The three sums appearing in this expression have been evaluated elsewhere for the present model.[9] They have the values

$$\sum_{\mathbf{k}j}\frac{e_x^2(\mathbf{k}j)}{\omega_j^2(\mathbf{k})}\sin^2[\tfrac{1}{2}\pi a_0(k_x+k_y)]=\frac{N}{\mu_2}(0.5811),$$

$$\sum_{\mathbf{k}j}\frac{e_z^2(\mathbf{k}j)}{\omega_j^2(\mathbf{k})}\sin^2[\tfrac{1}{2}\pi a_0(k_x+k_y)]=\frac{N}{\mu_2}(0.6323), \quad \text{(A18)}$$

$$\sum_{\mathbf{k}j}\frac{e_x(\mathbf{k}j)e_y(\mathbf{k}j)}{\omega_j^2(\mathbf{k})}\sin^2[\tfrac{1}{2}\pi a_0(k_x+k_y)]=\frac{N}{\mu_2}(-0.0811).$$

With these results, we can finally express S_2 as

$$S_2=-\frac{N^2\kappa^2}{\mu_2^2}(17.414). \quad \text{(A19)}$$

Combining Eqs. (A1), (A13), and (A18), we find that M_1 equals

$$M_1=-\frac{kT}{2\mu_2^2M^2}\kappa^2[2Aa_0^2+17.414B]. \quad \text{(A20)}$$

If we compare this result with that obtained using Ludwig's approximation,

$$M_1=-\frac{kT}{2\mu_2^2M^2}\kappa^2[2Aa_0^2+12B], \quad \text{(A21)}$$

we see that the coefficient of Aa_0^2 is given exactly by Ludwig's approximation, while the coefficient of B is 31% low in this approximation. These results are consistent with those obtained elsewhere[4] in cases where exact calculations could also be carried out. In all cases studied so far, Ludwig's approximation gives results which are somewhat smaller than the true values.

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Certain General Order-Disorder Models in the Limit of Long-Range Interactions*

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We investigate in the limit as the range of the interparticle interactions becomes indefinitely great, but is still small compared to the size of the system, the behavior of a large class of order-disorder models. This class includes, for example, the Ising model, the spherical model, and the Gaussian model. We show that when certain general conditions hold and the interparticle interaction is chosen to be the same for all models, but otherwise arbitrary, the energy per particle above the critical temperature has the same limiting value through terms of order $1/R$, where R is a measure of the number of spins in the range of the interaction. We further show why the behavior above the critical point in this limit does not necessarily provide information about the behavior below the critical point. Some examples are worked out which illustrate the above results.

1. INTRODUCTION

THE purpose of this paper is to investigate the behavior of a certain class of order-disorder models as a function of the range of the interaction. We investigate the limiting behavior of these models as the range of interaction becomes indefinitely great, but is still much smaller than the total size of the system. In the second section of this paper we show that above the critical point, when certain general conditions are met, the details of the model (probability distribution of the various states of the system) but not the shape of the interaction are unimportant in the limit of indefinitely long-range forces. The class of models considered is general enough to contain the Ising model, the spherical model, and the Gaussian model. Below the critical point no such general result is obtained. We see that in this region the details of the model affect the energy per spin in leading order.

In the third section we compute the energy per spin for the one-dimensional spherical model (and Gaussian model) with exponential interactions between spins. We verify explicitly the results of the second section for this type of interaction by comparing the results of the third section with the previously known results for the Ising model. We also verify explicitly that the behavior below the critical point is different for the spherical and Ising models.

In the last section of this paper we evaluate the energy per spin of the three-dimensional spherical model as a function of the range of a force which drops off approximately exponentially with distance. Here we may follow the behavior of the known third-order transition with range.

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We find that the thermal and magnetic properties are the same below the transition point for the one-, two-, and three-dimensional spherical models in the long-range interaction limit.

2. LIMIT OF LONG-RANGE FORCES

We shall illustrate a more general result on the behavior of various order-disorder models by three examples: the Gaussian model, the spherical model, and the Ising model. For these three models (and in fact for a whole class of models) the partition function is the same above the critical point through terms of order $1/R$, where R is the range of the force. Below the transition point there is little or no relationship. We shall use methods introduced previously by Kac[1] and extended by us[2] to show this result. Previously Brout[3] has essentially obtained, among other results, a special case of this result by diagrammatic analysis.

The first step in our program is to rewrite the partition function in a form to permit the sum over all states of the system to be performed easily. Let us suppose that the energy of the system given by a quadratic form (A is assumed symmetric)

$$E/kT=-\tfrac{1}{2}\sum_{i,j}\nu_i A_{ij}\nu_j. \tag{2.1}$$

If we make use of the well-known integration formula,[4]

$$\exp\Big(\tfrac{1}{2}\sum_{i,j}\nu_i A_{ij}\nu_j\Big)=(2\pi)^{-N/2}\int_{-\infty}^{+\infty}\cdots\int \exp\Big(-\tfrac{1}{2}\sum_{i=1}^{N}x_i^2-\sum_{i,j}x_i(A^{\frac{1}{2}})_{ij}\nu_j\Big)\prod_{j=1}^{N}dx_j, \tag{2.2}$$

which holds for any symmetric A_{ij}, then the partition function

$$Z=\sum_{\text{all states}}\exp(-E/kT) \tag{2.3}$$

may be written as

$$Z=(2\pi)^{-N/2}\sum_{\text{all states}}\int_{-\infty}^{+\infty}\cdots\int \exp\Big(-\tfrac{1}{2}\sum_{i=1}^{N}x_i^2-\sum_{i,j}x_i(A^{\frac{1}{2}})_{ij}\nu_j\Big)\prod_{j=1}^{N}dx_j$$
$$=(2\pi)^{-N/2}\int_{-\infty}^{+\infty}\cdots\int \exp\Big(-\tfrac{1}{2}\sum_{i=1}^{N}x_i^2\Big)\mathfrak{M}(x_i)\prod_{j=1}^{N}dx_j, \tag{2.4}$$

where $\mathfrak{M}(x_i)$ is defined by

$$\mathfrak{M}(x_i)\equiv\sum_{\text{all states}}\exp\Big(-\sum_{i,j}x_i(A^{\frac{1}{2}})_{ij}\nu_j\Big). \tag{2.5}$$

Eq. (2.4) is valid whenever (2.3) is. For the examples we are considering, $\mathfrak{M}(x_i)$ is simply evaluated.

In the Ising model the states of the system are given by $\nu_j=\pm1$, for all j. Thus we have at once, normalizing $\mathfrak{M}(0)=1.0$,

$$\mathfrak{M}_I(x_i)=\prod_{j=1}^{N}\{\cosh[\sum_{i=1}^{N}x_i(A^{\frac{1}{2}})_{ij}]\}. \tag{2.6}$$

In the Gaussian model[5] the ν_j are independently and normally distributed with mean zero and unit variance. Thus, again normalizing $\mathfrak{M}(0)=1.0$, we obtain

$$\mathfrak{M}_G(x_i)=\int_{-\infty}^{+\infty}\cdots\int \exp\Big(-\tfrac{1}{2}\sum_{i=1}^{N}\nu_i^2-\sum_{i,j}x_i(A^{\frac{1}{2}})_{ij}\nu_j\Big)\prod_{j=1}^{N}\frac{d\nu_j}{(2\pi)^{\frac{1}{2}}}=\exp\Big(\tfrac{1}{2}\sum_{i,j}x_iA_{ij}x_j\Big). \tag{2.7}$$

In the spherical model[5] all sets of values of the ν_j which satisfy

$$\sum_{j=1}^{N}\nu_j^2=N \tag{2.8}$$

are equally likely. We impose this restriction in the now standard way by use of the Laplace transform formula

$$\delta(x-y)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}e^{q(x-y)}dq. \tag{2.9}$$

[1] M. Kac, Phys. Fluids **2**, 8 (1959).
[2] G. A. Baker, Jr., Phys. Rev. **122**, 1477 (1961).
[3] R. Brout, Phys. Rev. **118**, 1009 (1960); and **122**, 469 (1961).
[4] See, for example, M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics* (Hafner Publishing Company, New York, 1958), Vol. 1, Sec. 15.2.
[5] Introduced by T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).

Thus, approximately normalizing $\mathfrak{M}(0)=1.0$, we have

$$\mathfrak{M}_S(x_i)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\int_{-\infty}^{+\infty}\cdots\int \exp\Big(Nq-\sum_{j=1}^{N} q\nu_j^2-\sum_{i,j} x_i(A^{\frac{1}{2}})_{ij}\nu_j\Big)\prod_{i=1}^{N}\frac{d\nu_i}{(2\pi e)^{\frac{1}{2}}}$$

$$=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{Nq}\exp\Big(\frac{1}{4q}\sum_{i,j} x_iA_{ij}x_j\Big)(2eq)^{-N/2}dq. \quad (2.10)$$

The value of the q integral is known[6] to be

$$\mathfrak{M}_S(x_i)=\Bigg(\frac{4N}{\sum_{i,j} x_iA_{ij}x_j}\Bigg)^{\frac{1}{2}N-\frac{1}{2}} I_{\frac{1}{2}N-1}([N\sum_{i,j} x_iA_{ij}x_j]^{\frac{1}{2}})(2e)^{-\frac{1}{2}N}. \quad (2.11)$$

For N very large we may use the method of steepest descents[7] to obtain an asymptotic value for (2.10). It is

$$\mathfrak{M}_S(x_i)\sim\{\tfrac{1}{2}[1+(1+4\xi)^{\frac{1}{2}}]\}^{-\frac{1}{2}N} \times\exp\{\tfrac{1}{2}N[(1+4\xi)^{\frac{1}{2}}-1]\}, \quad (2.12)$$

where

$$\xi=(\sum_{i,j} x_iA_{ij}x_j)/N. \quad (2.13)$$

The next step in our program is to examine the behavior of the interaction matrix $A_{\mathbf{ij}}$. We shall now assume that the indices $\mathbf{i}$ and $\mathbf{j}$ are d-dimensional position vectors and that A is a function of $(\mathbf{i}-\mathbf{j})$ only. We may then diagonalize A by introducing the eigenvectors $Z_{\mathbf{q}}$ and eigenvalues $a(\mathbf{q})$,

$$Z_{\mathbf{q}}=\frac{1}{N^{\frac{1}{2}}}\sum_{\mathbf{j}}\exp(2\pi i\mathbf{q}\cdot\mathbf{j})x_{\mathbf{j}},$$
$$A_{\mathbf{ij}}=\frac{1}{N}\sum_{\mathbf{q}} a(\mathbf{q})\exp[2\pi i\mathbf{q}\cdot(\mathbf{j}-\mathbf{i})]. \quad (2.14)$$

If $\mathbf{r}$ is the vector between two points and $V(\mathbf{r})$ is the potential, then

$$a(\mathbf{q})=\frac{1}{N}\sum_{\text{lattice}} V(\mathbf{r})\exp[2\pi i\mathbf{r}\cdot\mathbf{q}]. \quad (2.15)$$

In order to maintain the same total strength for $V(\mathbf{r})$ for varying ranges, let

$$V(\mathbf{r})=v(\mathbf{r}/R^{1/d})/R, \quad (2.16)$$

where R is proportional to the number of spins in the range of the interaction. Using (2.16), Eq. (2.15) becomes

$$a(\mathbf{q})=\frac{1}{RN}\sum_{\text{lattice}} v(\boldsymbol{\zeta})\exp(2\pi iR^{1/d}\boldsymbol{\zeta}\cdot\mathbf{q}), \quad (2.17)$$

where $\boldsymbol{\zeta}=\mathbf{r}/R^{1/d}$. We may show by use of the same type of arguments which lead to Riemann's theorem on trigonometric functions[8] that

$$|a(\mathbf{q})|\leq M\prod_{j=1}^{d}(1+R^{1/d}q_j)^{-1}, \quad (2.18)$$

where q_j is the jth component of $\mathbf{q}$. We have also assumed that

$$\int|v(\boldsymbol{\zeta})|d\boldsymbol{\zeta}\leq M', \quad (2.19)$$

where M and M' are certain finite positive constants.

We are now in a position to show that for $T>T_c$ the partition function for each of the three examples we are considering is

$$Z=(2\pi)^{-\frac{1}{2}N}\int_{-\infty}^{+\infty}\cdots\int\exp\{-\tfrac{1}{2}\sum_{\mathbf{q}}|Z_{\mathbf{q}}|^2[1-a(\mathbf{q})]\}\prod_{\mathbf{q}} dZ_{\mathbf{q}}+O([\ln R]^{2d}/R^2). \quad (2.20)$$

For the Gaussian model, (2.20) follows at once from (2.4) and (2.7) when we note that

$$\sum_{i,j} x_iA_{ij}x_j=\sum_{\mathbf{q}} a(\mathbf{q})|Z_{\mathbf{q}}|^2. \quad (2.21)$$

Eq. (2.20) is exact for the Gaussian model.

For the spherical model we may, assuming ξ to be small, expand (2.12) as

$$\approx\exp(+\tfrac{1}{2}N\xi-\tfrac{1}{4}N\xi^2+\cdots). \quad (2.22)$$

The terms through order ξ in (2.22) together with (2.4) give (2.20). It remains to show that the remainder of the expression is neglectable to the required order. The weighting factor in (2.4) is a positive definite quadratic form. As (2.12) can increase asymptotically as $\xi\rightarrow\infty$ at a rate bounded by $\exp[N(\xi)^{\frac{1}{2}}]$ the whole integral converges. If in addition the maximum of $|a(\mathbf{q})|$ is less than 1.0, then the integrand has a single maximum at the origin ($|Z_{\mathbf{q}}|=0$). Leaving aside factors of the order of 1.0, we may estimate the mean value of ξ^2 by estimating the value of each term of ξ separately and adding them up. Thus, replacing sums with integrals over the

[6] G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1948), pair 650.0.

[7] See, for instance, H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, New York, 1950), Chap. 17.

[8] See, for example, P. Franklin, *A Treatise on Advanced Calculus* (John Wiley & Sons, Inc., New York, 1949), p. 480.

first Brillouin zone, we obtain

$$N\xi^2 \propto N\left[\int \frac{a(\mathbf{q})d\mathbf{q}}{1-a(\mathbf{q})}\right]^2 < \frac{N}{(1-a_{\max})^2}\left[\int a(\mathbf{q})d\mathbf{q}\right]^2. \quad (2.23)$$

By Eq. (2.18), we have

$$\frac{NM^2}{(1-a_{\max})^2}\left[\prod_{j=1}^{d}\int \frac{dq_i}{1+R^{1/d}q_j}\right]^2 = \frac{NM^2}{(1-a_{\max})^2}\left(\frac{\ln R}{R^{1/d}}\right)^{2d}, \quad (2.24)$$

plus terms of lower order, which is the required result. We denote the absolute value of the maximum $a(\mathbf{q})$ by $a_{\max}$. We remark that (leaving factors of order 1.0 aside) one can obtain corresponding bounds for ξ^3, ξ^4, etc. Equation (2.24) may not be the best bound obtainable, but it is sufficient to show that (2.20) is at least asymptotically correct. It is to be noted that this proof fails when $a_{\max}$ reaches 1.0. At this point the integrand of (2.4) no longer possesses a single peak at the origin. We identify

$$a_{\max} = 1.0 \quad (2.25)$$

as the equation for the critical point. For lower temperatures than it, there is little relationship between the models considered here. This point will be made clearer by several examples examined in the following sections of this paper.

We follow the same procedure to establish (2.20) for the Ising model as we did for the spherical model. Expanding the ln cosh terms in the exponent of the partition function in powers of their arguments, we get from (2.6)

$$\begin{aligned}\mathfrak{M}_I(x_i) &\approx \exp\{\tfrac{1}{2}\sum_j [\sum_i x_i(A^{\frac{1}{2}})_{ij}]^2 - \tfrac{1}{12}\sum_j [\sum_i x_i(A^{\frac{1}{2}})_{ij}]^4 + \cdots\} \\ &= \exp\{\tfrac{1}{2}\sum_{ij} x_i A_{ij} x_j - \tfrac{1}{12}\sum_j [\sum_i x_i(A^{\frac{1}{2}})_{ij}]^4 + \cdots\}. \quad (2.26)\end{aligned}$$

We again identify the first term in the exponent as the part of $\mathfrak{M}_I(x_i)$ which we need to complete (2.20). Our problem is again to bound the remainder. Transforming the fourth-order term to the $Z_{\mathbf{q}}$ basis, we obtain

$$\frac{N}{N^2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}{}' Z_{\mathbf{q}_1}a^{\frac{1}{2}}(\mathbf{q}_1)Z_{\mathbf{q}_2}a^{\frac{1}{2}}(\mathbf{q}_2)Z_{\mathbf{q}_3}a^{\frac{1}{2}}(\mathbf{q}_3)Z_{\mathbf{q}_4}a^{\frac{1}{2}}(\mathbf{q}_4). \quad (2.27)$$

The prime means the summation extends over only those $\mathbf{q}_i$ for which $\sum_i \mathbf{q}_i$ is an integral multiple of $\mathbf{N}$. On the assumption that (2.27) is small, we may compute its approximate magnitude by treating it as a perturbation to the integrand of (2.20). To avoid a spuriously low estimate we compute the root-mean-square value, and include all terms. Thus the error due to the fourth-order term is less than

$$N\left\{\left[\frac{1}{N}\sum_{\mathbf{q}} E(|Z_{\mathbf{q}}|^2)a(\mathbf{q})\right]^4\right\}^{\frac{1}{2}}, \quad (2.28)$$

plus terms of lower order in N. However, this is the same expression as we obtained (2.23) for the error in the spherical model. Hence the same bound (2.24) follows under the same restrictions ($a_{\max} < 1.0$). Corresponding bounds can be obtained for any order. Thus Eq. (2.20) also holds for the Ising model.

We see from the above special cases that for the asymptotic formula (2.20) to hold the important requirements are: (1) the integrand of (2.4) must have a single peak at the origin; (2) through second order in the x's, $\ln\mathfrak{M}(x)$ must agree with $\ln\mathfrak{M}_G(x)$; and (3) the higher order terms in $\ln\mathfrak{M}(x)$ must becrease more rapidly than $1/R$ in the limit as $R\rightarrow\infty$.

Although condition (3) is satisfied for a wide class of functions, it need not always be satisfied as can be seen from the following example. Suppose the states of the system are $\nu_{\mathbf{q}} = \pm 1$, where $\nu_{\mathbf{q}}$ are the Fourier transforms of the ν_{j}. Then

$$\mathfrak{M}(x) = \prod_{\mathbf{q}}\{\cosh Z_{\mathbf{q}}[a(\mathbf{q})]^{\frac{1}{2}}\}. \quad (2.29)$$

The partition function (2.4) may be easily evaluated to give

$$Z = \exp\left[\tfrac{1}{2}N\int a(\mathbf{q})d\mathbf{q}\right]. \quad (2.30)$$

Evaluating (2.20), we obtain

$$\begin{aligned}Z &= \exp\left[-\tfrac{1}{2}N\int \ln(1-a(\mathbf{q}))d\mathbf{q}\right] \\ &\approx \exp\left[\tfrac{1}{2}N\int a(\mathbf{q})d\mathbf{q} - \tfrac{1}{4}N\int a^2(\mathbf{q})d\mathbf{q} + \cdots\right]. \quad (2.31)\end{aligned}$$

From (2.18) we see that there is a volume of order R^{-1} near the origin of $\mathbf{q}$ space in which, in general, $a(\mathbf{q})$ is of order 1.0 rather than of order $1/R$. Thus there will be contributions from this region to the higher terms in (2.31) by terms of order $1/R$, which spoils (2.20).

3. EXAMPLE OF THE ONE-DIMENSIONAL SPHERICAL MODEL

The Ising model for one dimension with an exponential interaction between spins has previously been solved exactly and some of its properties analyzed in the long-range limit.[2] If the spin-spin interaction energy is given by

$$(1-r)J\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} r^{k-j-1}\nu_j\nu_k, \quad (3.1)$$

then the energy per particle is found to be asymptotically equal to

$$-\tfrac{1}{2}(1-r)J[-1+(1-2K)^{-\frac{1}{2}}], \qquad (3.2)$$

above the critical point, $K_c=\frac{1}{2}$. The constant J is equal to half the maximum possible interaction energy per spin, and $K=J/kT$. Below the critical point the energy is

$$E=-\tanh^2 z, \qquad (3.3)$$

where

$$z=2K\tanh z. \qquad (3.4)$$

In the Appendix of our previous paper[2] it was stated that the results of the Bragg-Williams approximation differed from these by a factor of two. That was incorrect. They are the same. Equation (A1) of that paper needs a factor of two to be consistent with Eq. (2.8), where K was defined as $\frac{1}{2}$ the maximum possible interaction energy divided by kT.

We shall work out the corresponding results for the spherical model.[5] We will verify explicitly that the results of the previous section are valid in this case of exponential interaction. If we define

$$B_{ij}=r^{|i-j|}+r^{N-|i-j|}, \qquad (3.5)$$

then we may write the interaction energy over kT as

$$A_{ij}=-(1-r)K[B_{ij}-(1+r^N)\delta_{ij}]/(2r), \qquad (3.6)$$

where δ_{ij} is the Kronecker delta and N is the number of spins in the system. The quantity K is taken to be half the maximum total interaction energy per spin, J, over kT as above. If we introduce the spherical constraint, (2.8), by means of (2.9), we may write the partition function, (2.3), as

$$Z=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{Nq}dq\frac{1}{(2\pi e)^{N/2}}\int_{-\infty}^{+\infty}\cdots \times\int \exp\Big(-q\sum_{j=1}^{N} v_j^2+\sum_{i,j} v_iA_{ij}v_j\Big)\prod_{j=1}^{N} dv_j. \qquad (3.7)$$

The integration over the v's is readily performed, yielding[4]

$$Z=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{N(q-\frac{1}{2})}dq\,\det|q\delta_{ij}-A_{ij}|^{-\frac{1}{2}}. \qquad (3.8)$$

Since the determinant is the product of the eigenvalues and the eigenvalues are, ignoring terms of order r^N,

$$q-(1-r)K\left[\frac{\cos(2\pi j/N)-r}{1+r^2-2r\cos(2\pi j/N)}\right], \quad j=1,\cdots,N, \qquad (3.9)$$

we may, where N is large enough to approximate sums with integrals, write (3.8) as

$$Z\approx\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} dq\exp\left[N(q-\tfrac{1}{2})-\frac{N}{4\pi}\int_0^{2\pi}d\omega \times\ln\left\{q-(1-r)K\left[\frac{\cos\omega-r}{1+r^2-2r\cos\omega}\right]\right\}\right]. \qquad (3.10)$$

For $r<1$, we may rewrite the ω integral in (3.10) as

$$\int_0^{2\pi}d\omega\ln\{(1+r^2)q+r(1-r)K-[(1-r)K+2rq]\cos\omega\} -\int_0^{2\pi}d\omega\ln(1+r^2-2r\cos\omega). \qquad (3.11)$$

By a standard integration formula[9] we may easily show that the second integral vanishes. Using the same standard formula to do the first integral, we may write for (3.10)

$$Z=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}dq\exp\left\{N(q-\tfrac{1}{2})-\tfrac{1}{2}N\ln\left[\frac{q(1+r^2)+r(1-r)K+\{[q(1+r^2)+r(1-r)K]^2-[(1-r)K+2rq]^2\}^{\frac{1}{2}}}{2}\right]\right\}.$$

In the limit as $N\to\infty$ we may evaluate $(\ln Z)/N$ exactly by means of the method of steepest descents.[7] The location of the saddle point is given by solving the following equation for q:

$$(2q-1)r=(1-r)\times\left\{-K+\tfrac{1}{2}K\left[q^2-\frac{2r}{1+r}qK-\left(\frac{1-r}{1+r}\right)K^2\right]^{-\frac{1}{2}}\right\}. \qquad (3.13)$$

Equation (3.13) was derived by setting the partial derivative of the logarithm of the integrand of (3.12) equal to zero and then simplifying. As (3.13) is a quartic equation it can be solved explicitly for q, but we will not carry out this step. For $r=0$ we recover the known[5] short-range solution,

$$q_s=(K^2+\tfrac{1}{4})^{\frac{1}{2}}. \qquad (3.14)$$

For r very near unity, the right-hand side of (3.13) is very small, so long as the radical does not vanish. When this condition is satisfied, we obtain $q_s=\frac{1}{2}$. For r near 1, the radical vanishes for $q\approx 0$ and $q\approx K$. Thus, in the limit as r tends to unity, we obtain

$$q_s=\tfrac{1}{2} \quad (K<\tfrac{1}{2}),\qquad q_s=K \quad (K>\tfrac{1}{2}). \qquad (3.15)$$

The energy per particle implied by the spherical

[9] B. O. Peirce, *A Short Table of Integrals* (Ginn and Company, Boston, 1910), No. 523.

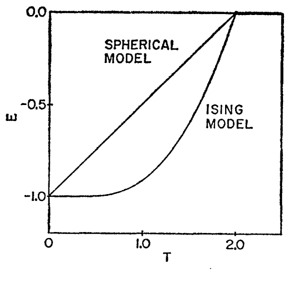

FIG. 1. Limiting energy per spin curves for the one-dimensional spherical and Ising models as the range of the interaction becomes indefinitely great.

model is[5]

$$E=-(q_s-\tfrac{1}{2})kT. \qquad (3.16)$$

If we carry the solution of (3.13) to the next higher order in $(1-r)$ above the critical point $(K=\frac{1}{2})$, then, substituting in (3.16) we obtain (3.2) for the energy per particle. Thus we see that the conclusions of Sec. 2 hold for the one-dimensional spherical and Ising models with exponential interactions. When we note that the Gaussian model may be obtained from the spherical model by setting $q=\frac{1}{2}$ instead of integrating over q, we see that the results are also identical for it above the critical point. It should be noted that the next order deviations can be obtained for these three cases are found to be of order $(1/R)^2$ rather than $(\ln R/R)^2$.

In Fig. 1 we illustrate the energies for the spherical and the Ising models below the critical point in the limit as $r\to 1$. One sees that agreement above the critical point does not imply anything about the behavior below the critical point. In fact, although the Gaussian model agrees with the other two models studied above the critical point, it is not even defined below the critical point.

It is important to note that neither the one-dimensional spherical nor the Ising[2] model possesses any critical point for $r<1$, but both possess a discontinuity in the specific heat at $K=\frac{1}{2}$ in the limit as $r\to 1$. Although the first-order term in the $(1-r)$ expansion has a singularity (and, in fact, so do all higher terms), one cannot conclude that the sum of all of them do. As we see from these examples, it need not. Hence caution must be exercised in attempting to draw conclusions about the nature of the transition for $r<1$ from the limiting nature of the transition.

While it is true that if one defines the spontaneous magnetization for $r=1$ as the limit of the spontaneous magnetization as $r\to 1$, one obtains zero below the transition point together with an infinite susceptibility, physically one should consider it to be

$$\mathfrak{M}=\lim_{H\to 0+}\lim_{r\to 1}\lim_{N\to\infty}\frac{\partial(\ln Z)/N}{\partial H}. \qquad (3.17)$$

This would correspond to the results of measurements at fixed magnetic field strengths as $r\to 1$. As the introduction of a magnetic field H corresponds to multiplying the integrand of (3.12) by

$$\exp[\tfrac{1}{4}\mu^2H^2N/(q-K)], \qquad (3.18)$$

one readily computes that $\mathfrak{M}=\mu[1-T/T_c]^{\frac{1}{2}}$ for $T<T_c$ and 0 for $T\geq T_c$. This result is in agreement with the results for $r\to 1$ in two dimensions and the results for all r in three dimensions.[10]

4. VARIATION WITH THE RANGE OF THE INTERACTION OF A THREE-DIMENSIONAL SPHERICAL MODEL

In the previous section we considered a one-dimensional model in which the interaction energy fell off exponentially with the distance between the spins. We shall generalize that model to three dimensions in the following way. Instead of having the interaction energy fall off exponentially with the distance we shall let it fall off exponentially with $|x|+|y|+|z|$, where x, y, and z are the three components of the separation vector between the two spins. We shall consider the spins to be placed on a simple cubic lattice. The interaction energy over kT may, for this model, be written as

$$A_{ij}=-(1-r)^3K[B\times B\times B-(1+r^N)^3\delta\times\delta\times\delta]/(6r+2r^3), \qquad (4.1)$$

where B is defined by (3.5) and $\times$ means the direct product. The quantity K is again taken to be half the maximum total interaction energy per spin, J, over kT. In this model we know from the work of Berlin and Kac[5] that there is a transition for nearest neighbor interactions only $(r=0)$ which corresponds to the onset of spontaneous magnetization. With the model discussed here we can follow the transition as a function of range all the way to the limit of infinite range (though small compared to the size of the system).

The analysis of this model follows very closely that for the one-dimensional problem which was given in the previous section. If we follow steps analogous to (3.7)–(3.11), we obtain for the partition function

$$Z\approx\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}dq\,\exp\Bigg\{N(q-\tfrac{1}{2})-\frac{N}{16\pi^3}\int\int_0^{2\pi}\int d\omega d\omega' d\omega'' \times\ln\bigg[(1+r^2-2r\cos\omega)(1+r^2-2r\cos\omega')(1+r^2-2r\cos\omega'')\bigg(q+\frac{(1-r)^3K}{6r+3r^3}\bigg)-\frac{(1-r)^3(1-r^2)^3K}{6r+2r^3}\bigg]\Bigg\}, \qquad (4.2)$$

where reference 9 has again been used and N denotes the total number of spins in the system. In the limit as

[10] The author is happy to acknowledge a fruitful discussion of this point with M. Kac and T. H. Berlin.

$N\to\infty$ we may evaluate $(\ln Z)/N$ exactly by means of the method of steepest descents,[7] if a normal saddle point exists. From the results of Berlin and Kac[5] we know that for $r=0$ a normal saddle point does exist above the critical point, and below the critical point the integrand possesses a cusp instead of a saddle point at the point of the maximum value on a path of steepest descents for the q integration. We shall find that the same situation holds for all $r<1$. If a normal saddle point exists, its location is given by solving the following equation for q:

$$0=1-\frac{1}{2}\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)^{-1}-\frac{1}{2}\frac{(1-r)^3(1-r^2)^3K}{6r+2r^3}\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)^{-1}\frac{1}{\pi^3}\int\int_0^\pi\int d\omega d\omega' d\omega''$$
$$\times\left[\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)(1+r^2-2r\cos\omega)(1+r^2-2r\cos\omega')(1+r^2-2r\cos\omega'')-\frac{(1-r)^3(1-r^2)^3K}{6r+2r^3}\right]^{-1}. \quad (4.3)$$

We may perform the integration over ω by means of a standard integration formula,[11] obtaining

$$0=1-\frac{1}{2}\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)^{-1}-\frac{1}{2}\frac{(1-r)^3(1-r^2)^3K}{6r+2r^3}\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)^{-1}\frac{1}{\pi^2}\int_0^\pi\int d\omega' d\omega''$$
$$\times\left\{\left[\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)(1+r^2)(1+r^2-2r\cos\omega')(1+r^2-2r\cos\omega'')-\frac{(1-r)^3(1-r^2)^3K}{6r+2r^3}\right]^2\right.$$
$$\left.-4r^2\left(q+\frac{(1-r)^3K}{6r+2r^3}\right)^2(1+r^2-2r\cos\omega')^2(1+r^2-2r\cos\omega'')^2\right\}^{-\frac{1}{2}}. \quad (4.4)$$

We may perform the integration over ω' by letting $w=\cos\omega'$, and factoring the denominator to find its zeros. If we confine our attention to $q\geq K>0$, the ferromagnetic-type interaction case, then we may use formula number 566 of reference 9 to perform the w integration. The corresponding analysis can easily be carried out for $K<0$, the antiferromagnetic case, but we shall not give it here. The result of the w integration is

$$0=1-\frac{3r+r^3}{(6r+2r^3)q+(1-r)^3K}-\frac{(1-r^2)(1-r)^3K}{(6r+2r^3)q+(1-r)^3K}$$
$$\times\frac{1}{\pi^2}\int_0^\pi d\omega''\frac{\mathbf{K}\left(\dfrac{16(1-r)^2K[rq+(1-r)^3K/(6+2r^2)](1+r-2r\cos\omega'')}{(1+r)(6+2r^2)\{q(1+r^2-2r\cos\omega'')+[(1-r)^3/(3+r^2)]K(r-\cos\omega'')\}^2}\right)}{|q(1+r^2-2r\cos\omega'')+[(1-r)^3/(3+r^2)]K(r-\cos\omega'')|}, \quad (4.5)$$

where $\mathbf{K}(k^2)=\mathrm{sn}^{-1}(1,k)$ is the complete elliptic integral of the first kind. One should note that (4.5) reduces in an obvious way to the corresponding result of Berlin and Kac[5] when r is set equal to zero. In order for the formula which corresponds to (3.8) to be valid in this derivation, q must be greater than K for the ν integrations to converge. In the one-dimensional case the term corresponding to the integral goes to infinity as $q\to K$ and thus there is a solution for all K and therefore no transition in the one-dimensional case for $r<1$. The same is true in two dimensions as was in one dimension. The equation analogous to (4.5) and (3.13) is, for two dimensions,

$$0=1-\frac{2r}{4rq+(1-r)^2K}-\frac{(1-r)^2K}{[4rq+(1-r)^2K]|q|\pi}\times\mathbf{K}\left(\frac{K[4rq+(1-r)^2K]}{(1+r)^2q^2}\right), \quad (4.6)$$

in which the appropriate term goes to infinity logarithmically as $q\to K$. The limit as $r\to 1$ is $q_s=\frac{1}{2}$ for $K<\frac{1}{2}$ and $q_s=K$ for $K>\frac{1}{2}$, which is exactly the same as for the one-dimensional case. The coefficient of $(1-r)^2$ is, of course, different than that of $(1-r)$ was in the one-dimensional case; hence the dimensionality makes a

[11] No. 300 in reference 9.

difference in the leading order (which vanishes) of the energy per spin above the critical point (critical only in limit as $r \to 1$).

In three dimensions we see from (4.5) that while the integrand is singular at $\omega''=0$, the integral, being essentially $\int \ln\omega'' d\omega''$ near $\omega''=0$, is finite in the limit $q \to K$. Thus, for $r<1$, we obtain a solution, q_s, of (4.5) for K less than K_s as given by

$$K_c=\frac{1}{2}\left(\frac{1-r}{1+r}\right)^3\left\{\frac{1}{2}-\frac{(3+r^2)(1-r^2)}{\pi^2}\int_0^\pi d\omega''\,\frac{\mathbf{K}\left(\dfrac{4(1-r^2)^2(1+r^2-2r\cos\omega'')}{[(3+r^2)(1+r^2)+r(1-r)^3-(1+r)^3\cos\omega'']^2}\right)}{[3+r^2(1+r^2)+r(1-r)^3-(1+r)^3\cos\omega'']}\right\}. \quad (4.7)$$

It follows for any $r<1$, in exactly the same way as Berlin and Kac[5] demonstrated for $r=0$, that $q_s=K$ for $K>K_c$. K_c varies from[5] $27+18\sqrt{2}-15\sqrt{3}-10.5\sqrt{6} \approx 0.7554396$ to 0.5 as r goes from 0 to 1.0. It should be noted that $(1-r^2)\int$ is of the order of unity and not of order $(1-r)$.

To establish the nature of the transition it will suffice to show that K near K_c is of the form

$$K=K_c+A(q-K_c)^{\frac{1}{2}}+O((q-K_c)). \quad (4.8)$$

By the arguments of Berlin and Kac[5] it then follows at once that the specific heat is continuous and its slope is discontinuous at the critical point for all $r<1$. Thus the transition is of the third order. In the limit as $r \to 1$, we again obtain the same limit as we did in the one- and two-dimensional cases. Again, of course, the coefficient of $(1-r)^3$ differs from that of $(1-r)$ in the one-dimensional case and so the dimensionality makes a difference in the leading order of the energy per spin. Also the spontaneous magnetization persists in the limits as $r \to 1$ in three dimensions. This result follows easily using the methods of Berlin and Kac.[5]

To establish (4.8), we use the expansion[12] for $\mathbf{K}(x)$:

$$\mathbf{K}(x)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2(1-x)^n \times\left[\ln(1-x)-4\ln 2+4\left(\frac{1}{1}-\frac{1}{2}+\cdots-\frac{1}{2n}\right)\right]. \quad (4.9)$$

It is easy to show that contributions to A can only come from the terms involving $\ln(1-x)$. As these terms are all of the same sign, we can obtain a lower bound on the magnitude of A by considering only the first one. A lower bound is all we need to differentiate A from zero. If we introduce $\delta=(q/K)-1$, substitute the $n=0$ coefficient of $\ln(1-x)$ into (4.5), drop several terms of order (δ) and order (1.0) we reduce the calculation of the lower bound to the magnitude of A to the evaluation of

$$-\frac{(1-r)^4(3+r^2)}{2\pi^2(1+r)^5}\int_0^\pi \frac{d\omega''\ln(\theta-\cos\omega'')}{\gamma-\cos\omega''}, \quad (4.10)$$

where

$$\theta \approx 1+(1-r)^2(3+r^2)(1+r)^{-3}\delta, \qquad \gamma=(3-2r+3r^2)(1+r)^{-2}. \quad (4.11)$$

By the use of some known integrals and series expansions[13] we can compute that

$$K \leq K_c-\left(\frac{1-r}{1+r}\right)^3\frac{(3+r^2)}{4\pi(1+r)}\left[\left(\frac{3+r^2}{1+r}\right)\delta\right]^{\frac{1}{2}}+O(\delta). \quad (4.12)$$

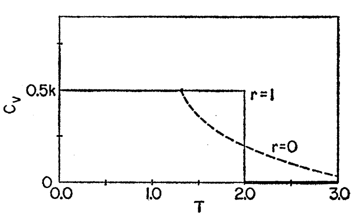

FIG. 2. Specific heat curves for nearest neighbor ($r=0$) and indefinitely long range ($r=1$) interactions for the three-dimensional spherical model. Intermediate values of the range interpolate between the two curves shown and have a discontinuity in slope which becomes progressively sharper as r goes from 0 to 1.

It is to be noted that in the limit as $r \to 0$ we obtain about 6% larger magnitude for A than Berlin and Kac[5] did. This is because in order to demonstrate the proper dependence on $(1-r)$ we had to include a slightly more complicated term than they did to establish the result for $r=0$.

In Fig. 2 we give a sketch of the specific heat curve for the three-dimensional problem for $r=0$ and $r=1$.

[12] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1927), Ex. 20, 21, p. 299.

[13] W. Gröbner and N. Hofreiter, *Integraltafel, Zweiter Teil, Bestimmte Integrale* (Springer-Verlag, Berlin, 1958), Nos. 332.24 and 338.13.