High-Energy Behavior of Nucleon Electromagnetic Form Factors*

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Theoretical implications of the suggestion that the observed nucleon electromagnetic form factors indicate the existence of a nucleon "core" are discussed. On the basis of physical arguments concerning the nature of such a core, it is shown that, for the neutron, both the charge form factor, $F_{ch}^n(q^2)$, and the magnetic form factor, $F_{mag}^n(q^2)$, must vanish as q^2, the invariant momentum transfer, increases without limit. On the other hand, for the proton $F_{ch}^p(q^2) \to Z_2^{(s)}$ and $F_{mag}^p(q^2) \to Z_2^{(s)}/2M$, where $Z_2^{(s)}$ is the wave function renormalization constant for strong interactions, which is a measure of the probability of the "core state." In terms of the Dirac form factor, $F_1(q^2)$, and the Pauli form factor, $F_2(q^2)$, these results read $F_1^n(q^2) \to 0$, $F_1^p(q^2) \to Z_2^{(s)}$, and $q^2F_2(q^2) \to 0$ for both neutron and proton. The results for $F_1(q^2)$ are the same as those obtained by Hiida, Nakanishi, Nogami, and Uehara. The other result implies the existence of a relationship which may be used to eliminate one parameter in the analysis of F_2. The generality of the interpretation of F_{ch} and F_{mag} as Fourier transforms of distributions of charge and magnetization, respectively, is demonstrated in the Appendix.

1. INTRODUCTION

RECENT experimental results[1] on the electromagnetic form factors of the nucleon have given some indication that the Dirac form factors $F_1^n(q^2)$, and $F_1^p(q^2)$ of the neutron and proton may become constant for large values of the invariant momentum transfer. q^2. This has been related[1,2] to the existence of a core in the charge distribution of the nucleon, especially by Olson, Schopper, and Wilson, and by Littauer *et al.*[2] It is the purpose of this note to examine more closely the theoretical aspects of this behavior of the form factors and their physical interpretation. We shall find that if $F_1^p(q^2)$ does indeed become constant, the value may be used to determine the wave-function renormalization constant $Z_2^{(s)}$ for strong interactions. On the other hand, our interpretation of $F_1^n(q^2)$ indicates that it must vanish for very large q^2, a result which may not disagree with the data even at presently available values of q^2 since the determination of F_1^n at large q^2 is still rather uncertain, and the suggested decrease in F_1^n may take place very slowly.

We also find that the Pauli form factors $F_2(q^2)$ vanish *more* strongly than q^{-2} for both neutron and proton. This somewhat surprising result implies a useful relationship between the parameters which are often used in analyzing the data, as is indicated in Sec. 4.

2. HIGH-ENERGY LIMITS OF FORM FACTORS

The form factors provide a measure of the charge and current distribution in the nucleon. The behavior of $F(q^2)$ in the limit as $q^2 \to \infty$ is then related to the behavior at the origin of the distribution in configuration space. If

$$\lim_{q^2\to\infty} F(q^2) = \text{const} \neq 0 \qquad (1)$$

the spatial distribution contains a δ function.

The existence of such a δ function in the distribution may be understood as follows: If the nucleon state is expressed as a superposition of products of nucleon and pion states, in the sense described in Appendix 1, the only term that can give rise to a δ function is the term corresponding to a single nucleon. For every other term the charges and currents are spread out due to the relative motion of the particles in the center-of-mass system.

The constant appearing in Eq. (1) is therefore expected to be determined by the probability for the occurrence of a single "bare" nucleon in the physical nucleon state. This probability is given by the wave-function renormalization constant $Z_2^{(s)}$, where the superscript (s) indicates that only strong interaction effects are included. Electromagnetic effects are taken into account here only to the lowest order.

If $F_{ch}(q^2)$ is the form factor measuring the charge distribution, we then expect that

$$\lim_{q^2\to\infty} F_{ch}(q^2) = QZ_2^{(s)}, \qquad (2)$$

where $Q=0$ or 1 is the charge of the bare nucleon in units of the proton charge. The charge is not subject to renormalization because electromagnetic corrections are not to be included. Furthermore, the magnetic form factor should satisfy the condition

$$\lim_{q^2\to\infty} F_{mag}(q^2) = Z_2^{(s)}Q/2M, \qquad (3)$$

where $eQ/2M$ is the magnetic moment of the bare nucleon.

The use of the physical mass in Eq. (3) is justified in Appendix 1. In effect, mass renormalization counter terms have been introduced for the strong interactions in such a way that the bare nucleon has the same mass as the physical nucleon.

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[1] D. N. Olson, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters 6, 286 (1961); R. Hofstadter, C. de Vries, and R. Herman, Phys. Rev. Letters 6, 290 (1961); R. M. Littauer, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters 7, 141 (1961); F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. 124, 1623 (1961).

[2] R. Hofstadter and R. Herman, Phys. Rev. Letters 6, 293 (1961); R. M. Littauer, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters 7, 144 (1961).

It is interesting to note that Eqs. (2) and (3) suggest the possibility of a measurement of the renormalization constant $Z_2^{(s)}$. Furthermore this, in turn, suggests a way to determine whether or not the nucleon is a fundamental particle or a composite particle. If $Z_2^{(s)}$ is finite, we would be inclined to the former view whereas, if it is zero, the latter would seem more likely.

There will certainly be some difficulty about deciding what values of q^2 are large enough to justify the use of Eqs. (2) and (3). One may try to use the purely empirical criterion that q^2 is large enough when the form factor seems to be constant. However, this condition may be misleading. Let us suppose that the charge distribution contains an extended core of radius $R \ll m_\pi^{-1}$. We expect that for $q^2 \approx R^{-2}$, this core will manifest itself by a variation in the form factor with q^2. But the rate of variation will be very small, and becomes smaller with smaller R^2. This can be seen from the relationship

$$R^2 \approx -6F'(q^2)/F(q^2), \quad (4)$$

where the prime denotes differentiation with respect to q^2. Equation (4) serves to define the radius of the core which governs the variation of F at the given value of q^2. If, for example, we insert a radius $R \approx (2M)^{-1}$, which might be associated with nucleon pair effects, we find

$$F'(q^2)/F(q^2) \approx -2\times 10^{-3}\ \mathrm{f}^2. \quad (5)$$

With measurement errors of the order of those characterizing the recent data, it would be very difficult indeed, to detect the existence of so small a rate of change of $F(q^2)$ as that indicated by Eq. (5) without extending the experiments to considerably higher values of q^2. If, on the other hand, one goes to very large values of q^2, $q^2 \gg 4M^2$, the electromagnetic radiative corrections are expected to become large so that it may no longer be possible to extract reliable values of the form factors from the data. We may hope[3,4] that there is a region of q^2 between these two so that Eqs. (2) and (3) have a domain of validity for q^2 small enough that radiative corrections can be ignored. In fact, Eqs. (2) and (3) have a clear meaning only in a theory containing a cutoff.[5] If the cutoff is denoted by Λ^2, then F_{ch} and F_{mag} are both functions of Λ^2 as well as q^2, and $Z_2^{(s)}$ is also a function of Λ^2. In the sense of current theories, the former functions are regular for $\Lambda^2 \to \infty$ but the latter is not. The meaning of $q^2 \to \infty$ here is then that

$$q^2 \gg \Lambda^2,$$

a condition which may be satisfied even for values of q^2 for which the electromagnetic corrections may be neglected, if the physical cutoff of the strong coupling is not too large. It is on this basis that the conclusions to be drawn below would be useful.

3. INTERPRETATION FOR NEUTRON AND PROTON

It has been indicated by Ernst, Sachs, and Wali[6] that the distribution of charge and magnetization in the nucleon is given by the form factors

$$F_{\mathrm{ch}}(q^2) = F_1(q^2) - (q^2/2M)F_2(q^2), \quad (6)$$

and

$$F_{\mathrm{mag}}(q^2) = F_1(q^2)/2M + F_2(q^2), \quad (7)$$

where F_1 and F_2 are the usual Dirac and Pauli form factors. The distributions given by Eqs. (6) and (7) are a measure of the interaction of the nucleon with weak, inhomogeneous, static, electric and magnetic fields. This is demonstrated in Appendix 2 by showing, by means of the method of reference 6, that $F_{\mathrm{ch}}(\mathbf{q}^2)$ and $F_{\mathrm{mag}}(\mathbf{q}^2)$ are, respectively, the Fourier transforms of the distribution of charge and the distribution of magnetization. It is to be noted that to obtain the Fourier transform of the spatial distribution, the invariant variable q^2 is replaced by the square of the three-vector, $\mathbf{q}^2$. This is equivalent to evaluating the form factors in the Breit frame. The relativistic behavior manifests itself here by the behavior at very small distances, or large $\mathbf{q}^2$, as expected. Thus, the higher the degree of the inhomogeniety in the static field, the larger the value of the argument (q^2) at which the form factors must be known.

From Eq. (A27) it follows that the interaction of the nucleon with a static electric field is completely determined by $F_{\mathrm{ch}}(q^2)$ and the interaction with a static magnetic field, by $F_{\mathrm{mag}}(q^2)$. We note in particular that the condition $F_1(\infty) = \text{constant}$ is *not* sufficient to establish the existence of a core in the charge distribution as is often assumed. The required condition is $F_{\mathrm{ch}}(\infty) = \text{const}$.

Let us first apply the conditions Eq. (2) and Eq. (3) to the neutron. Since the charge and magnetic moment of the bare neutron are expected to vanish ($Q=0$), we find that

$$\lim_{q^2\to\infty} [F_1^n(q^2) - (q^2/2M)F_2^n(q^2)] = 0, \quad (8)$$

and

$$\lim_{q^2\to\infty} [F_1^n(q^2)/2M + F_2^n(q^2)] = 0. \quad (9)$$

The corresponding conditions on F_1^n and F_2^n are then, as $q^2 \to \infty$,

$$F_1^n(q^2) \to 0, \quad (10)$$

and

$$q^2 F_2^n(q^2) \to 0. \quad (11)$$

Thus, F_2^n is required to vanish quite strongly while F_1^n may vanish very slowly. The scatter in the data and the ambiguities in their interpretation[7] for the case of

[3] S. D. Drell and S. Fubini, Phys. Rev. **113**, 741 (1959); Yung-Sui Tsai, Phys. Rev. **122**, 1898 (1961).

[4] See also the discussion following Eq. (27) of G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958).

[5] See, for example, M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[6] F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960).

[7] L. Durand, III, Phys. Rev. Letters **6**, 631 (1961); Phys. Rev. **123**, 1393 (1961). N. K. Glendenning and G. Kramer, Phys. Rev. Letters **7**, 471 (1961); W. R. Theis, Phys. Rev. Letters **8**, 45 (1962).

the neutron are so great that it cannot be said whether or not the conditions Eq. (10) and Eq. (11) are being met by the recent experimental results.

In the case of the proton, the conditions [Eqs. (2) and (3)] read ($Q=1$)

$$\lim_{q^2\to\infty}[F_1{}^p(q^2)-(q^2/2M)F_2{}^p(q^2)]=Z_2{}^{(s)}, \qquad (12)$$

and

$$\lim_{q^2\to\infty}[F_1{}^p(q^2)/2M+F_2{}^p(q^2)]=Z_2{}^{(s)}/2M. \qquad (13)$$

It is evident that as $q^2\to\infty$

$$q^2F_2{}^p(q^2)\to 0. \qquad (14)$$

Furthermore,

$$F_1{}^p(\infty)=Z_2{}^{(s)}. \qquad (15)$$

4. DISCUSSION

The results of Eqs. (10) and (15) for $F_1(q^2)$ are equivalent to those obtained by Hiida *et al.*[8] They obtain the results in a quite different manner but also give an interpretation which is essentially the same as the one presented here. In doing so, however, they have made the incorrect assumption that the charge distribution is given by $F_1(q^2)$, hence they obtain no corresponding information about $F_2(q^2)$.

The results for $F_1(q^2)$ are also similar to those obtained by Gell-Mann and Zachariasen[9] who show that if $Z_2(\Lambda^2)$ is the value of Z_2 for a theory with cutoff at Λ^2, then

$$\lim_{q^2\to\infty}[F_1(q^2)-Z_2(q^2)]=0. \qquad (16)$$

On the other hand, in these terms our results read

$$\lim_{q^2\to\infty}F_1(q^2,\Lambda^2)=Z_2(\Lambda^2), \qquad (17)$$

where the connection between the $F_1(q^2)$ appearing in Eq. (16) and $F_1(q^2,\Lambda^2)$ appearing in Eq. (17) is simply

$$F_1(q^2)=\lim_{\Lambda^2\to\infty}F_1(q^2,\Lambda^2). \qquad (18)$$

The possibility of reordering the limits to obtain Eq. (17) seems to be implicit in the work of Gell-Mann and Zachariasen, from which it follows that their results are also equivalent to ours for F_1 although they were also obtained in a quite different manner. Again, they do not obtain a condition on $F_2(q^2)$.

The result, Eq. (14), for $F_2(q^2)$ appears to be new and is somewhat surprising. It is stronger than necessary for the validity of unsubtracted dispersion relations.[10] In fact, it implies that

$$\int dq^2\, \mathrm{Im}F_2(q^2)=0, \qquad (19)$$

if $F_2(q^2)$ does indeed satisfy a dispersion relation.

In terms of the analysis of $F_2(q^2)$ as a sum of pole contributions, as in the Clementel-Villi[11] analysis used by Hofstader and Herman,[2] Eq. (19) imposes a condition on the constants: The sum of the residues must vanish. This condition is imposed in addition to the requirement that $F_2(q^2)$ should contain no constant term for either the proton or neutron. Neither of these requirements is satisfied by the Hofstader-Herman or Littauer, Schopper, and Wilson[2] analyses of the data, nor is it implied by these authors that the analysis is to be extended to larger values of q^2. However, an analysis of even the available data in terms of several pole terms might turn out to agree with our conclusions. On the other hand, it must be kept in mind that, particularly in view of the uncertainties in the data, the experiments may not have been carried to high enough values of q^2 to allow us to ascertain the asymptotic behavior.

The Hofstader-Herman analysis of $F_1{}^n(q^2)$ is also inconsistent with Eq. (10), but that of Littauer *et al.* in terms of a spread-out core is consistent with this condition. The fact that both seem to fit the data is a good illustration of the ambiguity in the analysis.

In view of the inadequacies of the Clementel-Villi analyses noted above, the results obtained in this way for $F_2{}^p(q^2)$ cannot be taken very seriously. However, it is just by such an analysis that, in principle at least, we can hope to determine $Z_2{}^{(s)}$. Therefore we note, with many reservations, that the results given by Hofstader and Herman would indicate

$$Z_2{}^{(s)}\approx 0.12.$$

We may remark that the asymptotic conditions obtained here are not directly related to those obtained by Evans,[12] and by Drell and Zachariasen,[13] who show that $F_1(q^2)\to 0$ if $Z_3{}^{(\mathrm{em})}$, the renormalization constant of the photon propagator, is different from zero. Since the electromagnetic radiative corrections are essential to their argument, it presumably involves values of q^2 much larger than those considered here, i.e., large enough for electromagnetic corrections to the form factors to be important.[4]

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[8] I. K. Hiida, N. Nakanishi, Y. Nogami, and M. Uehara, Progr. Theoret. Phys. (Kyoto) **22**, 247 (1959).
[9] M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065 (1961).
[10] G. F. Chew, University of California Radiation Laboratory Report UCRL-8194 (unpublished)· G. F. Chew *et al.*, reference 4; P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).
[11] E. Clementel and C. Villi, Nuovo cimento **4**, 1207 (1956).
[12] L. E. Evans, Nuclear Phys. **17**, 163 (1960).
[13] S. D. Drell and F. Zachariasen, Phys. Rev. **119**, 463 (1960).

APPENDIX I

It is the purpose of this Appendix to demonstrate in what sense we are considering the expansion of the nucleon state vector in products of nucleon and pion states. We use here a method introduced by Ernst.[14]

The form factors are defined in terms of the matrix element $\langle p'|j_\mu(0)|p\rangle$, where $j_\mu(x)$ is the current density operator in Heisenberg representation and $|p\rangle$, is the physical nucleon state of four-momentum p. We may use the method of Yang and Feldman[15] to define a free-field operator $\psi^\tau(x)$ which is equal to the Heisenberg field $\psi(x)$ when x is on τ. We denote the unitary transformation from τ to τ' by

$$\psi^{\tau'}(x)=U(\tau',\tau)\psi^\tau(x)U(\tau,\tau'). \quad \text{(A1)}$$

Now, since $j_\mu(x)=j_\mu{}^\tau(x)$ when x is on τ, we may write

$$j_\mu(0)=j_\mu{}^0(0). \quad \text{(A2)}$$

Furthermore

$$j_\mu{}^0(0)=U(0,-\infty)j_\mu{}^{\text{in}}(0)U(-\infty,0), \quad \text{(A3)}$$

where $j_\mu{}^{\text{in}}$ is the "in-field" operator referring to $\tau=-\infty$. Thus

$$\langle p'|j_\mu(0)|p\rangle = \langle p'|U(0,-\infty)j_\mu{}^{\text{in}}(0)U(-\infty,0)|p\rangle. \quad \text{(A4)}$$

We may define a complete set of Heisenberg states, $|\alpha,\text{in}\rangle$, in accordance with the procedure of Lehmann, Symanzik, and Zimmermann.[16] Note that $|p\rangle=|p,\text{in}\rangle$ is one of these states. If we define an auxiliary state

$$\Psi_p=U(-\infty,0)|p\rangle, \quad \text{(A5)}$$

we may expand in terms of the complete set

$$\Psi_p=\textstyle\sum_\alpha C_{\alpha p}|\alpha,\text{in}\rangle. \quad \text{(A6)}$$

It is in this sense that the nucleon state may be described as a superposition of products of physical nucleon and pion states, since the $|\alpha,\text{in}\rangle$ are just such products. Furthermore, we note that

$$\langle p'|j_\mu(0)|p\rangle=\langle\Psi_p{}'|j_\mu{}^{\text{in}}(0)|\Psi_p\rangle, \quad \text{(A7)}$$

and that $j_\mu{}^{\text{in}}(0)$ is just a combination of creation and annihilation operators for the one-particle states comprising $|\alpha,\text{in}\rangle$. Hence, by means of the expansion Eq. (A6), the matrix elements may be described directly in terms of the currents associated with the Fock wave functions which are superpositions of the states $|\alpha,\text{in}\rangle$. In this way we may obtain a description of the charge and current distribution in configuration space.

There is a direct relationship between the expansion coefficients Eq. (A6) and the expansion of the physical nucleon state in terms of bare states. The bare state Φ_α is here defined as that solution of the non-interacting field problem having the same quantum numbers as the physical state $|\alpha,\text{in}\rangle$. In particular, mass and energy counter terms are introduced[17] so that Φ_α has the same energy spectrum as $|\alpha,\text{in}\rangle$.

To obtain the desired relationship we note that the state $|\alpha,\text{in}\rangle$ in interaction representation is

$$\Psi_\alpha{}^I(t)=U(0,t)|\alpha,\text{in}\rangle. \quad \text{(A8)}$$

Now

$$\lim_{t\to-\infty}\Psi_\alpha{}^I(t)=\Phi_\alpha. \quad \text{(A9)}$$

Hence,

$$\Phi_\alpha=U(0,-\infty)|\alpha,\text{in}\rangle, \quad \text{(A10)}$$

or

$$|\alpha,\text{in}\rangle=U(-\infty,0)\Phi_\alpha. \quad \text{(A11)}$$

Thus, if the physical nucleon state $|p\rangle$ is expanded in bare states

$$|p\rangle=\textstyle\sum_\alpha C_{\alpha p}{}'\Phi_\alpha, \quad \text{(A12)}$$

we have

$$\Psi_p=U(-\infty,0)|p\rangle=\textstyle\sum_\alpha C_{\alpha p}{}'|\alpha,\text{in}\rangle, \quad \text{(A13)}$$

according to Eq. (A11). Therefore, comparing Eq. (A6), we find

$$C_{\alpha p}{}'=C_{\alpha p}. \quad \text{(A14)}$$

In particular,

$$C_{pp}=(Z_2)^{\frac{1}{2}},$$

is the overlap of the physical one-nucleon state with the bare one-nucleon state.

APPENDIX II

The generality of the interpretation of $F_{\text{ch}}(q^2)$ and $F_{\text{mag}}(q^2)$ as Fourier transforms of spatial distributions of charge and magnetization is demonstrated by considering an arbitrary moment of the charge or current distribution. As noted in reference 6, it is convenient to determine the distribution in a nucleon state described by a wave packet $g(\mathbf{p})$ in momentum space. The moments are calculated for an arbitrary packet and then the proton is brought to rest by taking the limit

$$|g(\mathbf{p})|^2\to\delta(\mathbf{p}). \quad \text{(A15)}$$

The terms resulting from the structure of the wave packet rather than the structure of the nucleon are not of interest here and are therefore dropped. Any term involving a derivative of $g(\mathbf{p})$ is of this type.

We define a moment of Nth order of the distribution, with $N=\alpha+\beta+\gamma$, by

$$M_{\alpha\beta\gamma}{}^\mu=\int d^3p'\int d^3p\, g^*(\mathbf{p}')g(\mathbf{p}) \times\int d^3x\, x_1{}^\alpha x_2{}^\beta x_3{}^\gamma\langle p'|j_\mu(x)|p\rangle, \quad \text{(A16)}$$

where $j_\mu(x)$ is the current density operator of the system

[14] F. J. Ernst, Jr., thesis, University of Wisconsin, 1958 (unpublished).

[15] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[16] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

[17] It is at this point that the mass renormalization is used in defining the bare states so that they have the physical mass.

in Heisenberg representation. The usual expression for the matrix element takes the form[18]

$$\langle p'|j_\mu(x)|p\rangle=(2\pi)^{-3}\times\exp(-iq\cdot x)\bar{u}(\mathbf{p}')F_\mu(\mathbf{q},q_0)u(\mathbf{p}),\quad\text{(A17)}$$

with

$$q_\lambda=p_\lambda'-p_\lambda,\quad\text{(A18)}$$

and

$$F_\mu(\mathbf{q},q_0)=ie[\gamma_\mu F_1(q^2)-\sigma_{\mu\nu}q_\nu F_2(q^2)].\quad\text{(A19)}$$

The spinor $u(\mathbf{p})$ is normalized so that $\bar{u}(\mathbf{p})u(\mathbf{p})=1$. The spin symbol has been suppressed for simplicity. Equation (17) may be inserted into Eq. (A16) and the substitutions $\mathbf{p}'=\mathbf{P}+\frac{1}{2}\mathbf{q}$, $\mathbf{p}=\mathbf{P}-\frac{1}{2}\mathbf{q}$ made to give

$$M_{\alpha\beta\gamma}{}^\mu=(2\pi)^{-3}\int d^3P\int d^3q\, g^*(\mathbf{P}+\tfrac{1}{2}\mathbf{q})\times g(\mathbf{P}-\tfrac{1}{2}\mathbf{q})\int d^3x\, x_1{}^\alpha x_2{}^\beta x_3{}^\gamma\exp(-iq\cdot x)\times\bar{u}(\mathbf{P}+\tfrac{1}{2}\mathbf{q})F_\mu(\mathbf{q},q_0)u(\mathbf{P}-\tfrac{1}{2}\mathbf{q}).$$

By the usual procedure of differentiating the exponential with respect to q_i to obtain a factor x_i, and integrating by parts, we find

$$M_{\alpha\beta\gamma}{}^\mu=i^{-N}\int d^3q\,\delta(\mathbf{q})\int d^3P\frac{\partial^\alpha}{\partial q_1{}^\alpha}\frac{\partial^\beta}{\partial q_2{}^\beta}\frac{\partial^\gamma}{\partial q_3{}^\gamma}\times[e^{iq_0t}g^*(\mathbf{P}+\tfrac{1}{2}\mathbf{q})g(\mathbf{P}-\tfrac{1}{2}\mathbf{q})\times\bar{u}(\mathbf{P}+\tfrac{1}{2}\mathbf{q})F_\mu(\mathbf{q},q_0)u(\mathbf{P}-\tfrac{1}{2}\mathbf{q})].\quad\text{(A20)}$$

Now, for the reasons given above, all terms involving derivatives of $g(\mathbf{p})$ are dropped and we find

$$M_{\alpha\beta\gamma}{}^\mu\rightarrow i^{-N}\int d^3q\,\delta(\mathbf{q})\int d^3p|g(\mathbf{P})|^2\frac{\partial^\alpha}{\partial q_1{}^\alpha}\frac{\partial^\beta}{\partial q_2{}^\beta}\frac{\partial^\gamma}{\partial q_3{}^\gamma}\times[e^{iq_0t}\bar{u}(\mathbf{P}+\tfrac{1}{2}\mathbf{q})F_\mu(\mathbf{q},q_0)u(\mathbf{P}-\tfrac{1}{2}\mathbf{q})],\quad\text{(A21)}$$

or by Eq. (A15),[19]

$$M_{\alpha\beta\gamma}{}^\mu\rightarrow i^{-N}\left\{\frac{\partial^\alpha}{\partial q_1{}^\alpha}\frac{\partial^\beta}{\partial q_2{}^\beta}\frac{\partial^\gamma}{\partial q_3{}^\gamma}\times[\bar{u}(\tfrac{1}{2}\mathbf{q})F_\mu(\mathbf{q},0)u(-\tfrac{1}{2}\mathbf{q})]\right\}_{\mathbf{q}=0}.\quad\text{(A22)}$$

[18] The invariant normalization of the nucleon state $|p\rangle$, obtained by multiplying the usual state (normalized to a δ function) by $p_0{}^{\frac{1}{2}}$, is used here.

[19] Note that

$$q_0=p_0'-p_0=[(\mathbf{P}+\tfrac{1}{2}\mathbf{q})^2+M^2]^{\frac{1}{2}}-[(\mathbf{P}-\tfrac{1}{2}\mathbf{q})^2+M^2]^{\frac{1}{2}}=0\quad\text{for}\quad\mathbf{P}=0.$$

The expression in square brackets is just the matrix element of the current in the Breit frame and it may be determined directly in terms of the form factors by substituting from Eq. (A19) for F_μ. Thence[6]

$$\bar{u}(\tfrac{1}{2}\mathbf{q})F_4(\mathbf{q},0)u(-\tfrac{1}{2}\mathbf{q})=ieF_{\text{ch}}(\mathbf{q}^2),\quad\text{(A23)}$$

and

$$\bar{u}(\tfrac{1}{2}\mathbf{q})\mathbf{F}(\mathbf{q},0)u(-\tfrac{1}{2}\mathbf{q})=ie(\boldsymbol{\sigma}\times\mathbf{q})F_{\text{mag}}(\mathbf{q}^2),\quad\text{(A24)}$$

where F_{ch} and F_{mag} are given by Eqs. (6) and (7).

The moments given by Eq. (A22) serve to define charge and current distributions $J_\mu(\mathbf{r})$ in configuration space. By virtue of the relationships Eqs. (A22), (A23), and (A24), the Fourier transforms of these distributions are given by

$$J_4(\mathbf{r})=ie(2\pi)^{-3}\int d^3q\,F_{\text{ch}}(\mathbf{q}^2)\exp(-i\mathbf{q}\cdot\mathbf{r}),\quad\text{(A25)}$$

and

$$\mathbf{J}(\mathbf{r})=ie(2\pi)^{-3}\int d^3q(\boldsymbol{\sigma}\times\mathbf{q})F_{\text{mag}}(\mathbf{q}^2)\exp(-i\mathbf{q}\cdot\mathbf{r}).\quad\text{(A26)}$$

The interaction of a nucleon with a static external field described by potentials $A_\mu(\mathbf{r})$ in the rest frame of the nucleon may be obtained by taking the expectation value, in the above described wave packet, of the operator $\int d^3r\; j_\mu(x)A_\mu(\mathbf{r})$. If the potentials may be expanded in Taylor series, this expectation value may be expressed in terms of the moments $M_{\alpha\beta\gamma}{}^\mu$. Then, if effects of the structure of the wave packet are again ignored, the interaction energy takes the form

$$\int d^3r\,J_\mu(\mathbf{r})A_\mu(\mathbf{r})=ie\int d^3q\left\{F_{\text{ch}}(\mathbf{q}^2)A_4(\mathbf{q})+F_{\text{mag}}(\mathbf{q}^2)[\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{A}(\mathbf{q})]\right\},\quad\text{(A27)}$$

where the expression on the right-hand side, given in terms of the Fourier transform $A_\mu(\mathbf{q})$ of $A_\mu(\mathbf{r})$, is a direct consequence of Eqs. (A25) and (A26).

From Eq. (A27) it is clear that $F_{\text{ch}}(q^2)$ and $F_{\text{mag}}(q^2)$ serve to determine the distribution of charge and magnetization, and in fact that the Fourier transform of each of these distributions is obtained by evaluating the appropriate form factor in the Breit frame, i.e., at $q^2=\mathbf{q}^2$.