Measurement of Linear Photon Polarization by Pair Production*

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(Received November 7, 1961)

The cross section for production of electron positron pairs by linearly polarized high energy photons is calculated for the case in which the pair particles and the photon are nearly coplanar. The cross section is integrated over the polar angles θ_1 and θ_2 between the momenta of the positron and the photon, and the momenta of the electron and the photon, respectively, and over a small range $2\Delta\phi$ of the azimuthal angle $\varphi=\varphi_1-\varphi_2$ between the pair particles about $\varphi=\pi$. The asymmetry ratio, defined as the ratio of the number of pairs emitted parallel to the photon polarization plane to the number of pairs emitted perpendicular to this plane, is plotted as a function of $\Delta\phi$ for the cases of no screening and complete screening. It is shown that for very small $\Delta\phi$ the cross section is a very rapidly varying function of $\Delta\phi$, such that the pairs emitted within the angular region $2\Delta\phi$ are predominantly perpendicular to the polarization plane for $\Delta\phi<\Delta\phi_0$ [$\Delta\phi_0=1.23(Z^{\frac{1}{3}}/111)$ for complete screening, for example] and predominantly parallel to this plane for $\Delta\phi>\Delta\phi_0$.

A discussion and comparison of several possible experimental methods for measuring the asymmetry ratio is given, viz., observing nearly coplanar pairs, observing only one of the pair particles and, lastly, observing all pairs except those which are emergent over a narrow angular range $2\Delta\phi$. The asymmetry ratio for the last method is also calculated and curves of the asymmetry ratio, as a function of the angular range $2\Delta\phi$ over which pairs are rejected, are given. This last method gives higher asymmetry ratios than do the two others.

Coulomb corrections have been included.

I. INTRODUCTION

It was suggested by Yang[1] and by Berlin and Madansky[2] that pair production may be used to determine the polarization of high energy photons by utilizing the correlation between the photon polarization plane and the plane of production of the electron positron pair. Berlin and Madansky[2] calculated the polarization dependent Born approximation pair cross section for linearly polarized photons. With the photon direction as z axis, they denote the azimuth of the electron and positron by φ_- and φ_+ respectively, and consider the case in which the photon and the electron positron pair are exactly coplanar, viz., when $\varphi_+-\varphi_-=\pi$. In this case they found that the pairs are most likely to be emitted in a plane *perpendicular* to the polarization plane.

It was then pointed out by Wick[3] that one should not, however, consider the case in which $\varphi\equiv\varphi_+-\varphi_-$ is *exactly* equal to π, but rather average over a small range of φ close to π, since this is what is done in any experiment—for example when the pair particles are detected in counters or in an emulsion. Using the Weizsäcker-Williams method which seemed to effect such an average, he obtained a result opposite to that found by Berlin and Madansky,[2] namely that the pairs are produced predominantly *in* the plane of polarization.

The discrepancy between these two results may seem somewhat surprising in view of the fact that the range of the angle φ over which one averages is small. However, as we shall see below, for φ close to π the cross section is a rapidly varying function of φ, which makes its value at $\varphi=\pi$ considerably different from the average over a small range about $\varphi=\pi$.

The present paper is written in order to reconcile this discrepancy and to give a quantitative theory which may be applied to experiments currently in progress.

In Sec. 2 we give the polarization dependent and polarization independent parts of the differential cross section for the cases of no screening and complete screening, and these are integrated over the polar angle between the electron and photon directions as well as over a range $-\Delta\phi$ to $\Delta\phi$ of the azimuthal angle $\phi=\pi-\varphi$ for $\Delta\phi\ll1$.

In Sec. 3 the integration over the remaining angle variable—the polar angle between the positron and photon directions—is performed for the particularly simplifying assumptions $\Delta\phi\gg\delta$ for no screening and $\Delta\phi\gg\beta$ for complete screening (but still $\Delta\phi\ll1$ in both cases), δ being the minimum energy transfer to the nucleus in units of mc^2, β^{-1} the screening radius of the atom in Compton wavelengths. The asymmetry (the ratio of the number of pairs produced in the plane of polarization to the number of pairs produced perpendicular to the plane of polarization) is given under these simplifying assumptions, and plotted in Figs. 2 and 3 for the case in which the energies of the final particles are not observed.

In Sec. 4 we give a check on the results of Sec. 3, provided by integrating the cross section over the regions excluded in Sec. 3, viz., $-\pi\leq\phi\leq-\Delta\phi$ and $\Delta\phi\leq\phi\leq\pi$, thus obtaining the total cross section, which is shown to agree with that obtained from Eq. (10.3) of reference 4.

The results of Sec. 2, integrated over the polar angle between the positron and photon directions, but without the simplifying assumptions ($\Delta\phi\gg\delta$ and $\Delta\phi\gg\beta$) made in Sec. 3, are given in Sec. 5 for the case of complete screening and in Sec. 6 for the case of no screening. The

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[1] C. N. Yang, Phys. Rev. **77**, 722 (1950).
[2] T. H. Berlin and L. Madansky, Phys. Rev. **78**, 623 (1950). Note that the asymmetry ratio defined by Berlin and Madansky is the reciprocal of that defined here.
[3] G. C. Wick, Phys. Rev. **81**, 467 (1951).

corresponding asymmetry ratios are also given in these two sections and plotted in Figs. 2, 3, and 4.

In Sec. 7 we discuss the relation between the results obtained here and those of Berlin and Madansky and of Wick. Several possible experimental methods for measuring the asymmetry ratio are considered in some detail there. One of these methods, which involves the removal of the very nearly coplanar pairs and gives a high asymmetry ratio, is described in Secs. 7 and 8. Graphs pertinent to this method are given in Fig. 4.

II. DIFFERENTIAL CROSS SECTIONS FOR NO SCREENING AND COMPLETE SCREENING

We wish then to consider a photon of momentum and energy $\mathbf{k}$, k and associated linear polarization vector $\mathbf{e}$ which produces an electron positron pair having momenta and energies $\mathbf{p}_2$, ϵ_2 and $\mathbf{p}_1$, ϵ_1 respectively. Energy and momentum are measured in units of mc^2 and mc respectively. We shall always use the usual high energy approximations—replacing p_1 and p_2 by ϵ_1 and ϵ_2, respectively, and replacing the sine of the angle between $\mathbf{p}_1$ or $\mathbf{p}_2$ and $\mathbf{k}$ by its argument, thereby making errors of $O(1/\epsilon^2)$. Further, we consider nearly coplanar events, by which we mean that the angle $\phi=\pi-\varphi$ is small. Here $\varphi=\varphi_1-\varphi_2$, where φ_1 and φ_2 are the azimuthal angles referring to the positron and electron, respectively, measured from the direction of the polarization vector $\mathbf{e}$.

For high energies the cross section for pair production is given by[4]

$$d\sigma=\frac{1}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{\epsilon_2{}^2}{k}\sum_{\text{spins}}|\mathbf{A}\cdot\mathbf{e}|^2p_1{}^2dp_1d\Omega_1d\Omega_2, \quad (1)$$

where, summing over the spins of the electron and positron,

$$\sum_{\text{spins}}|\mathbf{A}\cdot\mathbf{e}|^2=4(\tfrac{1}{2}k^2J^2-2\epsilon_1\epsilon_2|\mathbf{J}\cdot\mathbf{e}|^2). \quad (2)$$

Here $\mathbf{A}\cdot\mathbf{e}=(\psi_{2,-},\boldsymbol{\alpha}\cdot\mathbf{e}e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{1,-})$ is the amplitude for pair production and $\mathbf{J}$ is defined in Eq. (3) below.

It is important to note that for nearly coplanar events only small momentum transfers to the nucleus ($q=|\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2|\ll 1$) give a non-negligible contribution to the cross section.[5] This has two important consequences. The first is that the Coulomb correction is negligible,[6] and thus $\mathbf{J}$ is given by the Born approximation value,[7] viz.,

$$\mathbf{J}_{\text{Born}}=\frac{4\pi a}{k}\{(\mathbf{u}\xi+\mathbf{v}\eta)+\hat{k}(\xi-\eta)\}\frac{\{1-F(q)\}}{q^2}. \quad (3)$$

[4] Haakon Olsen and L. C. Maximon, Phys. Rev. **114**, 887 (1959). Note p. 892 Eq. (4.5), p. 893 Eq. (4.10), and p. 889 Eq. (3.3) and text following Eq. (2.12). It should be noted that in the second term in Eq. (4.10), p. 893, the factor $|\mathbf{J}\cdot\mathbf{e}^*|^2$ should read $|\mathbf{J}\cdot\mathbf{e}|^2$.

[5] This follows from the fact that nearly coplanar events are those from a *small* angular region: φ close to π. Thus the cross section for such events will be negligible unless the momentum transfer to the nucleus is also small, the cross section being proportional to q^{-4}.

[6] Haakon Olsen, L. C. Maximon and Harald Wergeland, Phys. Rev. **106**, 27 (1957), p. 28, and text following Eq. (7a.10) on p. 40.

[7] Reference 4, p. 891 Eq. (3.21).

Here $a=Ze^2/\hbar c$, Z being the atomic number of the nucleus, $\mathbf{u}$ and $\mathbf{v}$ are respectively the components of $\mathbf{p}_1$ and $\mathbf{p}_2$ perpendicular to $\mathbf{k}$, and $\xi=1/(1+u^2)$, $\eta=1/(1+v^2)$. $F(q)$ is the atom form factor. The second consequence is that since q^4 appears in the denominator of the cross section, the cross section itself is a rapidly varying function of φ near $\varphi=\pi$. We are now going to prove this. If q_z and $\mathbf{q}_\perp$ denote the components of $\mathbf{q}$ in the direction of and perpendicular to $\mathbf{k}$, respectively, then q^2 may be written as $q^2=q_z{}^2+q_\perp{}^2=q_z{}^2+(\mathbf{u}+\mathbf{v})^2$. Further, for φ close to π we may write

$$q^2\approx q_z{}^2+(u-v)^2+uv(\pi-\varphi)^2,$$

noting that for small q, $u-v$ must also be small. Now for high energies the important contributions to the cross section come from small angles of $O(1/\epsilon)$ between $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{k}$, and hence u and v are of $O(1)$. Further,

$$q_z=\frac{1+u^2}{2\epsilon_1}+\frac{1+v^2}{2\epsilon_2}\geqq\frac{k}{2\epsilon_1\epsilon_2}=\delta,$$

and therefore of $O(1/\epsilon)$. Thus q as well as the cross section will vary considerably over an angular interval $\pi-\varphi$ of $O(1/\epsilon)$. Thus we cannot expect the value of the cross section at $\varphi=\pi$ to be equal to its average over a small range of φ about π, which is what is done in an actual experiment, as Wick pointed out.[3]

In the present range of small momentum transfers q, the convenient variables are u, $w=u-v$, and $\phi=\pi-\varphi$. In terms of these variables the differential cross section is given by substituting Eqs. (2) and (3) in (1):

$$d\sigma=\frac{4}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{d\epsilon_1}{k}\times d\varphi_1[\tfrac{1}{2}k^2J^2-2\epsilon_1\epsilon_2(\mathbf{J}\cdot\mathbf{e})^2]u^2dud\phi dw. \quad (4)$$

The angular differential $p_1{}^2\epsilon_2{}^2d\Omega_1d\Omega_2$ in (1) is now $u^2dud\phi dwd\varphi_1$. Instead of the differential $d\varphi_1d\varphi_2$ we have here $d\varphi_1d\phi$.

We are going to consider the cross section for the emission of nearly coplanar pairs, in a plane making an angle φ_1 with the polarization plane[8] (Fig. 1). The pertinent quantities in Eq. (4) follow from Eq. (3) on making the approximations valid for small momentum transfers $q\ll 1$ (viz., $u-v\ll 1$ and $\pi-\varphi=\phi\ll 1$) and denoting the angle between $\mathbf{u}$ and $\mathbf{e}$ by φ_1. (The angle between $\mathbf{v}$ and $\mathbf{e}$ is then $\varphi_1-\varphi$). They are

$$J^2=\left(\frac{4\pi a}{k}\right)^2\frac{\xi^2(w^2+u^2\phi^2)}{(q_z{}^2+w^2+u^2\phi^2)^2}, \quad \text{(No sc.)} \quad (5)$$

[8] Note that in fact in the analysis we keep φ_1, the positron azimuth, fixed, and integrate over a small range of values of $\varphi_2=\varphi_1-\varphi$. It is immaterial which one of the angles φ_1 and φ_2 is varied since the cross section is symmetric with respect to the positron and electron. However, it might appear that we should integrate over a small range of both φ_1 and φ_2. This is unnecessary since for a given ϕ the cross section is indeed a slowly varying function of φ_1.

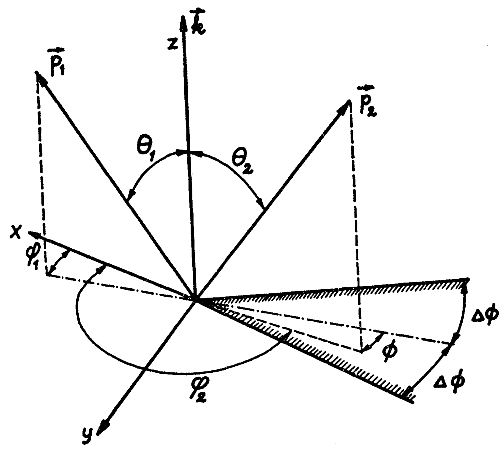

FIG. 1. Angles occurring in the integration of the differential cross section.

and

$$(\mathbf{J}\cdot\mathbf{e})^2=\left(\frac{4\pi a}{k}\right)^2\frac{\xi^2[w\xi(1-u^2)\cos\varphi_1+u\phi\sin\varphi_1]^2}{(q_z^2+w^2+u^2\phi^2)^2},\quad \text{(No sc.)} \quad (6)$$

in the case of no screening ($F(q)=0$).

The corresponding formulas for complete screening[9,10] $\{[1-F(q)]/q^2=1/(q^2+\beta^2),\ 2\epsilon_1\epsilon_2/k\gg 111Z^{-\frac{1}{3}}=\beta^{-1}\}$ are given by similar expressions, but where now q_z^2 is replaced by β^2,

$$J^2=\left(\frac{4\pi a}{k}\right)^2\frac{\xi^2(w^2+u^2\phi^2)}{(\beta^2+w^2+u^2\phi^2)^2},\quad \text{(Compl. sc.)} \quad (7)$$

$$(\mathbf{J}\cdot\mathbf{e})^2=\left(\frac{4\pi a}{k}\right)^2\frac{\xi^2[w\xi(1-u^2)\cos\varphi_1+u\phi\sin\varphi_1]^2}{(\beta^2+w^2+u^2\phi^2)^2}.\quad \text{(Compl. sc.)} \quad (8)$$

The quantities J^2 and $(\mathbf{J}\cdot\mathbf{e})^2$ are integrated over ϕ between the limits $\pm\Delta\phi$. The angular range $2\Delta\phi\ll 1$ over which pairs are accepted is determined by the conditions of the experiment. Since the cross section is negligible for $w=u-v\gg\Delta\phi$, we may take the limits on w to be $\pm\infty$. Similarly, the limits on u are taken to be 0 and ∞.

In the cross section, Eq. (4), the integrals which are needed are then the polarization independent integral

$$I=\int_0^\infty u^2du\int_{-\Delta\phi}^{\Delta\phi}d\phi\int_{-\infty}^{\infty}dw\, J^2, \quad (9)$$

and the polarization dependent integral

$$I_{\mathbf{e}}=\int_0^\infty u^2du\int_{-\Delta\phi}^{\Delta\phi}d\phi\int_{-\infty}^{\infty}dw\, (\mathbf{J}\cdot\mathbf{e})^2. \quad (10)$$

The integrated cross section of interest is then

$$d\sigma=\frac{4}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{d\epsilon_1}{k}d\varphi_1\{\tfrac{1}{2}k^2I-2\epsilon_1\epsilon_2I_{\mathbf{e}}\}. \quad (11)$$

We now note that since for q small, $q_z=(1+u^2)\delta$ is a function of u only, the w and ϕ integrations are the same in the case of no screening [Eqs. (5) and (6)] and complete screening [Eqs. (7) and (8)]. The integrals are elementary and the results are, for the case of no screening,

$$I=\left(\frac{4\pi a}{k}\right)^2\pi\int_0^\infty u\xi^2\left\{2\log\left[\frac{u\Delta\phi}{q_z}+\left(\left(\frac{u\Delta\phi}{q_z}\right)^2+1\right)^{\frac{1}{2}}\right]-\frac{u\Delta\phi}{q_z}\left(\left(\frac{u\Delta\phi}{q_z}\right)^2+1\right)^{-\frac{1}{2}}\right\}du. \quad \text{(No sc.)} \quad (12)$$

and

$$I_{\mathbf{e}}=\left(\frac{4\pi a}{k}\right)^2\pi\int_0^\infty u\xi^2\left\{[1-4\xi(1-\xi)\cos^2\varphi_1]\times\ln\left[\frac{u\Delta\phi}{q_z}+\left(\left(\frac{u\Delta\phi}{q_z}\right)^2+1\right)^{\frac{1}{2}}\right]-\sin^2\varphi_1\frac{u\Delta\phi}{q_z}\left(\left(\frac{u\Delta\phi}{q_z}\right)^2+1\right)^{-\frac{1}{2}}\right\}du. \quad \text{(No sc.)} \quad (13)$$

The corresponding expressions for complete screening are

$$I=\left(\frac{4\pi a}{k}\right)^2\pi\int_0^\infty u\xi^2\left\{2\ln\left[\frac{u\Delta\phi}{\beta}+\left(\left(\frac{u\Delta\phi}{\beta}\right)^2+1\right)^{\frac{1}{2}}\right]-\frac{u\Delta\phi}{\beta}\left(\left(\frac{u\Delta\phi}{\beta}\right)^2+1\right)^{-\frac{1}{2}}\right\}du, \quad \text{(Compl. sc.)} \quad (14)$$

and

$$I_{\mathbf{e}}=\left(\frac{4\pi a}{k}\right)^2\pi\int_0^\infty u\xi^2\left\{[1-4\xi(1-\xi)\cos^2\varphi_1]\times\ln\left[\frac{u\Delta\phi}{\beta}+\left(\left(\frac{u\Delta\phi}{\beta}\right)^2+1\right)^{\frac{1}{2}}\right]-\sin^2\varphi_1\frac{u\Delta\phi}{\beta}\left(\left(\frac{u\Delta\phi}{\beta}\right)^2+1\right)^{-\frac{1}{2}}\right\}du. \quad \text{(Compl. sc.)} \quad (15)$$

At this point we must consider the cases of no screening and complete screening separately. In Sec. 3 we shall, moreover, for simplicity, make the approximations $\Delta\phi\gg\delta$ and $\Delta\phi\gg\beta$ for the cases of no screening and complete screening respectively, (but still of course $\Delta\phi\ll 1$). This will generally be the condition of experimental interest at high energies.

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[9] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934). Note Sec. 8, p. 538.

[10] Reference 4, p. 901 Eq. (9.13) and p. 897 Eqs. (6.30)–(6.34).

III. CROSS SECTIONS AND ASYMMETRY RATIOS

a. No Screening and $\Delta\phi\gg\delta$

Introducing $\xi=(1+u^2)^{-1}$ as a variable and keeping only the largest terms in (12) and (13), these integrals become

$$I=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\int_0^1\left\{\ln[\xi(1-\xi)]+2\log\left(\frac{2\Delta\phi}{\delta}\right)-1\right\}d\xi$$
$$=(4\pi a/k)^2\pi[\ln(2\Delta\phi/\delta)-\tfrac{3}{2}], \qquad (16)$$

and

$$I_e=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{4}\int_0^1\left\{[1-4\xi(1-\xi)\cos^2\varphi_1]\right.$$
$$\left.\times\left[\ln(\xi(1-\xi))+2\log\left(\frac{2\Delta\phi}{\delta}\right)\right]-2\sin^2\varphi_1\right\}d\xi$$
$$=(4\pi a/k)^2(\pi/2)\{(1-\tfrac{2}{3}\cos^2\varphi_1)\ln(2\Delta\phi/\delta)$$
$$-2+(14/9)\cos^2\varphi_1\}. \qquad (17)$$

Substituting (16) and (17) in (11) we have the cross section for nearly coplanar events in the case of no screening and $\Delta\phi\gg\delta$:

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)[\ln(2\Delta\phi/\delta)-\tfrac{3}{2}]$$
$$+\epsilon_1\epsilon_2+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[\ln(2\Delta\phi/\delta)-7/3]\}. \qquad (18)$$

The ratio of the number of pairs produced in the plane of polarization to the number produced perpendicular to this plane (the asymmetry ratio) is, therefore, when the energy of the particles is also determined,

$$R(\epsilon_1)=\frac{d\sigma(\epsilon_1,\ \varphi_1=0)}{d\sigma(\epsilon_1,\ \varphi_1=\pi/2)}$$
$$=1+\frac{\tfrac{4}{3}\epsilon_1\epsilon_2[\ln(2\Delta\phi/\delta)-7/3]}{(\epsilon_1^2+\epsilon_2^2)[\ln(2\Delta\phi/\delta)-\tfrac{3}{2}]+\epsilon_1\epsilon_2}. \qquad (19)$$

In particular, for $\epsilon_1=\epsilon_2=\frac{1}{2}k$ we have $\delta=2/k$ and

$$R(\epsilon_1=\tfrac{1}{2}k)=1+\tfrac{2}{3}\frac{\ln(k\Delta\phi)-7/3}{\ln(k\Delta\phi)-1}. \qquad (19a)$$

If one does not distinguish between the energies of the pair particles, then the cross section (18) must be integrated over ϵ_1 from $\epsilon_1=1$ to $\epsilon_1=k-1$ (recall $\delta=k/2\epsilon_1\epsilon_2$), giving

$$d\sigma=(8/3)Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\varphi_1}{2\pi}\{\ln(4k\Delta\phi)$$
$$-41/12+\tfrac{1}{3}\cos^2\varphi_1[\ln(4k\Delta\phi)-4]\}. \qquad (20)$$

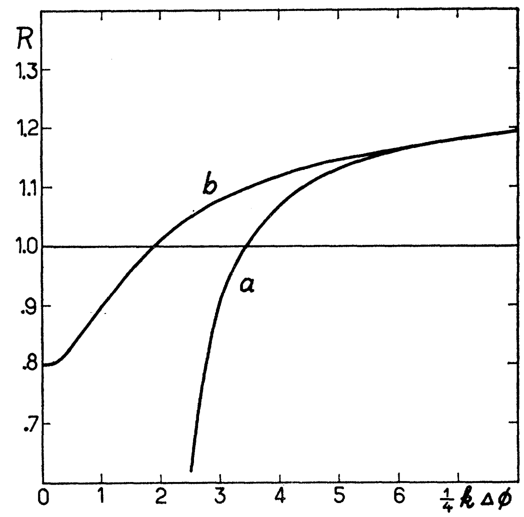

FIG. 2. The asymmetry ratio when the energies of the pair particles are not observed, $R=\int\sigma(\varphi_1=0)d\epsilon_1/\int\sigma(\varphi_1=\frac{1}{2}\pi)d\epsilon_1$, as a function of $\frac{1}{4}k\Delta\phi$ for the case of no screening. Curve a: Approximate calculation [Eq. (21)], valid for $\frac{1}{4}k\Delta\phi\gg1$, $\Delta\phi\ll1$. Curve b: Exact small angle calculation [Eq. (46)], valid for $\Delta\phi\ll1$.

In this case the asymmetry R is given by

$$R=\frac{d\sigma(\varphi_1=0)}{d\sigma(\varphi_1=\pi/2)}=1+\tfrac{1}{3}\frac{\ln(4k\Delta\phi)-4}{\ln(4k\Delta\phi)-41/12}. \qquad (21)$$

Comparing (21) with (19a) it is to be noted that considerably higher asymmetry ratios are obtainable by selecting particles of equal or almost equal energy.

R as given in Eq. (21) is plotted as a function of $\frac{1}{4}k\Delta\phi$ in Fig. 2. Note the comment at the end of Sec. 6.

b. Complete Screening and $\Delta\phi\gg\beta$

For the case of complete screening ($\beta^2\gg\delta^2$, $\beta=Z^{\frac{1}{3}}/111$) and also $\Delta\phi\gg\beta$ one finds, from (11), (14) and (15), corresponding to formulas (18) to (21),

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)[\ln(2\Delta\phi/\beta)-\tfrac{1}{2}]$$
$$+\epsilon_1\epsilon_2+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[\ln(2\Delta\phi/\beta)-\tfrac{3}{2}]\}, \qquad (22)$$

from which we obtain

$$R(\epsilon_1)=1+\frac{\tfrac{4}{3}\epsilon_1\epsilon_2[\ln(2\Delta\phi/\beta)-\tfrac{3}{2}]}{(\epsilon_1^2+\epsilon_2^2)[\ln(2\Delta\phi/\beta)-\tfrac{1}{2}]+\epsilon_1\epsilon_2}. \qquad (23)$$

In particular, for $\epsilon_1=\epsilon_2=\frac{1}{2}k$ we have

$$R(\epsilon_1=\tfrac{1}{2}k)=1+\tfrac{2}{3}\frac{\ln(2\Delta\phi/\beta)-\tfrac{3}{2}}{\ln(2\Delta\phi/\beta)}. \qquad (23a)$$

When the energies of the particles are not observed, the

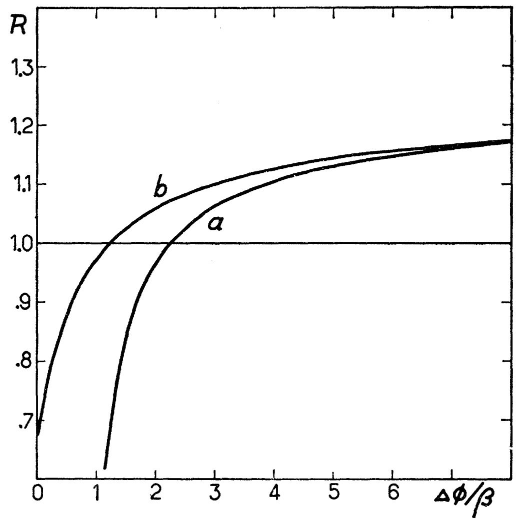

FIG. 3. The asymmetry ratio when the energies of the pair particles are not observed, $R=\int\sigma(\varphi_1=0)d\epsilon_1/\int\sigma(\varphi_1=\frac{1}{2}\pi)d\epsilon_1$, as a function of $\Delta\phi/\beta$ for the case of complete screening. Curve *a*: Approximate calculation [Eq. (25)], valid for $\Delta\phi/\beta\gg1$, $\Delta\phi\ll1$. Curve *b*: Exact small angle calculation [Eq. (36)], valid for $\Delta\phi\ll1$.

cross section is, [integrating (22) over ϵ_1],

$$d\sigma=(8/3)Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\varphi_1}{2\pi}\times\{\ln(2\Delta\phi/\beta)-\tfrac{1}{4}+\tfrac{1}{3}\cos^2\varphi_1[\ln(2\Delta\phi/\beta)-\tfrac{3}{2}]\},\quad(24)$$

and the asymmetry ratio is

$$R=1+\frac{1}{3}\frac{\ln(2\Delta\phi/\beta)-\frac{3}{2}}{\ln(2\Delta\phi/\beta)-\frac{1}{4}}.\quad(25)$$

R as given in Eq. (25) is plotted as a function of $\Delta\phi/\beta$ in Fig. 3. As in the case of no screening, it should be noted that appreciably higher asymmetry ratios may be obtained by selecting pairs of equal or almost equal energy.

IV. CROSS SECTION FROM THE REGION $\Delta\phi\leq|\phi|\leq\pi$

In order to check the results in the previous section we calculate here the contributions to the cross section from the regions $\Delta\phi\leq\phi\leq\pi$ and $-\pi\leq\phi\leq-\Delta\phi$. Adding the result to Eq. (18) or (22) for the cases of no screening and complete screening, respectively, we obtain the cross section for the emission of one of the pair particles in a plane making an angle φ_1 with the plane of polarization of the incident photon, irrespective of the direction of motion of the other pair particle. This cross section is then shown to agree with that obtained from Eq. (10.3) on p. 903 of reference 4, when the latter is integrated over u and summed over spin states.

The contributions to the cross section from the regions $\Delta\phi\leq\phi\leq\pi$ and $-\pi\leq\phi\leq-\Delta\phi$ are again given by Eq. (11), but where now

$$I=\int_0^\infty udu\int_0^\infty vdv\left\{\int_{-\pi}^{-\Delta\phi}+\int_{\Delta\phi}^{\pi}\right\}d\phi\, J^2,$$

and

$$I_{\mathbf{e}}=\int_0^\infty udu\int_0^\infty vdv\left\{\int_{-\pi}^{-\Delta\phi}+\int_{\Delta\phi}^{\pi}\right\}d\phi\, (\mathbf{J}\cdot\mathbf{e})^2.$$

$\mathbf{J}$ is here given by Eq. (3.20) p. 891 of reference 4. For simplicity, however, we shall leave out the Coulomb correction and use the Born approximation value, $\mathbf{J}_{\text{Born}}$, given in Eq. (3). Moreover, since in Sec. 3 we assumed $\Delta\phi\gg\beta$, so that all screening is contained in the region $-\Delta\phi<\phi<\Delta\phi$, we may now put $F(q)=0$ in the present integrals. The results in the present region are thus independent of the amount of screening present. Moreover, for the same reason, $q=q_\perp$ in the present range, so that J^2 reduces to

$$J^2=(4\pi a/k)^2\xi\eta/q_\perp^2.$$

In this way we obtain simply

$$I=2\left(\frac{4\pi a}{k}\right)^2\int_0^\infty udu\int_0^\infty vdv\int_{\Delta\phi}^{\pi}d\phi\,\frac{\xi\eta}{u^2+v^2-2uv\cos\phi}.$$

In the Appendix this is shown to be given by

$$I=(4\pi a/k)^2\pi\{-\ln(2\Delta\phi)+1\}.\quad(26)$$

In the same way we find

$$I_{\mathbf{e}}=2\left(\frac{4\pi a}{k}\right)^2\int_0^\infty udu\int_0^\infty vdv\int_{\Delta\phi}^{\pi}d\phi\,\frac{(\mathbf{u}\cdot\mathbf{e}\xi+\mathbf{v}\cdot\mathbf{e}\eta)^2}{(u^2+v^2-2uv\cos\phi)^2}$$
$$=(4\pi a/k)^2(\pi/2)\{-(1-\tfrac{2}{3}\cos^2\varphi_1)\times\ln(2\Delta\phi)+1-(2/9)\cos^2\varphi_1\}.\quad(27)$$

The contribution to the cross section from the present regions is thus, from Eqs. (11), (26), and (27),

$$d\sigma_{|\phi|>\Delta\phi}=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\times\{(\epsilon_1^2+\epsilon_2^2)[-\ln(2\Delta\phi)+1]+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[-\ln(2\Delta\phi)+\tfrac{1}{3}]\}.\quad(28)$$

Adding this to the contribution from small ϕ we obtain the cross section for emission of one of the pair particles in a plane making an angle φ_1 with the polarization plane, irrespective of the direction of motion of the other pair particle: Thus, for the case of no screening, from Eqs. (18) and (28),

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)[\ln(1/\delta)-\tfrac{1}{2}]+\epsilon_1\epsilon_2+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[\ln(1/\delta)-2]\}. \quad (29)$$

The dependence on $\Delta\phi$ has disappeared, as it should.

Formula (29) is in agreement with Eq. (10.3) of reference 4 (neglecting the Coulomb correction there) when the latter is integrated over u [recall $\xi=(1+u^2)^{-1}$] and summed over electron spin states,

$$d\sigma=2Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\int_0^1\{(\epsilon_1^2+\epsilon_2^2)[2\ln(1/\delta)-1]+2\epsilon_1\epsilon_2+16\epsilon_1\epsilon_2\cos^2\varphi_1u^2\xi^2[\ln(1/\delta)-2]\}d\xi$$

$$=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)[\ln(1/\delta)-\tfrac{1}{2}]+\epsilon_1\epsilon_2+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[\ln(1/\delta)-2]\},$$

which is identical to Eq. (29).

For the case of complete screening we find correspondingly, by adding Eq. (22) to Eq. (28),

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\{(\epsilon_1^2+\epsilon_2^2)[\ln(1/\beta)+\tfrac{1}{2}]+\epsilon_1\epsilon_2+\tfrac{4}{3}\epsilon_1\epsilon_2\cos^2\varphi_1[\ln(1/\beta)-7/6]\}, \quad (30)$$

which again may be shown to be identical to the result obtained from Eq. (10.3) of reference 4, using the value of Γ given there for complete screening (reference 4, p. 897, Eq. (6.34)) neglecting Coulomb correction:

$$\Gamma=\ln(1/\beta\xi)-2,$$

and integrating over u.

These results show that the calculations in Sec. 3 are correct.

V. CROSS SECTION AND ASYMMETRY RATIO FOR COMPLETE SCREENING AND $\Delta\phi\ll1$

In this section we shall calculate the cross sections and asymmetry ratio R, integrating over the region $-\Delta\phi\leq\phi\leq\Delta\phi$ without making use of the approximation $\Delta\phi\gg\beta$ of Sec. 3. For simplicity, we shall consider the case of complete screening here; the case of no screening is treated in the next section.

The quantities I and I_e given in Eqs. (14) and (15) assumed only that $q\ll1$ and $\Delta\phi\ll1$. A partial integration of the logarithmic term reduces the integral in Eq. (14) to

$$I=\left(\frac{4\pi a}{k}\right)^2\pi\alpha\int_0^\infty\frac{du}{(1+u^2)(1+\alpha^2u^2)^{\frac{1}{2}}}$$

$$=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2(\alpha^2-1)}[-\alpha^2+\alpha(2\alpha^2-1)f(\alpha)], \quad (31)$$

where

$$\alpha=\Delta\phi/\beta$$

and

$$f(\alpha)=\cosh^{-1}\alpha/(\alpha^2-1)^{\frac{1}{2}},\quad \alpha>1,$$
$$=1,\quad \alpha=1, \quad (32)$$
$$=\cos^{-1}\alpha/(1-\alpha^2)^{\frac{1}{2}},\quad \alpha<1.$$

Here we take $\cosh^{-1}\alpha\geq0$ and $0\leq\cos^{-1}\alpha\leq\frac{1}{2}\pi$.

By a similar partial integration of the logarithmic term we obtain from Eq. (15),

$$I_e=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2(\alpha^2-1)}\left\{-\alpha^2+\alpha^3f(\alpha)+\frac{(\alpha^2-\frac{3}{2})}{(\alpha^2-1)}\cos^2\varphi_1[\alpha^2-\tfrac{1}{3}\alpha(2\alpha^2+1)f(\alpha)]\right\}. \quad (33)$$

Substituting (31) and (33) in (11) we have the cross section corresponding to Eq. (22):

$$d\sigma=4Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2\frac{d\epsilon_1}{k^3}\frac{d\varphi_1}{2\pi}\frac{1}{2(\alpha^2-1)}\times\left\{(\epsilon_1^2+\epsilon_2^2)[-\alpha^2+\alpha(2\alpha^2-1)f(\alpha)]+2\epsilon_1\epsilon_2[\alpha^2-\alpha f(\alpha)]-4\epsilon_1\epsilon_2\times\frac{(\alpha^2-\frac{3}{2})}{(\alpha^2-1)}\cos^2\varphi_1[\alpha^2-\tfrac{1}{3}\alpha(2\alpha^2+1)f(\alpha)]\right\}. \quad (34)$$

It is easy to see that Eq. (34) goes over into Eq. (22) when $\alpha\gg1$. For equal energies of the pair particles, $\epsilon_1=\epsilon_2=\frac{1}{2}k$, we have, directly from (34), the asymmetry ratio

$$R(\epsilon_1=\tfrac{1}{2}k)=1+(\alpha^2-\tfrac{3}{2})\times[-\alpha+\tfrac{1}{3}(2\alpha^2+1)f(\alpha)]/[(\alpha^2-1)^2f(\alpha)]. \quad (35)$$

$R(\epsilon_1=\frac{1}{2}k)$ as given in (35) is plotted in Fig. 4 as a function of $\alpha=\Delta\phi/\beta$.

For the case in which the energies of the pair particles are not recorded, the integration over ϵ_1 is extremely simple with complete screening, since I and I_e are then independent of energy. Integrating (34) over ϵ_1 we find the asymmetry ratio

$$R=1+(2\alpha^2-3)[-\alpha+\tfrac{1}{3}(2\alpha^2+1)f(\alpha)]/(\alpha^2-1)[-\alpha+(4\alpha^2-3)f(\alpha)]. \quad (36)$$

A plot of R (Eq. (36)) as a function of α is given in Fig. 3, where it is compared with R obtained with the additional approximation $\alpha\gg1$ (Eq. (25)).

VI. CROSS SECTION AND ASYMMETRY RATIO FOR NO SCREENING AND $\Delta\phi\ll1$

The cross section without screening, and assuming only $\Delta\phi\ll1$, may be obtained from Eqs. (12) and (13). It is, however, more convenient to change the variable

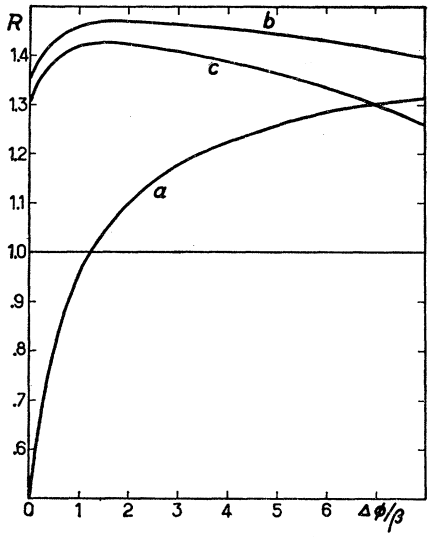

FIG. 4. The asymmetry ratio for equal energies, $\epsilon_1=\epsilon_2=\frac{1}{2}k$, as a function of $\Delta\phi/\beta$ for complete screening. Curve a: Exact small angle calculation [Eq. (35)] valid for $\Delta\phi\ll 1$. Pair particles observed emergent over angular region $2\Delta\phi$. Curves b and c: All pair particles observed except those absorbed by a wedge of angular width $2\Delta\phi\ll 1$. Curve b: $Z=29$ (Cu). Curve c: $Z=78$ (Pt).

of integration in these equations from u to x,

$$x=2\xi-1=(1-u^2)/(1+u^2),$$

and thus

$$q_z=2\delta/(x+1).$$

We then obtain

$$I=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\int_0^1\left\{2\ln\{\tau(1-x^2)^{\frac{1}{2}}+[1+\tau^2(1-x^2)]^{\frac{1}{2}}\}-\frac{\tau(1-x^2)^{\frac{1}{2}}}{[1+\tau^2(1-x^2)]^{\frac{1}{2}}}\right\}dx, \quad (37)$$

and

$$I_{\mathrm{e}}=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\int_0^1\left([1-(1-x^2)\cos^2\varphi_1]\ln\{\tau(1-x^2)^{\frac{1}{2}}+[1+\tau^2(1-x^2)]^{\frac{1}{2}}\}-\frac{\tau(1-x^2)^{\frac{1}{2}}\sin^2\varphi_1}{[1+\tau^2(1-x^2)]^{\frac{1}{2}}}\right)dx, \quad (38)$$

where $\tau=\Delta\phi/(2\delta)$. Integrating the ln term by parts, each of these integrals may be written in the form of elliptic integrals:

$$I=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\tau\int_0^1\frac{(3x^2-1)dx}{(1-x^2)^{\frac{1}{2}}[1+\tau^2(1-x^2)]^{\frac{1}{2}}}, \quad (39)$$

$$I_{\mathrm{e}}=\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\tau\int_0^1\frac{(2x^2-1)dx}{(1-x^2)^{\frac{1}{2}}[1+\tau^2(1-x^2]^{\frac{1}{2}}}+\left(\frac{4\pi a}{k}\right)^2\frac{\pi}{2}\tau\cos^2\varphi_1\int_0^1\frac{(1-2x^2+\frac{1}{3}x^4)dx}{(1-x^2)^{\frac{1}{2}}[1+\tau^2(1-x^2)]^{\frac{1}{2}}}. \quad (40)$$

We have thus

$$I=\left(\frac{4\pi a}{k}\right)^2\left(\frac{\pi}{2}\right)^2[3\mu^{-1}(F_+-F_-)-\mu F_+], \quad (41)$$

$$I_{\mathrm{e}}=\left(\frac{4\pi a}{k}\right)^2\left(\frac{\pi}{2}\right)^2\{2\mu^{-1}(F_+-F_-)-\mu F_+ +\cos^2\varphi_1[\tfrac{1}{9}\mu^{-3}((2+\mu^2)F_+-(2+2\mu^2)F_-)-2\mu^{-1}(F_+-F_-)+\mu F_+]\}, \quad (42)$$

where $\mu=\tau/(1+\tau^2)^{\frac{1}{2}}$ and F_+ and F_- are hypergeometric functions, simply related to the complete elliptic functions of the 1st and 2nd kind:

$$F_+\equiv F(\tfrac{1}{2},\tfrac{1}{2};1;\mu^2)=(2/\pi)K(\mu),$$
$$F_-\equiv F(-\tfrac{1}{2},\tfrac{1}{2};1;\mu^2)=(2/\pi)E(\mu). \quad (43)$$

If we wish to integrate the cross section equation (11)

$$d\sigma=\frac{4}{(2\pi)^4}\frac{e^2}{mc^2}\frac{\hbar}{mc}\frac{d\epsilon_1}{k}d\varphi_1[\tfrac{1}{2}k^2I-2\epsilon_1\epsilon_2I_{\mathrm{e}}], \quad (44)$$

over ϵ_1 $(1\le\epsilon_1\le k-1)$ it is more convenient to return to (39) and (40) than to integrate (41) and (42) directly. Substituting

$$\epsilon_1/k=\tfrac{1}{2}(1+z),$$
$$\tau=\tfrac{1}{4}k\Delta\phi(1-z^2), \quad (45)$$

in (39) and (40) and expanding the square root in an infinite series for $\frac{1}{4}k\Delta\phi<1$, the integrals are all of the form $\int_0^1(1-x^2)^\nu dx=\frac{1}{2}\pi^{\frac{1}{2}}\Gamma(\nu+1)/\Gamma(\nu+\frac{3}{2})$. The resulting infinite series can be summed in closed form in terms of the generalized hypergeometric functions ${}_3F_2$ to give the cross section integrated over final energies:

$$\sigma=\tfrac{1}{8}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 d\varphi_1\{-2\gamma^{-1}[{}_3F_2(-1/2,-1/2,3/2;3/4,5/4;-\gamma^2)-1]$$
$$-4\gamma^{-1}[{}_3F_2(-1/2,-1/2,1/2;3/4,5/4;-\gamma^2)-1]+\tfrac{1}{9}\gamma^{-3}\cos^2\varphi_1[{}_3F_2(-3/2,1/2,1/2;-1/4,1/4;-\gamma^2)-1]$$
$$+\tfrac{1}{9}\gamma^{-3}\cos^2\varphi_1[{}_3F_2(-3/2,-1/2,1/2;-1/4,1/4;-\gamma^2)-1]$$
$$+(512/945)\gamma^3\cos^2\varphi_1\,{}_3F_2(3/2,3/2,5/2;11/4,13/4;-\gamma^2)\}, \quad (46)$$

where

$$\gamma=\tfrac{1}{4}k\Delta\phi.$$

In order to compare with the result of Berlin and Madansky,[2] who consider the case $\Delta\phi=0$, we may expand (46) for $\gamma=\frac{1}{4}k\Delta\phi\ll 1$, giving

$$\sigma=\tfrac{1}{6}Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 d\varphi_1\gamma[1-\tfrac{1}{5}\cos^2\varphi_1 - (1/15)\gamma^2(1-(52/21)\cos^2\varphi_1)+\cdots]. \quad (47)$$

Thus, as $\Delta\phi\rightarrow 0$, we find

$$R=\sigma(\varphi_1=0)/\sigma(\varphi_1=\tfrac{1}{2}\pi)=\tfrac{4}{5}.$$

Berlin and Madansky[2] gave $R=0.814$.

A plot of R (from Eq. (46)) as a function of $\frac{1}{4}k\Delta\phi$ is given in Fig. 2. However, since the functions in (46) converge only for $\frac{1}{4}k\Delta\phi\leq 1$, we have substituted (39) and (40) in (41), and performed the integrations over x and ϵ_1 by means of an IBM 704 computer. For the integration over ϵ_1 we used the variable z as given in (45). $R=\sigma(\varphi_1=0)/\sigma(\varphi_1=\frac{1}{2}\pi)$ was then computed, the values thus obtained demanding only $\Delta\phi\ll 1$ for their validity. In order to see for what values of $k\Delta\phi$ one may use the simple expression for R (Eq. (21), which assumes $1/k\ll\Delta\phi\ll 1$) we have also plotted R as given in Eq. (21) as a function of $\frac{1}{4}k\Delta\phi$ in Fig. 2.

It should be noted from Fig. 2 that the Eq. (21) is only valid for $\frac{1}{4}k\Delta\phi\gtrsim 5$, so that it must assume $k\gg 20$, i.e., $k\gg 10$ Mev. For actual experimental conditions, Eq. (21) is thus quite limited, since for $k\gg 20$ Mev there would in general be considerable screening.

VII. DISCUSSION OF EXPERIMENTAL METHODS

From Fig. 2, which gives the asymmetry when pair particles of all energies are included, the connection between the calculation of Berlin and Madansky[2] and that of Wick[3] is clearly seen. For very small values of $k\Delta\phi$ one is more likely to find a pair in a plane perpendicular to the plane of polarization. In particular, for $\Delta\phi=0$, which is the case considered by Berlin and Madansky, we find with no screening, $R=\frac{4}{5}$. Their result is 0.814. The reason for this slight discrepancy is not clear. For the case of complete screening we find, at $\Delta\phi=0$, $R=\frac{2}{3}$. As one increases the angle $2\Delta\phi$ over which particles are admitted, R increases monotonically, becoming larger than one for $\frac{1}{4}k\Delta\phi>1.89$ in the case of no screening and $\Delta\phi/\beta>1.23$ in the case of complete screening. It is seen that Wick's value, $R=\frac{4}{3}$, is an overestimate.

Wick's result may be understood in the following way: As is well known, the Weizsäcker-Williams method requires the predominance of arbitrarily small momentum transfers for its validity, and is therefore exact in the limit of extremely high photon energies and only when screening is neglected. Under these conditions, as the photon energy is increased more and more of the contribution to the integrated cross section comes from momentum transfers $q\ll 1$,[11] and therefore also from small values of the angle $\phi=\pi-\varphi$ (see Sec. 2). The Weizsäcker-Williams method, which replaces the pair production process in the field of the nucleus, in which there are three particles in the final state, by the pair production process by two photons, in which there are only two particles in the final state, has already affected the angular integration over one of the final state particles, viz. over θ_2 and φ. Since, as mentioned above, the contribution to the cross section comes only from small values of ϕ, the cross section integrated over φ will contain only almost coplanar pairs; thus $\mathbf{p}_2$ is closely confined to the plane of $\mathbf{k}$ and $\mathbf{p}_1$. Thus, to obtain the Weizsäcker-Williams approximation result, the region of integration over φ, viz. $2\Delta\phi$, must be so large that the entire contribution to the cross section is included; thus $\Delta\phi\gg 1/k$, but otherwise $\Delta\phi$ need not be specified. From the work in this paper we have $R>1$ for $\Delta\phi\gg 1/k$ in accordance with Wick's result. On the other hand, the actual numerical value of R in the Weizsäcker-Williams approximation, viz. $\frac{4}{3}$, is valid only in the limit of extremely high energies and by neglecting screening, as mentioned above.

Wick's result, $R=\frac{4}{3}$, would thus be obtained from Eq. (21) if we could put $\ln(4k\Delta\phi)\gg 1$. As this is never the case for real experimental situations one might be tempted to conclude that the method is not as useful as has been supposed, i.e., that for most experimental situations R is quite close to unity. In fact it is interesting to compare this method with the simpler method of recording only one of the pair particles, irrespective of the direction of motion of the other particle.

For lower photon energies, for which there is no screening, we have, integrating (29) over ϵ_1,

$$R'=\frac{\sigma(\varphi_1=0)}{\sigma(\varphi_1=\frac{1}{2}\pi)}=1+\tfrac{1}{3}\frac{\ln(2k)-11/3}{\ln(2k)-25/12}, \quad (48)$$

from which we find $R'=1.00$ at $k=10$ Mev and $R'=1.07$ at $k=15$ Mev.

At higher energies, for which there is complete screening, we have, integrating (30) over ϵ_1, the asymmetry ratio

$$R'=1+\tfrac{1}{3}\frac{\ln(1/\beta)-7/6}{\ln(1/\beta)+3/4}. \quad (49)$$

Using $\beta=Z^{\frac{1}{3}}/111$, R' is seen to be smallest for the heaviest elements. For uranium $R'=1.17$ and thus $R'\geq 1.17$.

From Fig. 3 it can be seen that at high energies this experimental method, i.e., observing only one of the particles, therefore gives a higher value for the asymmetry than can be obtained by observing coplanar pairs.[12] This may be understood if one considers curve

[11] Reference 9, Sec. 7, pp. 537–538.

[12] It should be noted that if one could observe pairs emergent over an extremely narrow angular range $2\Delta\phi$ one could obtain a larger asymmetry ratio than by observing only one of the particles: From Fig. 3, for $\Delta\phi/\beta<0.4$ we have $1.17\leq(1/R)\leq 1.5$. For $Z=29$ e.g., $\Delta\phi/\beta<0.4$ implies $\Delta\phi<0.63°$. The observation of pairs within such a small angular range seems, however, to meet with experimental difficulties.

(*a*) in Fig. 4 (complete screening and $\epsilon_1=\epsilon_2=\frac{1}{2}k$). We see that exactly coplanar pairs are predominantly perpendicular to the photon polarization plane. As the angle $2\Delta\phi$ over which pairs are accepted is increased beyond 2.45β, the pairs are mostly parallel to the polarization plane. This suggests the conclusion that is in fact borne out by a Born approximation calculation[13] of the ratio of the cross section differential in φ, $R(\phi)=d\sigma(\varphi_1=0)/d\sigma(\varphi_1=\frac{1}{2}\pi)$, namely that for complete screening $R(\phi)<1$ for $\phi<0.4\beta$, $R(\phi)>1$ for $0.4\beta<\varphi<\frac{1}{2}\pi$, and again $R(\phi)<1$ for $\frac{1}{2}\pi<\phi<\pi$. In fact, for $\epsilon_1=\epsilon_2=\frac{1}{2}k$ and $Z=29$, $R(\phi)$ attains its maximum value of 2.05 at $\phi=32.3°$. Thus one obtains a larger asymmetry ratio by observing the two pair particles at a fixed azimuthal angle $\varphi_1-\varphi_2=147.7°$; the low intensity of such pairs would, however, make such an experiment difficult. The asymmetry ratio $R'=1.35$ (for $\epsilon_1=\epsilon_2=\frac{1}{2}k$, $Z=29$) obtained by observing only one of the pair particles, is an average of $R(\phi)$ properly weighted over the number of pairs at each angle. For an experiment what is needed is a value of $|R-1|/(R+1)$ as large as possible. This may be obtained in two ways: Either by accepting only those pairs coming from a very narrow angular region $2\Delta\phi$ for which $R(\phi)<1$, predominantly, or, if this is not feasible, by rejecting some or all of these pairs, thus leaving a predominance of pairs for which $R(\phi)>1$.

Thus we may consider three methods for measuring the polarization of the photon beam. First, as proposed by Yang, by observing coplanar pairs. This method has the difficulties first pointed out by Wick and discussed here in detail. Second, that of observing only one of the pair particles. This is undoubtedly the simplest from the experimental standpoint, but gives, as we have noted, an asymmetry ratio which is a weighted average of $R(\phi)$ rather than a value close to the largest that is theoretically obtainable. Finally, we may speak of the wedge technique suggested in the previous paragraph, by which some of the pairs are either absorbed by a narrow wedge or rejected by counters in anticoincidence. This method allows one to obtain a higher asymmetry ratio than is obtained by observing only one particle, but with the loss of only a fairly small fraction of the intensity. The calculations pertinent to this method are given in the following section.

VIII. WEDGE METHOD

The cross section for observing pairs outside the angular range $-\Delta\phi\leq\phi\leq\Delta\phi$ is obtained directly by subtracting Eq. (34) from Eq. (30). For simplicity we will consider here only the case of complete screening and equal energies, $\epsilon_1=\epsilon_2=\frac{1}{2}k$. The Coulomb correction is easily included since it only occurs for large angles ϕ. Thus the effect of the Coulomb correction is to replace $\ln(1/\beta)$ in Eq. (30) by[14] $\ln(1/\beta)-f(Z)$ where $f(Z)$ is given by Davies *et al.*[15] Thus one obtains the asymmetry ratio

$$R_w=1+\left\{\frac{2}{3}\left(\ln\frac{1}{\beta}-f(Z)-7/6\right)+\frac{(\alpha^2-\frac{3}{2})}{(\alpha^2-1)^2}\left[\alpha^2-\tfrac{1}{3}\alpha(2\alpha^2+1)f(\alpha)\right]\right\}\Big/\left\{\ln\frac{1}{\beta}-f(Z)+1-\alpha f(\alpha)\right\}. \quad (50)$$

R_w as a function of $\alpha=\Delta\phi/\beta$ is plotted in Fig. 4, curves (*b*) and (*c*), for Cu and Pt. The presence of a wedge is seen there to increase the value of R_w above that obtained when only one of the particles is observed, viz. $R_w(\alpha=0)$. It should be noted, however, that R_w decreases with increasing α above $\alpha=2$. This is to be expected since for large values of α increasing the angle of the absorbing wedge removes pairs which are nearly parallel to the polarization plane.

The effect of the Coulomb correction is to reduce the asymmetry ratio by an almost constant amount between $\alpha=0$ and $\alpha=2.0$, viz., 0.003 for Cu and 0.025 for Pt.

ACKNOWLEDGMENTS

We are grateful to Dr. H. Schopper who brought the apparent discrepancy between the results of T. H. Berlin and L. Madansky and those of G. C. Wick to our attention, to Dr. M. Danos for several very helpful discussions, and to the staff of the Laboratory of Nuclear Studies of Cornell University for discussions which, during the course of a Journal Club talk, led to the wedge method considered in the last two sections of this paper.

We are indebted to Dr. M. Berger for performing the numerical calculations.

APPENDIX

In the integral

$$I=2\left(\frac{4\pi a}{k}\right)^2\int_0^\infty udu\int_0^\infty vdv\int_{\Delta\phi}^{\pi}d\phi\frac{\xi\eta}{u^2+v^2-2uv\cos\phi},$$

occurring in Sec. 4 we introduce polar variables ρ and χ by

$$u=\rho\cos\chi,\quad v=\rho\sin\chi.$$

The ρ and ϕ integrations may then be performed

[13] This calculation was performed without using the small momentum transfer approximations of the present paper (viz., $u-v\ll1$ and $\pi-\varphi=\phi\ll1$).

[14] Reference 4, p. 897 Eq. (6.34).

[15] Handel Davies, H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 788 (1954). Note p. 791 Eqs. (36)–(39).

separately:

$$I=\left(\frac{4\pi a}{k}\right)^2\int_0^{\frac{1}{2}\pi}d\chi\,\sin 2\chi\int_0^\infty\frac{\rho d\rho}{(1+\rho^2\sin^2\chi)(1+\rho^2\cos^2\chi)}\times\int_{\Delta\phi}^{\pi}\frac{d\phi}{1-\sin 2\chi\cos\phi}$$

$$=-2\left(\frac{4\pi a}{k}\right)^2\int_0^{\frac{1}{2}\pi}d\chi\frac{\tan 2\chi}{|\cos 2\chi|}\times\ln\tan\chi\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{(1+\sin 2\chi)}{|\cos 2\chi|}\frac{\Delta\phi}{2}\right)\right\}.$$

Since $\Delta\phi\ll 1$ we have replaced $\tan(\frac{1}{2}\Delta\phi)$ by $\frac{1}{2}\Delta\phi$ in the last integrand.

Introducing the new variable $t=\tan\chi$ and noting that

$$\frac{\tan 2\chi}{\cos 2\chi}\ln\tan\chi d\chi=\frac{2t\ln t}{(1-t^2)^2}dt=-\frac{1}{2}\frac{d}{dt}\left\{\frac{t^2\ln t^2}{1-t^2}+\ln|1-t^2|\right\}dt,$$

a partial integration gives

$$I=-\left(\frac{4\pi a}{k}\right)^2\int_0^\infty dt\left\{\frac{t^2\ln t^2}{1-t^2}+\ln|1-t^2|\right\}\times\frac{\Delta\phi}{(1+t)^2(\frac{1}{2}\Delta\phi)^2+(1-t)^2}.$$

Since $\Delta\phi\ll 1$ the contribution to the integral comes only from a small region around $t=1$. Thus we obtain, with $1-t=y\Delta\phi$ and neglecting terms of relative order $\Delta\phi$,

$$I=-\left(\frac{4\pi a}{k}\right)^2\int_{-\infty}^\infty dy\,\{-1+\ln(2\Delta\phi)+\ln|y|\}\frac{1}{1+y^2}$$

$$=(4\pi a/k)^2\pi\{-\ln(2\Delta\phi)+1\},$$

which is the result used in Eq. (26) in the text. The same method may be used to obtain the integral I_e given in Eq. (27).

PHYSICAL REVIEW VOLUME 126, NUMBER 1 APRIL 1, 1962

Coordinate Covariance and the Particle Spectrum*

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(Received July 11, 1961, revised manuscript received November 20, 1961)

An attempt is made to find an analog, for the quantum mechanics of non-Euclidean space-time, to the classification of representations of the Lorentz group. The difficulty of obtaining any such classification in terms of curvilinear coordinates is pointed out, and the use of a higher-dimensional set of pseudo-Euclidean coordinates is chosen as an alternative mode of attack. A class of representations then follows easily. On the basis of an intuitive approximation it is found that spectra of elementary particles, with conservation of quantities of the nature of isotopic spin, seem to arise from these representations.

I. INTRODUCTION

Considerations of covariance have often been of value to the development of a theory[1,2]; perhaps this will also be the case for the generalization of quantum mechanics to allow for the curvature of space. It is, of course, true that such generalizations are always expressed in covariant language. However, there has been no attempt to enumerate the modes of covariance allowed to a physical quantity—that is, the set of representations of the group of coordinate transformations—such as has been done for the case of Minkowski space, where this group is the Lorentz group.[3] The purpose of the present article is to consider an approach which may shed light on this problem, and at the same time does suggest a possible origin for the observed multiplet structure of the elementary particle spectrum.

The following considerations are based on the familiar assumption:

> Any physical system (in particular, an elementary particle) can be representated by a wave function belonging to some irreducible representation of the group of all coordinate transformations.

This is a combination of the quantum-mechanical assumption that systems may be represented by wave functions, with the requirement that a system describable in one coordinate system be describable also in any other coordinate system[4]; the "Schrödinger picture" is

* This work has been supported by the Air Force Office of Scientific Research.
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[1] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1950).

[2] E. P. Wigner, Nuovo cimento **3**, 517 (1956).

[3] E. P. Wigner, Ann. Math. **40**, 149 (1939); Iu. M. Shirokov, JETP **6**, 919, 929 (1958).

[4] The requirement that the group of wave function transformations be a "representation" of the group of coordinate transformations means essentially that if wave function transformations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ correspond to coordinate transformations T_1, T_2,