Experimental Consequences of the Hypothesis of Regge Poles*

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In the nonrelativistic case of the Schrödinger equation, composite particles correspond to Regge poles in scattering amplitudes (poles in the complex plane of angular momentum). It has been suggested that the same may be true in relativistic theory. In that case, the scattering amplitude in which such a particle is exchanged behaves at high energies like $s^{\alpha(t)}[\sin\pi\alpha(t)]^{-1}$, where s is the energy variable and t the momentum transfer variable. When $t=t_R$, the mass squared of the particle, then α equals an integer n related to the spin of the particle. In contrast, we may consider the case of a field theory in which the exchanged particle is treated as elementary and we examine each order of perturbation theory. When $n>1$, we can usually not renormalize successfully; when $n\leq 1$ and the theory is renormalizable, then the high-energy behavior is typically $s^n(t-t_R)^{-1}\phi(t)$. Thus an experimental distinction is possible between the two situations. That is particularly interest-

ing in view of the conjecture of Blankenbecler and Goldberger that the nucleon may be composite and that of Chew and Frautschi that all strongly interacting particles may be composite dynamical combinations of one another. We suggest a set of rules for finding the high-energy behavior of scattering cross sections according to the Regge pole hypothesis and apply them to $\pi\text{-}\pi$, $\pi\text{-}N$, and $N\text{-}N$ scattering. We show how these cross sections differ from those expected when there are "elementary" nucleons and mesons treated in renormalized perturbation theory. For the case of $N\text{-}N$ scattering, we analyze some preliminary experimental data and find indications that an "elementary" neutral vector meson is probably not present. Various reactions are proposed to test the "elementary" or "composite" nature of other baryons and mesons. Higher energies may be needed than are available at present.

I. INTRODUCTION

IN conventional Lagrangian field theory, particles of spin higher than one give rise to difficulties.

If we treat a particle as "elementary," by analogy with the electron and photon in quantum electrodynamics, we assign it a field and consider a Lagrangian in which there is a free-field term for the particle and also coupling terms to other fields. We expand in a perturbation series, renormalizing masses and coupling strengths, and look at the behavior of each order. When the spin of the particle is higher than one (and, in some cases, when it equals one) the resulting theory is unrenormalizable or divergent in each order.[1] The divergences are connected, loosely speaking, with a singular behavior at high energies of scattering amplitudes in which the particle of high spin is exchanged.

Now objects of high spin obviously exist in nature, and therefore from the point of view of renormalizable field theory they have to be regarded as "composite." Somehow, when a composite particle of high spin is exchanged, the singular behavior of the scattering amplitudes is avoided. Regge,[2] investigating the nonrelativistic Schrödinger equation, has found what is no doubt the mechanism by which composite states of high spin make themselves respectable. This mechanism

can apply just as well to states of spin 0, $\frac{1}{2}$, or 1, and one is led naturally to the conjecture[3] that all dynamical bound and resonant states follow the Regge type of behavior.

For spins 0 and $\frac{1}{2}$, however, and sometimes for spin 1 as well, we have the alternative possibility of considering a bound or resonant state as coming from an "elementary" particle in the sense described above. In many cases, one can exhibit, in every order of the resulting renormalizable field theory, the high-energy behavior of amplitudes in which the "elementary" particle is exchanged. This perturbation theory behavior is very different from that of the Regge case. We shall use the words "elementary" and "composite" to describe the two situations, even though the applicability of these words depends on perturbation theory in one case and on conjecture in the other.

Recently, Chew and Frautschi[4] have suggested that *all* strongly interacting particles[5] may exhibit the Regge behavior that we believe to be typical of composite states. In a sense, then, all baryons and mesons would be bound states of one another. It is made plausible that under this hypothesis all the mass ratios and

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[1] For the case of the graviton, having mass zero and spin 2 and obeying Einstein's nonlinear equation that satisfies the gauge invariance of general relativity, the question of renormalizability has not been settled.

[2] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960).

[3] S. Mandelstam has suggested and emphasized repeatedly since 1960 that the Regge behavior would permit a simple description of dynamical states (private discussions). Similar remarks have been made by R. Blankenbecler and M. L. Goldberger and by K. Wilson.

[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961). See also reference 14.

[5] It is also possible that the other particles are composite in this sense. The most fascinating possibilities are those involving the electron, muon, and photon. Quantum electrodynamics would still be correct at low energies and momenta, but would be gradually cut off at high momentum transfers by the Regge mechanism without violating causality.

coupling constants of the strongly interacting particles could, in principle, be calculated.

We are concerned here with the possibility of testing directly by experiment the hypothesis that the various baryons and mesons obey the Regge conditions. It is often possible to compare the predictions of the "elementary" and "composite" pictures of particular baryons and mesons for the high energy behavior of scattering amplitudes in which they are exchanged. If s and t are the energy and momentum transfer variables, then the two predictions are essentially a form $s^{\alpha(t)}$ in the composite case and s^n in the elementary case, where n is a fixed integer depending on the spin of the exchanged particle, while $\alpha(t)$ is variable and smaller than n in the physical region for the scattering.

Let us consider these statements in more detail. Stable particles appear as poles in S-matrix elements at real values of energy or momentum transfer variables. Correspondingly, unstable particles (or resonances) give poles on unphysical sheets of the S matrix at complex values of the same variables. Consider a two-particle scattering process $a+b \rightarrow c+d$, for which the energy variable is s (center-of-mass energy squared), and the corresponding crossed reaction $a+\bar{c} \rightarrow \bar{b}+d$, for which the energy variable is t. We may speak of the s reaction and the t reaction, respectively. In the physical region for the s reaction, $s > s_{\text{threshold}} > 0$ and $t < t_{\max}$ where $t_{\max} \rightarrow 0$ as $s \rightarrow \infty$, while in the physical region for the t reaction we have $t > t_{\text{threshold}} > 0$ and $s < s_{\max}$ where $s_{\max} \rightarrow 0$ as $t \rightarrow \infty$. The cosine x_t of the scattering angle in the t reaction is linearly related to the energy variable of the s reaction. In particular, if q_t and p_t are the center-of-mass momenta of $a+\bar{c}$ and $\bar{b}+d$, respectively, then for large s we have

$$x_t \approx -s(2q_t p_t)^{-1}. \qquad (1.1)$$

Suppose, for simplicity, that a, b, c, and d are spinless and that a particle of spin l gives a pole in the t variable. In the invariant amplitude $T(s,t)$, the residue of the pole is then evidently a number times $P_l(x_t)$:

$$T(s,t) = [CP_l(x_t)/t - t_R] + \text{other terms.} \qquad (1.2)$$

Thus, in the s reaction, the contribution to the scattering amplitude of the pole (occurring at an *unphysical* value of the momentum transfer variable t) has the energy dependence s^l at large s.

As we indicated earlier, *it* is possible in many cases to show, for the *renormalizable* theories of elementary particles of spin ≤ 1, that in each order of perturbation theory the high energy behavior characteristic of the pole term persists for all values of t. (See Sec. VI for details.)

For fixed *physical* (i.e., negative) values of the momentum transfer t in the s reaction, if this energy dependence of the pole contribution is not cancelled by other terms, then any value of l greater than 1 gives

us a rate of energy variation of $T(s,t)$ at large s that is embarrassing for the following reasons:

(1) The experimental situation seems to be that the most singular behavior for $T(s,t)$ (or its analog for the case of particles with spin) is exhibited by the imaginary part of elastic scattering amplitudes for $t=0$ and that the variation in that case is exactly or approximately linear with s, corresponding (with the use of the optical theorem) to constant or approximately constant total cross section.

(2) Froissart,[6] using the Mandelstam representation, has shown (for the case of spinless particles a, b, c, d) that the invariant amplitude cannot grow faster than $s \ln^2 s$ for large s and fixed t.

The situation described by Regge avoids these difficulties. He treated the nonrelativistic Schrödinger equation for one particle in a potential that is a superposition of Yukawa potentials. Let t be the energy variable and $x_t = \cos\theta_t$. One may examine the behavior of the scattering amplitude for large x_t, even though this limit is not connected with high energy in a crossed reaction, since there is no nonrelativistic crossing relation. Regge has found that in this simple case there is a beautiful mechanism that reduces the singularity of the behavior of the scattering amplitude at large $\cos\theta_t$ as t decreases and becomes negative. If there are resonances or bound states, the scattering amplitude at large x_t is dominated by a sum of terms of the form[7]

$$\frac{\beta(t)}{\sin\pi\alpha(t)} P_{\alpha(t)}(-x_t) \xrightarrow{s\rightarrow\infty} \frac{c(t)s^{\alpha(t)}}{\sin\pi\alpha(t)}, \qquad (1.3)$$

where each term represents, in general, a family of resonances and/or bound states of *variable angular momentum*. We have used the asymptotic form (1.1) of x_t and the fact that $P_\alpha(y) \propto y^\alpha$ at large y.

We shall discuss (1.3) further in the next section but for the moment let us just note the relationship to the simple resonance formula (1.2). For values of t below threshold t_0 (that is, below zero kinetic energy in the Schrödinger problem), Regge's α is real and increasing with t. A bound state of angular momentum l occurs at a value $t_R < t_0$ if $\alpha(t_R) = l$, since near t_R we have

$$\frac{\beta(t_R)}{\pi\alpha'(t_R)} \frac{P_l(x_t)}{t - t_R}, \qquad (1.4)$$

which just corresponds to (1.2).

At the bound state, then, we have the same situation as always, with the scattering amplitude varying like s^l at large s. However, as t decreases from t_R, so does $\alpha(t)$, and the dependence on s at large s keeps getting less singular.

[6] M. Froissart, Phys. Rev. **123**, 1053 (1961). Froissart's proof does not apply to the exchange of massless particles like the graviton.

[7] Below threshold, each term may actually be proportional to $(-1)^\alpha P_\alpha(x_t)$ or some other function that is *asymptotically* the same as $P_\alpha(-x_t)$.

In the relativistic problem, if Regge's mechanism operates, it can give precisely the desired effects. By the time we reach negative values of t and enter the physical region for the crossed reaction, α can have decreased to a value ≤ 1 so that we have an acceptable high energy behavior in the s reaction even though the spin of the resonance at $t = t_R$ is greater than one.

Moreover, even for spin 0 (and similarly for spin $\frac{1}{2}$ and spin 1), where the asymptotic law s^l for fixed l leads to no trouble, the still less singular Regge situation is an alternative possibility. These two situations are just the ones we described earlier under the names "elementary" and "composite," respectively. Evidently they can be distinguished by experiments. In the physical region, the "elementary" picture makes the pole contributions persist at high energies with the same energy dependence as at the pole; this is in the spirit of the "peripheral model" of high-energy scattering. The "composite" picture, in contrast, makes the peripheral terms much weaker at high energies. The Regge description also makes important predictions for diffraction scattering.

In Sec. II, we discuss the Regge mechanism in detail and make specific conjectures as to how it enters the relativistic problem. We present these conjectures as a set of rules for calculating the high-energy behavior of scattering amplitudes.

In Sec. III, we illustrate the use of the rules by treating π-π scattering; the problem of diffraction scattering arises here, as elsewhere, and we discuss it.

In Sec. IV, we apply the rules to π-N scattering and show how the nature of the nucleon pole can be tested by experiment.

In Sec. V, we treat N-N scattering and analyze some experimental data, which seem consistent with the "composite" hypothesis for mesons and, in particular, seem to be difficult to reconcile with the existence of an "elementary" neutral vector meson.

In Sec. VI, we treat the "elementary particle" situation that is contrasted with the Regge pole hypothesis; we base our discussion on the field theory perturbation expansion and explore the connection with the "peripheral model." Finally, we list tests of the Regge property of various baryons and mesons, including strange particles.

II. REGGE POLES

We have mentioned the problem of extending the scattering amplitude for the nonrelativistic Schrödinger equation to large values of $x_t = \cos\theta_t$. Regge solved this problem by the mathematical method of Watson and Sommerfeld, involving complex angular momenta. Let us describe the method briefly.

The usual phase shift expansion,

$$T(x_t,t) = \sum_{l=0}^{\infty} (2l+1)P_l(x_t)A(l,t), \qquad (2.1)$$

where $A(l,t)$ is proportional to $\sin\delta_l \exp i\delta_l$, does not converge for large x_t. To obtain an expression that does converge, one considers the solution of the radial Schrödinger equation for arbitrary complex l, obtaining an analytic continuation of $A(l,t)$. The phase shift expansion can now be rewritten in the form of a contour integral,

$$T(x_t,t) = \frac{1}{2\pi i} \oint dl$$
$$\times (2l+1)P_l(-x_t)A(l,t)\pi(\sin\pi l)^{-1}, \qquad (2.2)$$

over a contour just surrounding the positive real l axis. The residues from the poles of $\pi(\sin\pi l)^{-1}$ give back the terms of the sum (2.1).

For a superposition of Yukawa potentials, Regge shows that one may distort the contour in (2.2) to the vertical line from $l = -\frac{1}{2} - i\infty$ to $l = -\frac{1}{2} + i\infty$ without encountering any singularities other than simple poles of $A(l,t)$, when t is above threshold t_0. These "Regge poles" occur at complex values of l, called $\alpha_n(t)$, at which the Schrödinger equation (for energy variable $= t$) has solutions corresponding formally to resonant states with zero width. The position $\alpha_n(t)$ in the complex l plane of a given Regge pole (the nth one) varies continuously with t. We use here only values of α_n to the right of the vertical line at $\text{Re}\, l = -\frac{1}{2}$.

For each $t > t_0$, we distort the contour, pick up the Regge poles, and obtain in place of (2.2) the expression

$$T(x_t,t) = \frac{-1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl$$
$$\times (2l+1)P_l(-x_t)A(l,t)\pi(\sin\pi l)^{-1}$$
$$+ \sum_n \beta_n(t)P_{\alpha_n}(-x_t)(\sin\pi\alpha_n)^{-1}, \qquad (2.3)$$

which represents the scattering amplitude for all values of x_t and allows us to extract the asymptotic form that we want at large x_t. Note the Regge pole contributions have the form (1.2); if they are present they dominate the line integral in (2.3), which is bounded by a constant times $x_t^{-\frac{1}{2}}$ at large x_t.

For energies below threshold the specific representation (2.3) is not quite correct, but $A(l,t)$ continues to have simple poles at positions $\alpha_n(t)$ in the complex l plane; these positions are now on the real axis and represent formally the angular momenta of bound states at value t of the energy variables. The *asymptotic* behavior of $T(x_t,t)$ for large x_t is presumably still dominated by the Regge terms:

$$T(x_t,t) \underset{x_t \to \infty}{\approx} \sum_n \beta_n(t)P_{\alpha_n}(-x_t)(\sin\pi\alpha_n)^{-1}. \qquad (2.4)$$

To get a bound state more and more below threshold, we need more and more attraction. For real l between $-\frac{1}{2}$ and 0, $l(l+1)$ is negative and gives a "centrifugal

attraction" that is greatest at $l=-\frac{1}{2}$, where $l(l+1)\hbar^2/2mr^2$ just balances the kinetic energy $(\hbar/2r)^2$ $\times(1/2m)$ coming from the uncertainty principle. Thus if the potential gives attraction at small distances, a bound state should be possible at $l=-\frac{1}{2}$ for some value of t below threshold. As t increases, we need less "centrifugal attraction" to supplement the attractive potential, and so each $\alpha_n(t)$ should emerge[8] from the vertical line at $l=-\frac{1}{2}$ for some value of t below threshold and move to the right along the real l axis as t increases. At threshold, α_n is continuous, although it has a cusp.

For t above threshold, $\alpha_n(t)$ represents the angular momentum of a resonance of zero width and so must be complex. We shall see that α_n in fact acquires a positive imaginary part above threshold. When $\mathrm{Im}\,\alpha_n$ is small, then $\mathrm{Re}\,\alpha_n$ represents approximately the angular momentum of a resonance of positive width.

Genuine physical bound states and resonances are now very easy to discuss. As a given $\alpha_n(t)$ increases from $-\frac{1}{2}$ along the real l axis while t increases (below threshold) it may reach zero; there is a genuine bound s state at this value of t, say t_R. The contribution of this Regge term to the asymptotic scattering amplitude (2.4) near $t=t_R$ is just

$$\frac{\beta_n(t_R)}{\pi\alpha'(t_R)}\frac{P_0(x_t)}{t-t_R},\qquad(2.5)$$

as in (1.3). If α_n attains higher integral values below threshold, these correspond to bound p states, d states, etc., all belonging to a single family with a given number of radial nodes in the wave function.

Above threshold, if $\mathrm{Re}\,\alpha_n$ continues to increase and rises through integral values while $\mathrm{Im}\,\alpha_n$ is still small, then there are resonances in the family. Say $\mathrm{Re}\,\alpha_n$ rises through $\alpha_n=l$ at $t=t_R$ above threshold with $\mathrm{Re}\,\alpha_n'(t_R)$ $=\epsilon_R$; and say $I_R=\mathrm{Im}\,\alpha_n$ is small there. Nearby, the contribution of the Regge term to the scattering amplitude (2.3) is approximately

$$\frac{\beta_n(t_R)}{\pi\epsilon_R}\frac{P_l(x_t)}{t-t_R+iI_R\epsilon_R^{-1}},\qquad(2.6)$$

which is just what we expect.

Before we pass on to the relativistic problem, we must consider a slight generalization of the non-relativistic case, namely the addition of an exchange potential to the direct potential in the Schrödinger equation. The potentials for the radial Schrödinger equation are then different for states of even and odd angular momentum. Each of the two mathematical problems can be treated à la Regge and continued to arbitrary l. However, when the solution of the even-wave Schrödinger equation has a bound or resonant

state at odd integral l, or vice versa, we must not expect this to lead to a pole in t in the physical scattering amplitude $T(x_t,t)$. We show in the Appendix that the necessary cancellation comes about as follows: with exchange scattering, each asymptotic Regge term takes on the form

$$\beta_n(t)(\sin\pi\alpha_n)^{-1}\tfrac{1}{2}[P_{\alpha_n}(-x_t)\pm P_{\alpha_n}(x_t)],\qquad(2.7)$$

instead of (2.4). The Regge terms corresponding to physical states of even l take the $+$ sign in (2.7); we shall refer to these terms as having positive signature. Likewise, the terms corresponding to physical states of odd l have negative signature. If the exchange scattering disappears, then two Regge terms of opposite signature coalesce, giving back the form of (2.3).

We now suppose that for the relativistic problem the behavior of the invariant scattering amplitude $T(x_t,t)$ is likewise dominated by terms like (2.6). For the general case of the t reaction $a+\bar{c}\to\bar{b}+d$ (and the corresponding s reaction $a+b\to c+d$), with arbitrary spins for the particles involved, we conjecture the following rules for finding the form of a given Regge term:

(1) Consider a complete set of linearly independent invariant scattering amplitudes $A_i(s,t)$ free of kinematic singularities and zeros in s and t. For example, in π-π scattering there are three of these, for the three isotopic spin states; in π-N scattering there are four, since there are two values of the isotopic spin and also the possibility of spin flip or no spin flip.

(2) For the t reaction, take any set of values of the conserved quantum numbers except j, the total angular momentum. Then, as a function of j, construct the contribution to the amplitudes A_i of a hypothetical exchanged particle with these quantum numbers; the "particle" is introduced for mathematical convenience only and may occur at any value M^2 of t. For each A_i, this contribution will be a sum of terms containing Legendre functions of x_t (or derivatives thereof) with indices depending on j. At large s, each such function of x_t is asymptotic to a power of s, where the exponent varies with j like $j+$const. Thus the contribution to A_i takes the form

$$c_i s^{(j-\nu_i)}/(t-M^2),$$

asymptotically in s; there may, of course, be constraints on the c_i.

(3) Write $j=\alpha$ for integral spin in the t reaction or $j=\alpha+\frac{1}{2}$ for half-integral spin and continue to complex α. Then each Regge term has, asymptotically in s, a dependence on s such as described in rule (2), with α depending on t, and with $[(1\pm e^{-i\pi\alpha})/2\sin\pi\alpha]c_i(t)$ appearing as an over-all factor in place of $c_i(t-M^2)^{-1}$. The reason for choosing this form is clear from (2.6) and (1.1). (See also the discussion in the Appendix.) Each Regge term is associated with a definite set of conserved quantum numbers in the t reaction (except j)

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[8] By using other representations of the scattering amplitude, it may be possible to follow the poles and the corresponding α's into the region to the left of $\mathrm{Re}\,l=-\frac{1}{2}$. M. Froissart, M. Goldberger, and S. Mandelstam (private communication).

and with a definite signature, which may be ± 1 for any set of these quantum numbers.[9] We see that when the signature is positive (negative), there is no pole for odd (even) integral α.

A Regge term having the *same* α will then appear in *each* reaction in which this set of quantum numbers can occur; that is analogous to the fact that a given resonance occurs in many reactions. The coefficients, generalizations of β in (2.6), will of course vary from one process to another. We always exhibit explcitly the factor

$$\frac{1 \pm e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)},$$

which is independent of the particular process in which the quantum numbers are exchanged.

It is attractive to suppose, as in the nonrelativistic potential problem, that each $\alpha(t)$ eventually becomes negative as t decreases. There are, however, some cases in which this assumption raises difficult questions. For example, there are many known systems, such as nuclei, for which the ground state spin is greater than $\frac{3}{2}$. Take the case of spin 2. The corresponding Regge term must have positive signature and $\alpha = 2$ at the energy of the ground state. As t decreases further, α will have to pass through zero if it is to attain a negative value. That would put us in the absurd position of having a state of spin zero below the ground state, unless *all* coefficients $c(t)$ for this Regge term vanish at the point where α is zero.

From the point of view of nonrelativistic quantum mechanics, it is presumably the Pauli principle that prevents the existence of the spin zero state, given the nuclear dynamics responsible for the ground state of spin two. That would suggest that perhaps it *is* possible to continue the Regge pole down to a point where $\alpha = 0$, but the c's vanish because the corresponding wave function vanishes after antisymmetrization.

We shall consider, in our discussion of diffraction scattering, a Regge α (called α_P) for a state of positive signature such that $\alpha_P = 1$ at $t = 0$. If α_P is to reach negative values, it must pass through zero at a *negative* value of t, giving a physical state of negative mass squared; again, we can be saved if all the c's vanish at the same point.

Whether or not the α's become negative, they are presumably ≤ 1 in the physical region $t < 0$ for the s reaction, even though Froissart's proof may not apply when there are anomalous thresholds such as exist in nuclei, and there are no known experimental limitations in the case when a heavy nucleus is exchanged. There are, however, nuclei in which the ground state spin is 3, for example. Hence the α's must pass through 1 at a value of t above zero but below that of the spin 3 ground state, but since no spin 1 state exists, the c's

should in this case all vanish at the place where $\alpha = 1$. This indicates it is not absurd to expect that the c's may vanish in other situations when α's pass through zero.

Let us now apply our rules (1), (2), and (3) to some particular scattering problems.

III. PION-PION SCATTERING

For π-π scattering, the independent amplitudes of rule (1) can be taken to be the three isotopic spin amplitudes $T^I(s,t)$ with $I = 0$, 1, 2 for the reaction in which s is the total energy squared.

A prominent feature of the scattering process is the $I = 1$, $J = 1^-$ resonance at about 750 Mev, called ρ. Let us start by considering, for the purposes of rule (2), the quantum numbers in the t reaction of this state: $I = 1$, $P = -1$, $G = +1$. Suppose the ρ meson is a physical manifestation of a Regge term with these quantum numbers and signature -1. The exchange of a particle with $I = 1$, $P = -1$, $G = +1$, spin j ($j = 1$, 3, 5, etc.) and mass M, contributes to (T^0, T^1, T^2) a term

$$(-2, -1, 1)P_j(x_t)C/(t - M^2), \tag{3.1}$$

and rule (3) gives us for the Regge term for large x_t the form

$$(-2, -1, 1)s^\alpha \frac{1 - \exp[-i\pi\alpha(t)]}{2\sin\pi\alpha(t)}c(t), \tag{3.2}$$

where we might use the more explicit notation $\alpha_\rho(t)$ and $c_{\pi\pi\rho\pi\pi}(t)$. To avoid having variable dimensions for the quantity c, it is useful to put

$$c(t) = 4m_\pi^2 (2m_\pi^2)^{-\alpha_\rho(t)} b(t). \tag{3.3}$$

Near $t = m_\rho^2$, we have information about α and b. We may, if we like, define m_ρ^2 to be the value of t for which $\text{Re}\,\alpha_\rho = 1$. Setting

$$\begin{aligned}\epsilon_\rho &= \text{Re}\,\alpha_\rho'(m_\rho^2), \\ I_\rho &= \text{Im}\,\alpha_\rho(m_\rho^2),\end{aligned} \tag{3.4}$$

and treating the imaginary part as small, then in the neighborhood of m_ρ^2 the expression (3.2) gives us approximately

$$(-2, -1, 1)s\frac{2b(m_\rho^2)}{\pi\epsilon_\rho}\frac{(-1)}{t - m_\rho^2 + iI_\rho\epsilon_\rho^{-1}}, \tag{3.5}$$

to be compared with the contribution to π-π scattering of the exchange of a single virtual, slightly unstable ρ particle in field theory or dispersion theory:

$$(-2, -1, 1)P_1(x_t)[-2\gamma_{\rho\pi\pi}^2(m_\rho^2 - 4m_\pi^2)]$$
$$\times \frac{(-1)}{t - m_\rho^2 + i\Gamma_\rho m_\rho}, \tag{3.6}$$
$$\xrightarrow[s\to\infty]{} (-2, -1, 1)8\gamma_{\rho\pi\pi}^2 s\frac{(-1)}{t - m_\rho^2 + i\Gamma_\rho m_\rho},$$

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[9] The signature has sometimes been confused with parity in the literature. Even when parity is not conserved, the poles are still absent at every other integral value of α because of the signature.

where $\gamma_{\rho\pi\pi}{}^2/4\pi$ is an effective coupling constant of ρ to π and π. If we make the approximation that $\rho \to 2\pi$ is the dominant decay mode of ρ, then the width Γ_ρ is given[10] by

$$\Gamma_\rho m_\rho = \tfrac{1}{3}(\gamma_{\rho\pi\pi}{}^2/4\pi)(m_\rho{}^2 - 4m_\pi{}^2)^{\frac{3}{2}} m_\rho{}^{-1}, \qquad (3.7)$$

so that the experimental value of $\gamma_{\rho\pi\pi}{}^2/4\pi$ is around unity. Evidently in (3.5) and (3.6) we make the identifications (for small Γ_ρ)

$$b(m_\rho{}^2)/\pi\epsilon_\rho = 4\gamma_{\rho\pi\pi}{}^2, \qquad (3.8)$$
$$I_\rho\epsilon_\rho{}^{-1} = \Gamma_\rho m_\rho.$$

Using (3.7) and (3.8) we obtain a result that comes just from the assumption of the pure decay $\rho \to 2\pi$:

$$\frac{\pi I_\rho}{b(m_\rho{}^2)} = \frac{(m_\rho{}^2 - 4m_\pi{}^2)^{\frac{3}{2}} m_\rho{}^{-1}}{48\pi}. \qquad (3.9)$$

If the charge exchange amplitudes for π-π scattering are dominated by the Regge term containing the ρ meson for large s, then a charge exchange cross section will be, for example,

$$\frac{d\sigma^{I=0}}{dt} - \frac{d\sigma^{I=2}}{dt} \xrightarrow[s\to\infty]{} 3F_{\rho\pi\pi}(t)\left(\frac{s}{2m_\pi{}^2}\right)^{2\alpha_\rho(t)-2}, \quad (3.10)$$

where

$$F_{\rho\pi\pi}(t) = \frac{1}{16\pi}\left| b_{\rho\pi\pi}(t)\left(\frac{1-e^{-i\pi\alpha_\rho(t)}}{\sin\pi\alpha_\rho(t)}\right)\right|^2 \qquad (3.11)$$

and we have restored some of the subscript indices for b.

In the physical region for the s reaction $\alpha(t)$ is no doubt <1. This is quite different from the situation for an elementary ρ in the lowest order of perturbation theory; however, it seems unlikely that the ρ could be elementary in any case, since the perturbation expansion of an elementary $J=1$, $I=1$ particle is not renormalizable. Later, for example, in Sec. V where we discuss N-N scattering, such distinctions will take on more importance, since in N-N scattering the exchange of a $J=1$, $I=0$ particle, which can be renormalized, is possible.

We have already indicated that the Regge approach can provide an explanation of the experimental result that total cross sections become constant at high energies if we assume the existence of a particular α, called α_P, with even signature and such that $\alpha_P(0)=1$. Let us associate α_P with the set of quantum numbers describing the vacuum. Then its existence also guarantees the validity of the Pomeranchuk theorems, which state that particle and antiparticle total cross sections become equal at high energies, and that all two-body inelastic cross sections vanish, provided the α_P Regge term dominates the amplitude for small momentum

[10] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

transfers. Froissart has shown that no α may be greater than one[6] for $t \leqslant 0$; to assure the Pomeranchuk statement, then, we must assume no other α, associated with a different set of quantum numbers, equals one for $t \leqslant 0$.

The form in which this Pomeranchuk Regge term will appear at high energies in the π-π problem is, by rule (3),

$$(1,1,1)(s/2m_\pi{}^2)^{\alpha_P(t)}$$

$$\times \left\{\frac{1+\exp[-i\pi\alpha_P(t)]}{2\sin\pi\alpha_P(t)}\right\} 4m_\pi{}^2 b_{P\pi\pi}(t). \quad (3.12)$$

Near $t=0$, we have $\alpha_P(t) \approx 1$, while other α's from other Regge terms such as that associated with the ρ meson which was discussed before, are presumably less than 1. The entire π-π amplitude is then dominated by (3.12); hence, as $t \to 0$ we find the amplitude becomes pure imaginary, and

$$T^I(s,0) \xrightarrow[s\to\infty]{} -(1,1,1)isb_{P\pi\pi}(0). \qquad (3.14)$$

The optical theorem for π-π scattering states that

$$\mathrm{Im}T^I(s,0) \xrightarrow[s\to\infty]{} -s\sigma_T{}^I, \qquad (3.15)$$

where $\sigma_T{}^I$ is the asymptotic total π-π cross section in the isotopic spin I channel. Therefore

$$\sigma_T{}^I = b(0). \qquad (3.16)$$

The differential cross section for π-π scattering at high energies which results from (3.13) may be written

$$\frac{d\sigma^I}{dt} \to F_{P\pi\pi}(t)\left(\frac{s}{2m_\pi{}^2}\right)^{2\alpha_P(t)-2}, \qquad (3.17)$$

if we define

$$F_{P\pi\pi}(t) = \frac{1}{16\pi}\left| b_{P\pi\pi}(t)\left(\frac{1+e^{-i\pi\alpha_P(t)}}{\sin\pi\alpha_P(t)}\right)\right|^2. \qquad (3.18)$$

These equations are valid for all t for which the "Pomeranchuk" Regge term dominates; therefore, they should certainly be valid for small t. For larger negative t however, there is in principle nothing to stop a different α from being bigger than α_P. If this happens, the form (3.17) is still valid, but with a different $F_{\pi\pi}(t)$ and the newly dominant α replacing α_P.

The "Pomeranchuk" Regge term can be exchanged in all elastic scattering amplitudes, since it goes with the quantum numbers of the vacuum. Therefore, all elastic differential cross sections will, for sufficiently large energies, and for momentum transfers at which α_P dominates all other α's, have the energy dependence of (3.17) with the same exponent $\alpha_P(t)$. The coefficient

$F_{P\pi\pi}(t)$ in (3.17) will, of course, not be the same for different processes.[11]

For small t, we may write $\alpha_P(t) = 1 + t\alpha_P'(0)$, in which case (3.17) becomes

$$\frac{d\sigma^I}{dt} \to F_{P\pi\pi}(t) \exp[-2|t|\alpha_P'(0)\ln(s/2m_\pi^2)]. \quad (3.19)$$

For very large energies, the most rapid variation with t will come from the exponential, so there will be an exponential diffraction peak with a width that decreases logarithmically with increasing energy.

IV. PION-NUCLEON SCATTERING

The discussion in this case is much the same as for the π-π problem in the forward direction at high energies; the only real difference arises from the nucleon spin. However, in the π-N case, because of the non-identity of the particles, we no longer have the symmetry between the forward and backward high

energy behavior that was true of the π-π problem. We must therefore discuss these two limits separately.

In accordance with rule (1) we may choose the amplitudes describing the π-N process to be $A^\pm$ and $B^\pm$ where the invariant amplitude is written

$$T = A^+\delta_{\sigma'\sigma} + A^-[\tau_{\sigma'},\tau_\sigma]/2$$
$$+ (B^+\delta_{\sigma'\sigma} + B^-[\tau_{\sigma'},\tau_\sigma]/2)(q'+q)/2. \quad (4.1)$$

Here σ', σ and q', q are the final and initial charges and 4-momenta of the pions.

First, we shall concentrate on the high energy forward limit $s \to \infty$, t fixed, where s is the total c.m. energy squared for π-N scattering, and t is the total c.m. energy squared for the process $\pi + \pi \to N + \bar{N}$. As in the π-π problem, the ρ meson will appear in the t reaction, as will the Pomeranchuk term. The quantum numbers of the ρ meson are $I = 1$, $P = -1$, $G = +1$, so using rule (2) of Sec. II and the known form of the partial wave expansion for the $\pi + \pi \to N + \bar{N}$ process,[12] we may write the form of the relevant meson pole term:

$$A^{(-)} = \frac{C^{(1)}P_j(x_t) + m_N(M^2 - 4m_\pi^2)^{\frac{1}{2}}(M^2 - 4m_N^2)^{-\frac{1}{2}}C^{(2)}x_t P_j'(x_t)}{t - M^2},$$

$$B^{(-)} = C^{(2)}P_j'(x_t)/(t - M^2). \quad (4.2)$$

There is no contribution to $A^{(+)}$ or $B^{(+)}$ since these amplitudes correspond to a pure $I = 0$ state in the t reaction. The cosine of the scattering angle in the t reaction is

$$x_t = -(s - m_N^2 - m_\pi^2 + \tfrac{1}{2}t)/2q_t p_t,$$

where for this problem we have

$$q_t^2 = \tfrac{1}{4}t - m_\pi^2, \quad p_t^2 = \tfrac{1}{4}t - m_N^2.$$

Asymptotically, we still have (1.1).

At large s, $P_j(x_t)$ goes like s^j and $x_t P_j'(x_t)$ like js^j, while $P_j'(x_t)$ goes like js^{j-1}. Applying rule (3) and taking out some factors for convenience, we have for the asymptotic Regge term the form:

$$-A^{(-)} \to \frac{1 - e^{-i\pi\alpha_\rho(t)}}{2\sin\pi\alpha_\rho(t)}\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho(t)}$$
$$\times 2m_\pi[b_{\pi\pi\rho NN}^{(1)}(t) - \alpha_\rho(t)b_{\pi\pi\rho NN}^{(2)}(t)] + \cdots,$$
$$-B^{(-)} \to \frac{1 - e^{-i\pi\alpha_\rho(t)}}{2\sin\pi\alpha_\rho(t)}\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho(t)-1}$$
$$\times 2\alpha_\rho(t)b_{\pi\pi\rho NN}^{(2)} + \cdots. \quad (4.3)$$

For simplicity, we shall usually drop the π and N subscripts on $b^{(1)}$ and $b^{(2)}$.

The appearance of two unknown functions of t, $b_\rho^{(1)}(t)$ and $b_\rho^{(2)}(t)$, reflects the fact that there are two possible states of the N-$\bar{N}$ system in the t reaction, for example $^3S_1$ and $^3D_1$, and without a complete solution to the dynamics of the process, there is an unknown mixing parameter between these two.

The α_ρ here is, of course, the same α_ρ as we found in the π-π problem. The position and width of the ρ meson are expressed in terms of this α. The functions $b_\rho^{(1)}$ and $b_\rho^{(2)}$, on the other hand, are characteristic of the particular process in question. As in the π-π problem, these functions when evaluated at m_ρ^2 may be related to the coupling constants $\gamma_{\rho\pi\pi}$ and $\gamma_{\rho NN}$ and the "anomalous moment" $\mu_{\rho NN}$ of the ρ meson coupling to pions and nucleons. Specifically, we find

$$b_\rho^{(1)}(m_\rho^2)/\pi\epsilon_\rho = 2\gamma_{\rho NN}\gamma_{\rho\pi\pi},$$
$$b_\rho^{(2)}(m_\rho^2)/\pi\epsilon_\rho = 2\gamma_{\rho NN}\gamma_{\rho\pi\pi} - 4m_N\mu_{\rho NN}\gamma_{\rho\pi\pi}. \quad (4.4)$$

At large s and for $t < 0$, so that we are in the physical region for the s reaction, this Regge term contributes to the no-spin-flip and spin-flip scattering amplitudes as follows:

$$f^- \to \frac{m_N}{4\pi s^{\frac{1}{2}}}\left(\frac{1 - e^{-i\pi\alpha_\rho}}{2\sin\pi\alpha_\rho}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho}2m_\pi b_\rho^{(1)} + \cdots,$$

$$\tilde{f}^- \to \frac{1}{16\pi}\left(\frac{1 - e^{-i\pi\alpha_\rho}}{2\sin\pi\alpha_\rho}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho}2m_\pi(b_\rho^{(1)} - \alpha b_\rho^{(2)})$$
$$+ \cdots. \quad (4.5)$$

---

[11] Note the presence of the factor $1/2m_\pi^2$ in the quantity raised to the power $2\alpha_P(t) - 2$ is purely arbitrary. In π-N and N-N scattering, we shall use $2m_\pi m_N$ and $2m_N^2$, which are equally arbitrary. In general, any constant raised to the power $2\alpha_P(t) - 2$ can be absorbed into $F(t)$.

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

The no-spin-flip amplitude f is defined, for large s, by

$$f^{\pm} = -\frac{m_N}{4\pi s^{\frac{1}{2}}}\left(A^{\pm} + \frac{s}{2m_N}B^{\pm}\right),$$

and the spin-flip amplitude by

$$\tilde{f}^{\pm} = \frac{1}{16\pi}(-A^{\pm} + s^{\frac{1}{2}}B^{\pm}). \tag{4.6}$$

Hence, the differential cross section is

$$\frac{d\sigma^{\pm}}{d\Omega} = \frac{1}{16\pi^2}$$

$$\times\left(\frac{m_N{}^2}{s}\left|A^{\pm} + \frac{s}{2m_N}B^{\pm}\right|^2 + \frac{\sin^2\theta}{16}|A^{\pm} - s^{\frac{1}{2}}B^{\pm}|^2\right). \tag{4.7}$$

The asymptotic charge exchange π-N cross section is thus

$$\frac{d\sigma^-}{dt} \to \left(\frac{s}{2m_\pi m_N}\right)^{2\alpha_\rho - 2} F_{\rho\pi N}(t), \tag{4.8}$$

where we have

$$F_{\rho\pi N}(t) = \frac{1}{16\pi}\left\{|b^{(1)}|^2 - \frac{t}{4m_N{}^2}|b^{(1)} - \alpha b^{(2)}|^2\right\}$$

$$\times\left|\frac{1 - e^{-i\pi\alpha_\rho}}{\sin\pi\alpha_\rho}\right|^2. \tag{4.9}$$

In the t reaction without charge exchange we expect to find the "Pomeranchuk" Regge term. The form which this term takes is, according to our rules,

$$A^{(+)} \to \frac{1 + e^{-i\pi\alpha_P(t)}}{2\sin\pi\alpha_P(t)}\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_P(t)}$$

$$\times 2m_\pi[b_{\pi\pi PNN}{}^{(1)}(t) - \alpha_P(t)b_{\pi\pi PNN}{}^{(2)}(t)] + \cdots,$$

$$B^{(+)} \to \frac{1 + e^{-i\pi\alpha_P(t)}}{2\sin\pi\alpha_P(t)}\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_P(t) - 1} \tag{4.10}$$

$$\times 2\alpha_P(t)b_{\pi\pi PNN}{}^{(2)}(t),$$

much as in (4.3).

At high energies in the physical region for the s reaction, this Regge term contributes to the no-flip and flip amplitudes as follows:

$$f^{(+)} \to -\frac{m_N}{4\pi s^{\frac{1}{2}}}\left(\frac{1 + e^{-i\pi\alpha_P}}{2\sin\pi\alpha_P}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_P}$$

$$\times 2m_\pi b_P{}^{(1)} + \cdots,$$

$$\tilde{f}^{(+)} \to -\frac{1}{16\pi}\left(\frac{1 + e^{-i\pi\alpha_P}}{2\sin\pi\alpha_P}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_P} \tag{4.11}$$

$$\times 2m_\pi(b_P{}^{(1)} - \alpha b_P{}^{(2)}) + \cdots.$$

As $t \to 0$, we have $\alpha_P(0) = 1$ and therefore

$$f^{(+)} \to (s^{\frac{1}{2}}/8\pi)ib_P{}^{(1)}(0) + \cdots. \tag{4.12}$$

Assuming as always that the "Pomeranchuk" Regge term dominates, we may then use the optical theorem, which states for large s that

$$\text{Im}f^{(+)} = (s^{\frac{1}{2}}/8\pi)\sigma_T{}^{(+)}, \tag{4.13}$$

to relate $b_P{}^{(1)}(0)$ to the asymptotic total π-N cross section. Thus, we find

$$b_P{}^{(1)}(0) = \sigma_T{}^{(+)}. \tag{4.14}$$

At any $t < 0$ for which the "Pomeranchuk" Regge term dominates the entire amplitude, we may write

$$\frac{d\sigma^{(+)}}{dt} = F_{\pi NP}(t)\left(\frac{s}{2m_\pi m_N}\right)^{2\alpha_P(t) - 2}, \tag{4.15}$$

where we define

$$F_{\pi NP}(t) = \frac{1}{16\pi}\left\{|b_P{}^{(1)}|^2 - \frac{t}{4m_N{}^2}|b_P{}^{(1)} - \alpha b_P{}^{(2)}|^2\right\}$$

$$\times\left|\frac{1 + e^{-i\pi\alpha_P}}{\sin\pi\alpha_P}\right|^2. \tag{4.16}$$

Let us now turn to a discussion of backward high-energy Regge terms. These will be Regge terms associated with the u reaction, where u is the crossed momentum transfer. The u reaction is then also π-N scattering. The partial wave expansion for this process is well known,[13] and in accordance with rule (2) we consider the hypothetical pole terms

$$A = C\left(\frac{W + m_N}{E + m_N}P_{j+\frac{1}{2}}{}'(x_u) + \frac{W - m_N}{E - m_N}P_{j-\frac{1}{2}}{}'(x_u)\right)(u - M^2)^{-1},$$
$$\tag{4.17}$$

$$B = C\left(\frac{1}{E + m_N}P_{j+\frac{1}{2}}{}'(x_u) - \frac{1}{E - m_N}P_{j-\frac{1}{2}}{}'(x_u)\right)(u - M^2)^{-1},$$

for states with $j = l + \frac{1}{2}$, and

$$A = C\left(\frac{W + m_N}{E + m_N}P_{j-\frac{1}{2}}{}'(x_u) + \frac{W - m_N}{E - m_N}P_{j+\frac{1}{2}}{}'(x_u)\right)(u - M^2)^{-1},$$
$$\tag{4.18}$$

$$B = C\left(\frac{1}{E + m_N}P_{j-\frac{1}{2}}{}'(x_u) - \frac{1}{E - m_N}P_{j+\frac{1}{2}}{}'(x_u)\right)(u - M^2)^{-1},$$

for states with $j = l - \frac{1}{2}$.

In this reaction, a single C suffices to describe each pole term since there is no mixing between states possible as long as parity is conserved. For the moment, we are ignoring isotopic spin. The notation in (4.17)

[13] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

and (4.18) is the following:

$$x_u = -(s - m_N^2 - m_\pi^2 + 2E_u\omega_u)/2q_u^2,$$
$$q_u^2 = u/4 - \tfrac{1}{2}(m_N^2 + m_\pi^2) + (m_N^2 - m_\pi^2)^2/4u,$$
$$E_u = (u + m_N^2 - m_\pi^2)/2u^{\frac{1}{2}},$$
$$\omega_u = (u - m_N^2 + m_\pi^2)/2u^{\frac{1}{2}},$$
$$W_u = E_u + \omega_u = u^{\frac{1}{2}}.$$

(4.19)

Applying rule (3) to a Regge family with positive signature and negative parity or negative signature and positive parity (counting the intrinsic nucleon parity as $+1$ and that of the pion as -1), we use (4.17) and obtain for the high-energy contribution to A and B something of the form

$$A \rightarrow \frac{[1 \pm e^{-i\pi\alpha(u)}]}{2\,\sin\pi\alpha(u)}\left(\frac{s}{2m_N m_\pi}\right)^{\alpha(u)}(W_u + m_N)b(u)$$
$$+\cdots, \quad (4.20)$$
$$B \rightarrow \frac{[1 \pm e^{-i\pi\alpha(u)}]}{2\,\sin\pi\alpha(u)}\left(\frac{s}{2m_N m_\pi}\right)^{\alpha(u)}b(u)+\cdots.$$

Correspondingly, for a Regge family with positive signature and positive parity or negative signature and negative parity, we use (4.18) and find

$$A \rightarrow \frac{[1 \pm e^{-i\pi\alpha(u)}]}{2\,\sin\pi\alpha(u)}\left(\frac{s}{2m_N m_\pi}\right)^{\alpha(u)}(W_u - m_N)b(u)+\cdots,$$
$$(4.21)$$
$$-B \rightarrow \frac{[1 \pm e^{-i\pi\alpha(u)}]}{2\,\sin\pi\alpha(u)}\left(\frac{s}{2m_N m_\pi}\right)^{\alpha(u)}b(u)+\cdots.$$

There are a number of stable and unstable states in the π-N process which we may associate with Regge terms of this sort, namely the nucleon itself and the various π-N resonances: the 33 resonance, the presumed $d_{\frac{3}{2}}$, $I = \frac{1}{2}$ resonance at 1520 Mev, and the presumed $f_{\frac{5}{2}}$, $I = \frac{1}{2}$ resonance at 1680 Mev.

If the nucleon is due to a Regge term,[14] this term must be of the form (4.21) with even signature and $\alpha(m_N^2) = 0$. Furthermore, the nucleon has $I = \frac{1}{2}$, so the Regge term must appear with coefficient $(1, -1)$ in the $((+),(-))$ amplitudes. Let us take $b(u)$ to have the sign given by (4.21) for the $(+)$ amplitude. It is easy to compare this Regge term near $u = m_N^2$ with the usual nucleon pole and relate $b(m_N^2)$ to the usual pion-nucleon coupling constant:

$$b(m_N^2)/\pi\epsilon = g_{NN\pi}^2, \quad (4.22)$$

where we define $\epsilon = \alpha'(m_N^2)$. Since $u = m_N^2$ is below threshold in the u reaction α is real there. The contribution of this Regge term to the high energy no-spin-flip and spin-flip amplitudes in the physical region[15] for the

s reaction is easily seen to be, for large s and fixed u,

$$f^{\pm} \rightarrow \pm\frac{1}{8\pi}\left(\frac{1 + e^{-i\pi\alpha}}{2\,\sin\pi\alpha}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha}b(u)(W_u - 2m_N)+\cdots,$$
$$(4.23)$$
$$\tilde{f}^{\pm} \rightarrow \mp\frac{s^{\frac{1}{2}}}{16\pi}\left(\frac{1 + e^{-i\pi\alpha}}{2\,\sin\pi\alpha}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha}b(u)+\cdots.$$

Note that we expect α in (4.23) to be less than zero for u in the physical region for the s reaction, while if the nucleon is elementary it can be shown in each order of perturbation theory[16] that the form (4.23) is correct but with exponent $= 0$ for all u. If the nucleon Regge term were to dominate in the limit, then we would have an immediate method for testing whether the nucleon is "elementary." In general, however, we cannot be sure that this particular Regge term will dominate, because there is no reason to believe that the Regge α's associated with the various π-N resonances are smaller than the nucleon α in the physical region. Of course, if *all* these Regge α's are considerably less than zero in the physical region, then the difference from the "elementary" case will still be obvious. In any case, the variation of the α's with t can still distinguish the Regge situation.

Of the higher resonances, we might expect that the $f_{\frac{5}{2}}$, $I = \frac{1}{2}$ resonance is associated with the *same* α as the nucleon,[14] since the quantum numbers of the two states are the same. Thus we should have not only $\alpha(m_N^2) = 0$ but also $\mathrm{Re}\,\alpha(m_{f\frac{5}{2}}^2) = 2$. This requires a rate of change of α of about $\alpha' \approx 1(\mathrm{Bev})^{-2}$, which, as we shall see in Sec. V, is of the same order of magnitude as the slope which seems to be experimentally indicated for the Pomeranchuk α_P.

There should then be two additional α's, associated with the $p_{\frac{3}{2}}$, $I = \frac{3}{2}$ and the $d_{\frac{3}{2}}$, $I = \frac{1}{2}$ resonances. We shall close this section by indicating the form of the Regge term associated with the 33 resonance. We use (4.20) with odd signature and $\alpha(m_{33}^2) = 1$. The form (4.20) must appear in the $[(+),(-)]$ amplitudes with coefficients $(2,1)$. The width of the 33 resonance is given for small Γ, by

$$m_{33}\Gamma_{33} = I_{33}/\epsilon_{33}, \quad (4.24)$$

where $\epsilon_{33} = \mathrm{Re}\,\alpha'(m_{33}^2)$, $I_{33} = \mathrm{Im}\,\alpha(m_{33}^2)$. We can evaluate the coefficients b at resonance by using unitarity and the condition of a single dominant decay mode $(33) \rightarrow N + \pi$. We obtain a relation analogous to (3.9)

[14] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[15] Backward scattering in the s reaction corresponds to a maximum value of u that decreases from $u = (m_N - m_\pi)^2$ at thresh-

old to $u = 0$ at infinite s. R. Blankenbeder, L. Cook, and M. L. Goldberger have pointed out to us that between this maximum value of u and $u = 0$ the quantity $|x_u|$ is less than unity. But asymptotic expansions such as (4.23) are valid in the limit of large s for fixed u and we can take such a limit only for negative u; in that case $|x_u|$ becomes large at high energies.

[16] M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065 (1961).

for $\rho \rightarrow 2\pi$, namely

$$\frac{\pi I_{33}}{b_{33}} = -\frac{1}{4\pi} \frac{q_u{}^3(E_u + m_N)}{m_\pi m_N}\bigg|_{u=m_{33}^2}. \qquad (4.25)$$

It should be remarked that for this Regge term one cannot say that we must have $\alpha < 0$ in the physical region, but only $\alpha < 1$ on the basis of assuming decreasing α's. Nevertheless, we may hope that for sufficiently large negative u this α will go negative as well.

V. NUCLEON-NUCLEON SCATTERING: EXPERIMENTAL DATA

From the preceding analysis of the π-π and π-N cases, we can easily see the general features of the N-N problem without going through the details.

The scattering amplitude without isotopic spin exchange ($I=0$ in the t reaction) contains the Pomeranchuk term and dominates the amplitude in which $I=1$ is exchanged; the latter can be studied conveniently only in backward n-p scattering, unless "elementary" mesons contribute.

Consider the nonexchange amplitude for fixed $t < 0$ and large s. Suppose only Regge terms contribute. Because of the dominance of α_P (at least near $t=0$) the main phenomenon is the diffraction peak and the cross section has the form

$$d\sigma/dt \rightarrow F_{NNP}(t)(s/2m_N{}^2)^{2\alpha_P(t)-2}, \qquad (5.1)$$

where $F(t)$ is relatively slowly varying.[11] The corresponding scattering amplitude goes like $s^{\alpha_P(t)}$.

Now suppose there is an elementary neutral vector meson with $I=0$, which might be identified with the observed ω^0. In field theory we may couple it to a conserved vector current and construct a renormalizable perturbation series; in each order, the diagrams corresponding to the exchange of an ω^0 with a dressed propagator and a dressed vertex at each end give a contribution to the scattering amplitude that persists at large s with the form

$$s(t - m_\omega{}^2)^{-1}\phi(t), \qquad (5.2)$$

where $\phi(t)$ includes the effect of the vertex function at each end and the modification of the propagator by interactions. We see that the amplitude has a real part going like s^1 at high energies with the exponent unity independent of t. This behavior is in sharp contrast with that of a "Pomeranchuk" Regge term, which is pure imaginary in the forward direction and goes like $s^{\alpha(t)}$ with α decreasing from unity away from the forward direction.

If we describe the ω^0 (or other vector meson) as a member of a Regge family with parameter $\alpha_\omega(t)$, then the term contributed to the N-N amplitude by the exchange of this family goes like $s^{\alpha_\omega(t)}$ at high energies. But $\mathrm{Re}\alpha_\omega$ equals unity only at $t=m_\omega{}^2$ and as t decreases to zero (to reach the physical region for the s reaction)

α_ω falls well below unity, so that in the region of the diffraction peak the exchange of the ω family is overshadowed by the Pomeranchuk amplitude.

There are various experimental ways to test for an ω^0 acting as an elementary particle does in perturbation theory. The thoroughness with which these tests must be carried out depends on the effective strength of the ω^0 coupling to nucleons.

First, one can compare the high-energy forward scattering cross section (eliminating the Coulomb effect for p-p collisions) with that calculated from the optical theorem for the imaginary part of the amplitude alone.[17] A real part of the forward amplitude with the same linear behavior in s as the imaginary part would come from the exchange of an "elementary" vector meson. In fact, that is just the behavior we expect for the exchange of a photon, treating it as elementary.

Second, one may search for a persistent real part of the nuclear forward scattering amplitude by looking for interference with the Coulomb amplitude, especially in nucleon-nucleus collisions.

Third, one may examine the form of the diffraction peak for a fixed high energy. In field theory, there is no known reason for the function $\phi(t)$ in (5.2) to fall off very rapidly (e.g., exponentially) as t decreases from zero. If we look at the cross section and see a diffraction peak like that given by (5.1), which for small t is $d\sigma/dt = F(t)\exp[-2|t||\alpha_P'(0)|\ln(s/2m_N{}^2)]$, we can set a rough limit on the strength with which a term decreasing approximately like $(m_\omega{}^2 - t^2)^{-2}$ could be present.

A fourth slightly different approach to the data, which can in principle test for "elementary" mesons of either spin zero or spin one (and of either isotopic spin), is the following. For each fixed momentum transfer t, we examine the s dependence of the cross section at large s and try to find the dominant power law. This method improves rapidly with energy at high momentum transfers. At two sufficiently high energies s_1 and s_2 we should have

$$\frac{d\sigma}{dt}(s_1,t) \bigg/ \frac{d\sigma}{dt}(s_2,t) \approx (s_1/s_2)^{2[L(t)-1]}, \qquad (5.3)$$

where $L(t)$ is the dominant power at momentum transfer t, whether that is a Regge $\alpha(t)$ or the *fixed* angular momentum of an "elementary" meson (1 for a vector and 0 for a scalar or pseudoscalar meson). It would be desirable to have higher energies than are at present

[17] See, for example, B. Cork, W. A. Wenzel, and C. W. Causey, Jr., Phys. Rev. **107**, 859 (1957), and G. A. Smith, H. Courant, E. C. Fowler, H. Kraybill, J. Sandweiss, and H. Taft, *ibid.* **123**, 2160 (1961). These authors find extrapolated forward cross sections consistent with a pure imaginary forward amplitude between 2 and 6 Bev. W. M. Preston, R. Watson, and J. C. Street, *ibid.* **118**, 579 (1960), find some evidence for a real part.

available,[18] but a preliminary analysis has been made of the existing data.

For the first two methods discussed above, we do not have good enough data available. The experiment of Cocconi et al.[19] permits some application of the third and fourth methods. The diffraction peak seems to be quite clearly exponential in shape. It is evident that (5.3) gives a direct measure of $L(t)$. At small $|t|$ the errors in the experimental cross sections at different energies overlap, so $L(t)$ cannot yet be accurately determined in this range. For larger $|t|$, however, in the range 1–3 Bev^2, the cross sections at different energies are clearly separated, and yield roughly $L(-1 \text{ Bev}^2) \approx 0$, $L(-2.7 \text{ Bev}^2) \approx -0.7 \pm 0.3$. Supposing that $L(t)$ is in fact $\alpha_P(t)$ in this range, we have a crude estimate of the rate at which $\alpha_P(t)$ changes. If this rate of change is maintained for positive t, we may expect α_P to pass through 2 at about $t \sim (1 \text{ Bev})^2$. There would then be a spin two object with a mass around one Bev, and $I=0$, $P=+1$, $G=+1$.[20]

The data also seem to indicate that α_P has passed through zero near $t=-1$ Bev^2. Because of the even signature of the α_P terms, that means there is a ghost of mass squared around -1 Bev^2. As we have remarked before, the difficulty may be overcome by the vanishing of all the b's coupling the "Pomeranchuk" Regge term to any particle at this point.

An alternative possibility would be to separate the $I=0$, $J=0$ state in the t channel and determine it by the N/D method in such a way that the ghost does not appear. Still another possibility is that $\alpha_P \to 0$ as $t \to -\infty$. In either of these cases, however, we would have to ignore the slight indication from experiment that the leading α passes through zero near $t=-1$ Bev^2.

Using the above estimate for $\alpha_P(t)$, or $L(t)$, one may calculate from the data the variation with t of the coefficient $F_{NNP}(t)$ in Eq. (5.1). We find $F(t)$ decreases only by a factor of ~ 3 between $t=0$ and $t=-2.7$ Bev^2; that is almost nothing compared to the decrease of $d\sigma/dt$ due to the factor $(s/2m_N^2)^{2L-2}$, which is of the order 10^5.

This encourages the hope that most of the exponential behavior in t for small t comes from the coefficient $(s/2m_N^2)^{2L-2}$, and little from the $F(t)$. If we make that assumption, putting[11]

$$d\sigma/dt \approx F(0) \exp[-2|t|\alpha_P'(0) \ln(s/2m_N^2)] \quad (5.4)$$

for small t, we find $\alpha_P'(0) \ln(s/2m_N^2) \approx 3.75$ from the data $0 < -t \lesssim 1$ Bev^2, 30 Bev$^2 \lesssim s \lesssim 40$ Bev^2. Thus we

get $\alpha_P'(0) \sim 1.3$ Bev^{-2}, which is roughly consistent with our earlier estimates.

The above discussion has all been for noncharge-exchange scattering. It is evident, however, that similar statements may be made for charge exchange scattering. The differences will be the absence of the "Pomeranchuk" Regge term and of other Regge terms corresponding to $I=0$ exchange. There remain terms for the exchange of $I=1$ vector or pseudoscalar mesons, such as ρ or π. If we define forward scattering to be the case where the proton is undeflected in angle, then $\rho^\pm$ and $\pi^\pm$ Regge terms will show up in the backward charge exchange scattering at high energies.

It is important to remark that the "Pomeranchuk" Regge term occurs in the scattering whenever a state with the quantum numbers (other than j) of the vacuum can be exchanged, even in spin-flip and genuinely inelastic processes. For example, we can see from (4.11) and (4.16) that in spin-flip π-N scattering without isotopic spin-flip, the contribution to the high energy cross section is of the form

$$\frac{d\sigma}{dt} \approx F(t)\left(\frac{s}{2m_\pi m_N}\right)^{2\alpha_P(t)-2}, \quad (5.5)$$

but with $F(t) \propto -t$ near $t=0$, since the angular distribution contains a factor $\sin^2\theta_s$. In N-N scattering also, there is a contribution of the "Pomeranchuk" Regge term to spin-flip scattering, but that is not yet of great experimental interest.

A phenomenon that has been studied experimentally[18] is inelastic diffraction scattering of protons on protons. Consider, for example, a reaction

$$N+N \to N+N^*, \quad (5.6)$$
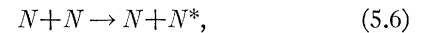
where N^* is an unstable nucleon isobar. In such a reaction, the maximum value of t in the physical region is not zero, but a negative quantity t_{max}. For large s, we have

$$t_{max} \approx -m_N^2(m_{N^*}^2 - m_N^2)^2/s^2.$$

Now whenever N^* is such that the "Pomeranchuk" Regge family can be exchanged, we have for (5.6) the contribution[21]

$$d\sigma/dt \approx F(t)(s/2m_N^2)^{2\alpha_P(t)-2}, \quad (5.7)$$

to the asymptotic cross section. Since t is less than zero in the physical region, $\alpha_P(t)$ is less than unity and $2\alpha_P(t)-2$ is less than zero. Thus inelastic scattering is reduced (at high energy in the forward direction) compared to elastic scattering.

In reaction (5.6), the 33 isobar can never be reached in the exchange of the Pomeranchuk channel, since the latter has $I=0$. The second and third resonances can,

[18] This is especially true of the application of the method to particles other than nucleons, since the range of presently available energies is then even lower.

[19] G. Cocconi, A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters 7, 450 (1961); we wish to thank Dr. Cocconi, Dr. Wetherell, and Dr. Taylor for illuminating discussions of this work. A theoretical discussion of inelastic N-N scattering is given by S. Drell and Z. Hiida, ibid. 7, 199 (1961).

[20] C. Lovelace (to be published).

[21] When a reaction involves particles that are unstable or have anomalous thresholds, certain complications may arise in the complex angular momentum plane. See B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters 8, 346 (1962).

however, be reached. If they are $d_{\frac{3}{2}}$ and $f_{\frac{5}{2}}$ states respectively of π and N, then they require the exchange of at least 1 and 2 units of angular momentum, respectively, in the forward direction. As a consequence, $F(t)$ contains a factor $[\alpha_P(t)]^2$ for the second resonance and $[\alpha_P(t)]^2[\alpha_P(t)-1]^2$ for the third resonance near $t=t_{\max}$, while the characteristic diffraction peak function $(s/2m_N{}^2)^{2\alpha_P(t)-2}$ should appear in each case as in (5.7). For small t, then, the ratio of the lowest inelastic peak to the elastic one should be roughly constant, while the ratio of the next inelastic peak to the elastic one should go approximately like t^2. All these results are consistent with the observations of Cocconi et al.[19] but further experimental work is needed if the interpretation we have given is to be properly tested.

VI. REGGE POLES AND "ELEMENTARY" PARTICLES

We have discussed at some length the effect on scattering amplitudes of the Regge pole hypothesis. However, our treatment of the contrasting situation has not so far been very thorough; we shall now go into it in more detail.

Take, for example, the $B^{\pm}$ amplitudes in π-N scattering that we considered in Sec. IV. In pseudoscalar pion-nucleon field theory, to each order in the π-N coupling constant, the $B^{\pm}$ amplitudes obey the Mandelstam representation[22] in the following form[10,23]:

$$B^{\pm}=\frac{g_{NN\pi}{}^2}{s-m_N{}^2}\mp\frac{g_{NN\pi}{}^2}{u-m_N{}^2}+\frac{1}{\pi}\int\frac{b_1(s')ds'}{s'-s}$$
$$\mp\frac{1}{\pi}\int\frac{b_1(u')du'}{u'-u}+\frac{1}{\pi^2}\int\int\frac{B_{13}{}^{\pm}(s',t')ds'dt'}{(s'-s)(t'-t)}$$
$$\mp\frac{1}{\pi^2}\int\int\frac{B_{13}{}^{\pm}(u',t')dt'du'}{(t'-t)(u'-u)}$$
$$+\frac{1}{\pi^2}\int\int\frac{B_{12}{}^{\pm}(u',s')du'ds'}{(u'-u)(s'-s)}. \quad (6.1)$$

There are no subtractions, but there are pole terms and single integrals in addition to the double integrals. Using the fact that an unsubtracted dispersion integral vanishes as its argument approaches infinity, we may explore the behavior of (6.1) as $s\to\infty$ with u fixed (asymptotic scattering at backward angles, such as we discussed in Sec. IV). Since $t\to-\infty$ as $s\to\infty$, all the terms in (6.1) vanish except the pole term and single integral term in u:

$$B^{\pm}\xrightarrow[s\to\infty,\ u\text{ fixed}]{}\mp\frac{g_{NN\pi}{}^2}{u-m_N{}^2}\mp\frac{1}{\pi}\int\frac{b_1(u')du'}{u'-u}. \quad (6.2)$$

In the π-N field theory, the asymptotic form (6.2) has a very simple interpretation.[10] It is the contribution to $B^{(\pm)}$ of the sum of all crossed Feynman diagrams in which there is a stretch of bare nucleon line between the emission of the final pion and the absorption of the initial one. Thus, it may be written in terms of the renormalized nucleon propagator $S_{FC}(p)$ and the renormalized pion-nucleon vertex operator Γ_{5c}.

The pole term comes from the matrix element

$$\bar{u}_f(p_f)\gamma_5\frac{1}{p-m_N}\gamma_5 u_i(p_i)=\bar{u}_f(p_f)\gamma_5\frac{p+m_N}{p^2-m_N{}^2}\gamma_5 u_i(p_i), \quad (6.3)$$

where p_i and p_f are the initial and final nucleon four-momenta and p is the intermediate nucleon four-momentum, with $p^2=u$ and $(p_i-p)^2=(p-p_f)^2=m_\pi{}^2$. The complete expression (6.2) comes in a similar way from the matrix element

$$\bar{u}_f(p_f)\Gamma_{5c}(p_f,p)S_{FC}(p)\Gamma_{5c}(p,p_i)u_i(p_i)$$
$$\equiv\bar{u}_f(p_f)\gamma_5\frac{p\chi(p^2)+m_N\psi(p^2)}{p^2-m_N{}^2}\gamma_5 u_i(p_i), \quad (6.4)$$

so that we have

$$\frac{g_{NN\pi}{}^2}{u-m_N{}^2}+\frac{1}{\pi}\int\frac{b_1(u')du'}{u'-u}=\frac{g_{NN\pi}{}^2}{u-m_N{}^2}\chi(u). \quad (6.5)$$

The occurrence of the nucleon pole and single integral terms as additions to the double integrals is connected with the assumption of an elementary nucleon treated in perturbation theory. Unstable or bound dynamical isobars of the nucleon would not occur in each order of the expansion, but would appear only when infinite sets of terms of the expansion are summed. We can only conjecture how they would manifest themselves, but presumably it would be as Regge poles, with contributions at large s going like $s^{\alpha(u)}$ rather than s^0 as in (6.2). Moreover, at sufficiently large negative u, we expect the α's to become negative, so that the Regge contributions would vanish as $s\to\infty$, while a term like (6.2) persists.

It has been emphasized in reference 10 that the detection of terms such as (6.2) is a way of measuring off-energy-shell quantities in field theory, namely propagators and vertex functions. To the extent that all particles correspond to Regge poles, this possibility of measuring off-shell matrix elements disappears, as we might expect in a pure S-matrix theory.

Another point made in reference 10 is the connection between propagators or vertices and broken symmetries. In formal Lagrangian field theory, one conventionally describes a broken symmetry by the equality of bare quantities (for example, the bare masses of neutron and proton) when the physical quantities are not equal. In renormalized perturbation theory, one can convert the relation between formal bare quantities to a relation between the asymptotic forms of quantities like re-

[22] There may be extra terms if there are singularities such as those discussed by R. J. Eden, P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, J. Math. Phys. **2**, 656 (1961).

[23] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

normalized propagators and vertices for large values of their arguments. Thus it was proposed that comparison of quantities like $\chi(u)$ for large u in various reactions involving different baryons and mesons would provide a test of broken symmetries in strong interactions by means of measurements of S-matrix elements for strong processes. Again, if all baryons and mesons are just Regge poles, this possibility disappears and one must reconsider the whole question of the meaning of broken symmetry.[24] The same kind of argument applies even to familiar cases like isotopic spin conservation.

The most important aspect of high energy limits like (6.2), characteristic of many processes in renormalized perturbation theory, is their connection with the "peripheral model" of high energy collisions.[25] To discuss "peripheralism," let us choose another example, namely the charge exchange amplitude $P(s,t)$ in N-N scattering associated with the invariant $\gamma_5^{(1)} \gamma_5^{(2)}$, where the upper indices refer to the two nucleons. In the perturbation expansion of renormalized π-N field theory, we have, much as in (6.2), the result

$$P(s,t) \xrightarrow[s \to \infty, \ t \text{ fixed}]{} \frac{g_{NN\pi}^2}{t - m_\pi^2} + \frac{1}{\pi} \int \frac{b_1(t')dt'}{t' - t}, \qquad (6.6)$$

where the right-hand side may once again be interpreted as the product of a propagator and two vertex functions. This time we have the pion propagator $\Delta_{FC}(t)$ and the vertex $\gamma_5 V_c(t)$ for emission of an off-shell pion between two free nucleon lines:

$$\frac{g_{NN\pi}^2}{t - m_\pi^2} + \frac{1}{\pi} \int \frac{b_1(t')dt'}{t' - t} = g_{NN\pi}^2 [V_c(t)]^2 \Delta_{FC}(t). \quad (6.7)$$

Now the peripheral model emphasizes the dominance at high energies of this exchange of one off-shell pion. (Moreover, the same kind of term is assumed to dominate many other reactions.) But the situation is quite different if the pion belongs to a Regge family; in that case, we have an asymptotic amplitude proportional to $s^{\alpha(t)}/\sin\pi\alpha(t)$, which agrees with (6.7) only at $t = m_\pi^2$, where $\alpha = 0$. In the physical region for N-N scattering ($t \leq 0$), α is negative and the amplitude falls to zero at high energies instead of remaining constant. Of course, at moderate energies (~ 1 Bev) or even substantially higher if $|t|$ is kept small, the contribution of the one-pion pole is still expected to play an important role in the physical region.

Similar considerations apply to the amplitude $\mathcal{U}(s,t)$, in N-N scattering without isotopic spin exchange, that multiplies the invariant $\gamma_\alpha^{(1)} \gamma_\alpha^{(2)}$. Renormalized perturbation theory for a neutral vector meson such as ω^0, or else "peripheralism," suggests an

asymptotic form

$$\gamma_{NN\omega}^2 [V_{c\omega}(t)]^2 \Delta_{FC\omega}(t), \qquad (6.8)$$

analogous to (6.7). We saw in Sec. V that this asymptotic form is very different from what is produced by Regge poles alone.

There are other amplitudes (such as $A^\pm$ in π-N scattering) which obey the Mandelstam representation with subtractions, even in renormalized perturbation theory. For those cases, we cannot make any clear cut statement about the asymptotic behavior for large s. Moreover, if we consider field theory apart from the perturbation expansion, or merely allow for the possibility that the sum of the series acts differently from each individual term, then we do not know how many subtractions there are in the Mandelstam representation even for $B^\pm$ in π-N scattering or P or $\mathcal{U}$ in N-N scattering. If additional subtractions are necessary, then many new kinds of asymptotic behavior are possible, including the type characteristic of Regge poles. After all, the Regge pole hypothesis is only a special case of the situation with subtractions. Thus if the experiments show that the nucleon, pion, etc., are all members of Regge families, then we still cannot rigorously exclude a field theory that treats these particles as "elementary" in a broad sense. However, if the nonsingular character of the amplitudes according to the Regge hypothesis really permits the calculation of all coupling constants and mass ratios, then there is not much point in calling the particles "elementary."

In conclusion, let us list a number of reactions in which the Regge pole hypothesis can be tested for various baryons and mesons.

$$N: \quad \pi + N \to N + \pi,$$
$$\gamma + N \to N + \pi,$$
$$\pi + N \to N + \omega, \text{ etc.},$$
$$Y = \Lambda, \Sigma: \quad \pi + N \to Y + K,$$
$$K + N \to N + K,$$
$$\gamma + N \to Y + K, \text{ etc.},$$
$$\pi: \quad N + N \to N + N \text{ (charge exchange)},$$
$$\gamma + N \to \pi + N,$$
$$\pi + N \to \rho + N$$
$$K + N \to K^* + N, \text{ etc.},$$
$$K: \quad \pi + N \to K^* + Y,$$
$$\gamma + N \to K + Y, \text{ etc.},$$
$$\omega: \quad N + N \to N + N,$$
$$\pi + N \to \rho + N,$$
$$\gamma + N \to \pi^0 + N, \text{ etc.}$$

ACKNOWLEDGMENTS

[24] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[25] G. Salzman, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960); S. Drell, Revs. Modern Phys. **33**, 458 (1961).

Radiation Laboratory of the University of California at Berkeley for its hospitality to the HEPS group.

APPENDIX

We should like to discuss the effect of an exchange potential in the Schrödinger equation on the form of the Regge terms, and in particular to justify Eq. (2.7).

A trivial justification is the following: consider the scattering amplitude $T(q,x)$, where q is the momentum and x is the cosine of the scattering angle, and write it as the sum of an even part and an odd part in x:

$$T(q,x)=T_+(q,x)+T_-(q,x). \qquad (A1)$$

Now if $V_d(r)$ is the direct potential and $V_e(r)$ the exchange potential, then the effective potential for the even partial waves is $V_E=V_d+V_e$, while that for the odd partial waves is $V_O=V_d-V_e$. Now suppose we solve the Schrödinger equation for all partial waves with the potential V_E, obtaining the scattering amplitude $T_E(q,x)$; likewise, with V_O, we obtain the amplitude $T_O(q,x)$. Then we have

$$T_+(q,x)=\tfrac{1}{2}[T_E(q,x)+T_E(q,-x)],$$
$$T_-(q,x)=\tfrac{1}{2}[T_O(q,x)-T_O(q,-x)]. \qquad (A2)$$

Now each amplitude T_E and T_O has its own Regge pole terms, and these appear symmetrized or antisymmetrized in $T(q,x)$. Hence we have (2.7).

We may look at the same nonrelativistic problem in another way, which is more relevant to the relativistic theory. The scattering amplitude may be assumed to have certain analyticity properties, as a function of the energy and the momentum transfer, which are summarized by the statement that it satisfies the Mandelstam representation. These analyticity properties, together with the assumption that the amplitude also satisfies the usual unitarity condition, allow the construction of an integral equation for the amplitude, in which the potential itself appears as an inhomogeneous term. For potentials which are superpositions of Yukawa potentials, it has been shown[26] that the unitarity condition and the analyticity properties completely define the scattering amplitude, at least when the Mandelstam representation has no subtractions. This assumption is, presumably, just the statement that there are no bound states or resonances. If bound states or resonances do exist, they appear through Regge terms and the Mandelstam representation will require subtractions. Nevertheless, it is plausible to assume that the unitarity condition and the analyticity properties still completely determine the problem.

The unitarity condition for the scattering amplitude

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[26] R. Blankenbecler, M. Goldberger, N. N. Khuri, and S. Treiman, Ann. Phys. (New York) **10**, 62 (1960).

$T(q,x)$ may be written

$$\mathrm{Im}T(q,x)$$
$$=\frac{q}{4}\int_{-1}^{1}dx_1\int_{-1}^{1}dx_2\sum_l(2l+1)P_l(x)P_l(x_1)P_l(x_2)$$
$$\times T(q,x_1)^*T(q,x_2). \qquad (A3)$$

The normalization here is defined so that

$$T(q,x)=\sum_l(2l+1)P_l(x)(\sin\delta_l e^{i\delta_l}/q). \qquad (A4)$$

We now write $T(q,x)$ as in (A1), as a sum of even and odd parts in x. Using the facts that $P_l(-x)=(-1)^lP_l(x)$ and

$$\{\tfrac{1}{2}[1\pm(-1)^l]\}^2=\tfrac{1}{2}[1\pm(-1)^l],$$

it is easy to see that the unitarity condition is true separately for T_+ and T_-, so that we have

$$\mathrm{Im}T_\pm(q,x)$$
$$=\frac{q}{4}\int_{-1}^{1}dx_1\int_{-1}^{1}dx_2\sum_l(2l+1)P_l(x)P_l(x_1)P_l(x_2)$$
$$\times T_\pm(q,x_1)^*T_\pm(q,x_2). \qquad (A5)$$

Since the analyticity properties of $T_\pm$ are essentially the same as those of T itself, there are, as a result, two separate scattering problems which differ only in the potential term. First suppose only a direct potential exists. It is a function of $2q^2(1-x)-\Delta^2$ and we will write it $V_d(\Delta^2)$. Then the two potentials for T_+ and T_- are

$$V_\pm=\tfrac{1}{2}[V_d(\Delta^2)\pm V_d(\bar{\Delta}^2)], \qquad (A6)$$

where $\bar{\Delta}^2=2q^2(1+x)$. If the Regge terms are found for the T_+ and T_- amplitudes in this case, the same α's and β's must occur in each. For T_+, we find the Regge terms in the form

$$(\beta/\sin\pi\alpha)\tfrac{1}{2}[P_\alpha(-x)+P_\alpha(x)], \qquad (A7)$$

and for T_-,

$$(\beta/\sin\pi\alpha)\tfrac{1}{2}[P_\alpha(-x)-P_\alpha(x)], \qquad (A8)$$

with the same α and β, so that in $T=T_++T_-$ the Regge terms are simply

$$(\beta/\sin\pi\alpha)P_\alpha(-x). \qquad (A9)$$

If, however, an exchange potential is introduced as well, the situation changes. An exchange potential is a function of $\bar{\Delta}^2$, and we may call it $V_e(\bar{\Delta}^2)$. The two effective potentials for the T_+ and T_- amplitudes now become

$$V_\pm=\tfrac{1}{2}\{[V_d(\Delta^2)+V_e(\bar{\Delta}^2)]\pm[V_d(\bar{\Delta}^2)+V_e(\Delta^2)]\}$$
$$=\tfrac{1}{2}\{[V_d(\Delta^2)\pm V_e(\Delta^2)]\pm[V_d(\bar{\Delta}^2)\pm V_e(\bar{\Delta}^2)]\}. \qquad (A10)$$

Now the $V_\pm$ are no longer of the form (A6), in that the potential corresponding to $V_d(\Delta^2)$ in (A6) is no longer the same in V_+ as it is in V_-. Therefore, we can no longer expect the Regge α's and β's appearing in (A7)

to be the same as those in (A8), and as a result, the Regge terms in T will now be of the form

$$\frac{\beta_+}{\sin\pi\alpha_+}\left(\frac{P_{\alpha_+}(-x)+P_{\alpha_+}(x)}{2}\right)$$

$$+\frac{\beta_-}{\sin\pi\alpha_-}\left(\frac{P_{\alpha_-}(-x)-P_{\alpha_-}(x)}{2}\right),\quad\text{(A11)}$$

instead of as in (A9).

In the relativistic problem, if we discuss spinless particles and make the strip approximation, the mathematics is essentially identical with that we have gone through above. Without the strip approximation, in terms of a "generalized potential,"[27] the equations still look very similar and it is reasonable to expect that the results obtained here remain valid. We shall therefore assume that in the general case the form (A11) is correct at large x.

We will be interested in high energies in the crossed channel. If s is the square of the total c.m. energy in the crossed channel, and q is the initial and final c.m. momentum in the original channel (we take equal masses for convenience), then large s means $x\approx-s/2q^2$. The Regge terms in this limit then become

$$\frac{\beta}{(2q^2)^\alpha}\frac{1}{\sin\pi\alpha}\frac{[s^\alpha\pm(-s)^\alpha]}{2}.\quad\text{(A12)}$$

[27] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

Now β is essentially the residue of the lth partial wave amplitude at a pole $l=\alpha$ in the complex angular momentum plane; we may therefore expect that as a function of q, β behaves like $(2q^2)^\alpha$ near each threshold. It will be convenient to factor this dependence out of β, and furthermore to write $(-s)^\alpha$ as $s^\alpha e^{-i\pi\alpha}$. Thus, the Regge terms may be written

$$\frac{b}{\sin\pi\alpha}\left(\frac{s}{2m^2}\right)^\alpha\left(\frac{1\pm e^{-i\pi\alpha}}{1}\right),\quad\text{(A13)}$$

as in rule (3) of Sec. II.

The choice of phase, $(-s)^\alpha=s^\alpha e^{-i\pi\alpha}$ rather than $s^\alpha e^{+i\pi\alpha}$, is the one suggested by the analyticity of the scattering amplitude in the upper half of the complex u plane; note that the direct potential in the t channel is associated with the cut in u. Of course, we could have absorbed a factor $\pm e^{-i\pi\alpha}$ into b and gotten $(1\pm e^{+i\pi\alpha})/2$ instead of $(1\pm e^{-i\pi\alpha})/2$ in (A13). So our choice of phase reflects a belief that b as defined in (A13) has simple properties. In fact, we conjecture that it is real in a region extending down from threshold.

Note added in proof. We have been able to prove, assuming the Mandelstam representation, that $\alpha(t)$ and $b(t)$ are real analytic functions with only right-hand cuts. See also A. O. Barut and D. E. Zwanziger (to be published).