains as mutually commuting subgroups the Lorentz group and the group SU(3), currently in vogue in connection with the Sakata model.[15] On the other hand, the upper limit of ten on the dimension precludes (by one dimension) the inclusion of the seven-dimensional isospace (with its 8 "baryons" and 7 "mesons," here accompanied by the B_μ field) considered by several authors.[16] However, since fermions (physically) and spinors (mathematically) seem somewhat more basic than mesons and vectors, it may be worth noting in passing that $O^+(10)$ has two sixteen-dimensional spinor representations (the Γ representation being of dimension 32) which might be identified with the 32-component baryon, in a fashion analogous to the identification of the two two-dimensional representations of the Lorentz group with the four-component Dirac electron; the vector representation with six isospace components cannot then, of course, be identified with the seven familiar mesons.

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[13] Actually, it may fail significantly of being as good an approximation. For example, a two-dimensional surface may have corrugations in a region where the metric is Euclidean; at any point in such a region, the tangent plane is a very poor approximation. However, this objection can be removed by extending the considerations of the next section (relating to approximation by T_4) to the case that T_4 is replaced by the best pseudo-Euclidean approximation to R_4 (unpublished; copies on request).

[14] The simplest possible case would be $n=6$, according to reference 9. In this case the structure of "isospace" rotations, in a space of dimension $n-4=2$, is trivial: irreducible representations are simply of the form $\exp(i\alpha M)$ with M any number.

[15] See, e.g., A. Salam and J. C. Ward, Nuovo cimento **20**, 419 (1961), and papers cited there.

[16] J. Tiomno, Nuovo cimento **6**, 69 (1957); R. E. Behrends, Nuovo cimento **11**, 424 (1959); D. C. Peaslee, Phys. Rev. **117**, 873 (1960).

The discussion above has been given for the simplest case of a positive definite metric. Actually, not all of the numbers ϵ_i in (2) can be of the same sign,[9] so some further remarks must be added. Suppose that the ϵ_i are positive except for ϵ_a, ϵ_b, $\cdots\epsilon_c$, which are negative. The introduction of imaginary coordinates $z'^a=iz^a$, $z'^b=iz^b$, $\cdots$, $z'^c=iz^c$ produces the formal appearance of a positive definite metric, to the effect that the above list of representations can be carried over. The principal change is that if $\psi\rightarrow U\psi$, $\psi^\dagger\rightarrow\psi^\dagger U^\dagger$ under a (pseudo-)rotation, we no longer have $U^\dagger U=1$, since some of the parameters [the $\alpha_{\mu\nu}$'s in the expression $U=\exp(i\sum\alpha_{\mu\nu}M_{\mu\nu})$] are imaginary. But it is easily checked that $\psi^\dagger\Gamma_a\Gamma_b\cdots\Gamma_c\rightarrow\psi^\dagger\Gamma_a\Gamma_b\cdots\Gamma_c U^{-1}$, so that the scalar and vector quantities are now $\psi^\dagger\Gamma_a\Gamma_b\cdots\Gamma_c\psi$ and $\psi^\dagger\Gamma_a\Gamma_b\cdots\Gamma_c\Gamma_\mu\psi$, respectively.

So far, no specific requirements have been placed on the functional dependence of the wave functions. In the absence of any external interactions whatever, we would expect that $\sum\partial_{\mu\mu}\psi\rightarrow\text{const}\cdot\psi$, since $\sum\partial_{\mu\mu}$ is an operator commuting with all the generators ($M_{\mu\nu}$ and ∂_μ) of the group of rotations and translations under which ψ is to be irreducible. So a plausible assumption in the presence of interactions is that

$$\Box^2\psi\equiv\sum_{\mu=1}^{n}\frac{\partial^2\psi}{\partial x^{\mu 2}}=F\psi, \tag{7}$$

where F is some function (of the coordinates and, possibly, momenta) which determines the interaction with distant matter. Now suppose we can write $F=f(x,\partial_x)+g(y,\partial_y)$ in the tangent-space approximation; then we consider separated solutions:

$$\Box_x^2\psi_x(x)=(f+\kappa)\psi_x(x),\quad \Box_y^2\psi_y(y)=(g-\kappa)\psi_y(y),\quad \text{with}\quad \psi=\psi_x\cdot\psi_y. \tag{8}$$

Correspondence with conventional theory now requires f to be (very nearly) constant so that the first of these equations becomes the Klein-Gordon equation, and we have

$$[\Box_x^2-m^2]\psi_x(x)=0,$$
$$[\Box_y^2-g'(y,\partial_y)+m^2]\psi_y(y)=0. \tag{9}$$

The second of Eqs. (9) has the form of an eigenvalue equation; except for the number of dimensions, it is the time-independent Schrödinger equation for a particle subject to a potential $V\propto g'(y,\partial_y)$. For appropriate behavior of $g'(y,\partial_y)$, the eigenvalues will be discrete, and have a spacing given roughly by $\delta m\sim 1/l$, where l is a dimension characteristic of the function $g'(y,\partial_y)$. This suggests that the different mass states of elementary particles (or of nuclei) may be pictured as eigenstates of oscillation in a sort of potential well (of radius l) that restricts wave functions to a narrow region of the space E_n surrounding the space-time hypersurface R_4.[17] Such a restriction is certainly compatible with the physical requirement that particles must never be "found" far from R_4; and the maximum deviation, of the order of an elementary-particle Compton wavelength, seems reasonable. Unfortunately, practically nothing can be said about the resulting mass spectrum without some knowledge of $g'(y,\partial_y)$.

V. CONCLUSION

The principal idea that has been presented is that the possibility of embedding space-time in a higher-dimensional pseudo-Euclidean space offers an attack on the problem of finding all representations of the group of coordinate transformations in nonflat space-time. This problem may well be of significance for particle physics, as it is certainly not obvious that the conventional assumption of a pseudo-Euclidean space-time from the outset is justified. On the other hand, it is interesting that this approach implies the existence of an isospace, thus suggesting a fundamental connection between the macroscopic structure of space-time and internal symmetries of elementary particles. And although no predictions are possible without the development of a much more detailed theory, there is the possibility of a new origin for particle mass; thus (for example), particle interactions could show a higher degree of symmetry than is allowed by the usual assumption that they are responsible for the baryonic mass differences.

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[17] This is consistent with the "potential trough" picture mentioned in Sec. III as motivation for the present hyperplane approximation.