Hence

$$\mathbf{k}\cdot d\mathbf{x}=\mathbf{k}\cdot(\partial D_0/\partial\mathbf{k})d\tau$$
$$=-\omega(\partial D_0/\partial\omega)d\tau$$
$$=\omega dt \tag{157}$$

and therefore

$$S=\omega t. \tag{158}$$

If the medium is time independent ω is just a constant, so $\delta t=0$ follows from $\delta S=0$.

In more complicated problems there appear parameters, such as a plasma frequency ω_P, a collision frequency ω_c, a viscosity ν, etc., which destroy the homogeneity of D_0 and hence invalidate Fermat's principle. We can only use Fermat's principle if the medium is entirely characterized by a set of velocities, like the local speed of light, or of Alfvén waves, or of sound. (In particular, Fermat's principle could be applied to Sec. II but not in Sec. III of this paper; its use by Francis, Green, and Dessler[6] was actually justified.) At any rate, the only use usually made of Fermat's principle is in deriving Eqs. (146) and (147), and these equations are always valid.

ACKNOWLEDGMENTS

It is a great pleasure to thank Professor K. M. Watson, who suggested that the eikonal method would have a useful application to magnetohydrodynamics, and whose advice has been invaluable in carrying out this work. A number of discussions with Professor R. Blancenbecler, Professor S. Chandrasekhar, Dr. A. Dessler, and Professor R. Karplus proved extremely helpful.

PHYSICAL REVIEW VOLUME 126, NUMBER 6 JUNE 15, 1962

Lifetime Effects in Condensed Helium-3

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(Received February 13, 1962)

The condensation of a Fermion system by forming d-type bound pairs is discussed with the help of time-dependent correlation functions both at absolute zero and finite temperatures, for the purpose of applying this study to the case of liquid helium-3. We use essentially Gor'kov's method, suitably generalized to take into account the anisotropy of the bound pairs and also the finite lifetime of the quasi-particles which make up the pairs. The treatment proposed here goes one step further than the Hartree approximation in the sense that the finite decay rate of the quasi-particles is introduced by means of a model spectral density for the renormalized propagator (Green's function). This model features a single broad peak instead of the infinitely sharp peak which characterizes the Hartree approximation. Considerable care is taken to relate this microscopic model to the available experimental data on the scattering probability in liquid helium-3. It is concluded that the effect of scattering on the condensation can be adequately described by a cutoff Λ of the order of 1°K, limiting the domain in momentum space of the quasi-particles which participate effectively in the condensation process. This entails a reduction of the transition temperature estimated previously on the basis of the Hartree approximation, down to a value of the order of 0.02–0.03°K.

I. INTRODUCTION

THE idea of particle pairing, which is the basis of Bardeen, Cooper, and Schrieffer's (BCS) theory of superconductivity,[1] naturally leads us to ask whether this condensation could also obtain by forming pairs in a finite angular momentum state instead of the s state considered by BCS. Many authors[2–5] have agreed, on theoretical grounds, that this is indeed possible provided that the interaction potential is favorable, i.e., produces a larger binding energy in a finite l state than in the s state. The interaction potential between two bare helium-3 atoms comprises a strong repulsive core and a weak long-range attraction; consequently, the effective interaction in the liquid is thought to be attractive for two quasi-particles in a large relative angular momentum state.[6] Actual computations[3,7] showed that this interaction is indeed the most attractive in the d state ($l=2$), although it is repulsive for $l=0$ and 1. The existence of a condensed state of liquid helium-3, stable

* Part of this work was performed while this author was at the Bell Telephone Laboratories, Murray Hill, New Jersey.

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at low temperature, was, therefore, predicted; Emery and Sessler[3] estimated a transition temperature of the order of 0.07°K.

However, this condensation has not been found experimentally, even though the range of temperature investigated has been extended down to 0.008°K.[8] This experimental evidence led several authors[5,9] to suspect that the simple Hartree approximation used previously might be seriously inaccurate in the case of helium-3 on account of the finite lifetimes of the quasi-particle excitations of the system. Physically, one would expect the decay of the quasi-particles to limit their efficiency for participating in the condensation process and, therefore, reduce the transition temperature below that estimated on the basis of the Hartree approximation. This conclusion was qualitatively born out by the theory[9,10] so that we felt it would be useful to estimate as accurately as possible the decay rate of the quasi-particles in liquid helium-3 and to incorporate this information in a consistent treatment of the condensation problem. In contrast to the very detailed analysis of Kadanoff and Martin,[10] we endeavored not to introduce more theoretical parameters than can be readily computed from the experimental data. In view of the lack of detailed information on the collision probability in liquid helium-3, we had to restrict ourselves to the simplest approximation, i.e., isotropic scattering probability. On the other hand, we show by a detailed argument originally introduced by Betbeder-Matibet and Nozières[11] that the lifetimes of the quasi-particles are indeed the same in the normal and in the condensed fluid, within corrections of the order of the square of the gap. We are, therefore, justified to use the same spectral representation of the Green's function as in a normal Fermi system. Following the scheme proposed in reference 11, we approximate this spectrum by a Lorentz function with a finite width Γ_k proportional to the decay probability of the quasi-particle. A detailed computation of this probability proves that Γ_k is proportional to the square of the excitation energy ϵ_k in the low-temperature limit. We derive then the gap equation at absolute zero temperature on the basis of both the Hartree approximation used in earlier estimates and the present model (Sec. II). We extend this treatment to the case of finite temperatures following the method of Luttinger and Ward.[12] We show that the effect of the finite lifetime is essentially to cut off the domain (in momentum space) of the quasi-particles effectively contributing to the condensation energy. This cutoff is at a fixed energy Λ of the order of the Fermi energy and quite independent of the temperature. We then derive the expression for the transition temperature (Sec. III). Lastly, we estimate the (constant) collision probability both from first principles, using Landau's theory of Fermi liquids,[13,14] and from physical properties intimately related to scattering: thermal conductivity, viscosity and self-diffusion (using the expressions derived by Abrikosov and Khalatnikov.[14] We find that the values agree reasonably well in spite of our drastic approximations. Thus, we estimate a value of the cutoff Λ of the order of 1°K and a transition temperature $T_c=0.02$ to 0.03°K (Sec. IV).

II. CONDENSED GROUND STATE OF A NONIDEAL FERMI FLUID

Although the suitable Green's function formalism for describing a condensed Fermion system is readily available in the literature,[10,15] we want to derive it again and to describe in details the steps which lead us to the formulation of Dyson' equations; this will serve both purposes of introducing our notation and laying the base for the extension to the case of finite temperatures.

Basic Hypothesis of the Theory

It is well known that the difference between the normal (Fermi) state and the condensed state is the existence, in the latter state, of a condensed phase of paired particles. These pairs behave essentially like bosons and, therefore, any number of them can pile up in a unique ground state (ground pairs). We may then think of the condensed state φ_0 as describing not a fixed number of particles but rather an undetermined number of pairs. In other words, all physical quantities (particularly the Green's function) are unchanged, when two particles are added to the system or, more precisely, when one ground pair is added to the condensed phase. In order to take advantage of this continuity, we want to ignore the total number of particles N; we shall then consider the free energy $F=H-\mu N$, instead of the energy H (μ is the chemical potential) or, equivalently, we shall measure all particle energies relative to the Fermi energy. According to this view, one sees that the mean value of the interaction Hamiltonian in the condensed state includes the regular Hartree term

$$\langle\varphi_0|\psi(1)\psi^\dagger(4)|\varphi_0\rangle\langle\varphi_0|\psi(2)\psi^\dagger(3)|\varphi_0\rangle, \qquad (1)$$

as well as

$$\langle\varphi_0|\psi(1)\psi(2)|\varphi_0\rangle\langle\varphi_0|\psi^\dagger(3)\psi^\dagger(4)|\varphi_0\rangle. \qquad (2)$$

Both terms have an infinite range since each one can be split into two independent parts representing two

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unconnected processes, namely: (1) the propagation of two particles independently of each other; (2) the merging of two particles into the condensed phase and their subsequent emerging. Both should then be taken into account in the Hartree approximation. Clearly, the second term (2) is finite only in the condensed state and appears, therefore, as the main contribution to the condensation energy.

An equivalent way of presenting this view, using now Gor'kov's formalism, would be to treat on the same footing the normal propagator G:

$$G_{\sigma\sigma'}(x-x')=i\langle\varphi_0(N)|T\psi_\sigma(x)\psi_{\sigma'}^\dagger(x')|\varphi_0(N)\rangle, \quad (3)$$

and the "propagators" F and $\bar{F}$:

$$\begin{aligned}\bar{F}_{\sigma\sigma'}(x-x')&=i\langle\varphi_0(N)|T\psi_\sigma(x)\psi_{\sigma'}(x')|\varphi_0(N+2)\rangle,\\ F_{\sigma\sigma'}(x-x')&=i\langle\varphi_0(N+2)|T\psi_\sigma^\dagger(x)\psi_{\sigma'}^\dagger(x')|\varphi_0(N)\rangle.\end{aligned} \quad (4)$$

Here, T is the usual time-product; x is the four-vector $\mathbf{r}$, t and the spin coordinate is denoted by σ; $\varphi_0(N)$ and $\varphi_0(N+2)$ are the ground states (condensed state) for N and $(N+2)$ particles, respectively; we assume that $\varphi_0(N)$ and $\varphi_0(N+2)$ are indeed the same state within corrections of the order of $1/N$. Note that this approach leads us naturally to introduce only one function $\bar{F}$ and one F: we, thus, assume implicitly that there is only one kind of bound pairs undergoing Bose condensation. A detailed analysis carried out by Anderson and Morel[5] showed that this pair state is not a pure rotational eigenstate but a complex mixture of different spherical harmonics.

This view has been questioned by Gor'kov and Galitskii[4] on the basis of a different assumption about the nature of the ground state. These authors assume the existence of several condensed phases. Each phase is is made up with pairs in pure rotational eigenstates, with fixed angular momentum l but variable m. In the ground state, these $(2l+1)$ condensed phases are assumed to be equally populated, thus yielding an isotropic state. This assumption leads to very simple mathematics and looks very attractive. However, it is not clear that the concept of *independent* condensed phases has any meaning. To put it another way, a pair in the configuration $(\mathbf{k}, -\mathbf{k})$ may not remember to which m it belongs. As the above theory cannot be checked by the standard Bogoliubov or BCS techniques, its validity is still an open question. For this reason, we follow here the lines of Anderson and Morel's treatment, rather than Gor'kov and Galitskii's theory. As a matter of fact, this is not critical in any way, since our results could trivially be transposed to the other scheme.

Derivation of Dyson's Equations

Let us analyze the set of all processes which lead to the propagation of a quasi-particle (described by G) and the absorption of a pair into the condensed phase (described by F) (see Fig. 1). Summing up the contributions of these processes, we obtain the following equations which are really geometrical relations between classes of diagrams:

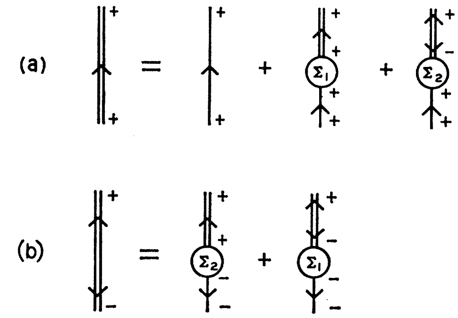

FIG. 1. Diagrammatic representation of Eqs. (5) and (6) (see text). The double lines represent the renormalized propagators G (one arrow) and F (two diverging arrows). The single lines represent unperturbed propagators G_0. Σ_1 and Σ_2 are self-energy diagrams. The small + and − signs refer to the particle spins.

$$G(\mathbf{k},\omega)=G_0(\mathbf{k},\omega)+G_0(\mathbf{k},\omega)\Sigma_1(\mathbf{k},\omega)G(\mathbf{k},\omega) -G_0(\mathbf{k},\omega)\bar{\Sigma}_2(\mathbf{k},\omega)F(\mathbf{k},\omega), \quad (5)$$

$$F(\mathbf{k},\omega)=G_0(\mathbf{k},-\omega)\Sigma_2(\mathbf{k},\omega)G(\mathbf{k},\omega) +G_0(\mathbf{k},-\omega)\Sigma_1(\mathbf{k},-\omega)F(\mathbf{k},\omega). \quad (6)$$

We have used here the energy-momentum representation and omitted the spin indexes for simplicity.[16] G_0 is the unperturbed propagator. Σ_1, Σ_2, and $\bar{\Sigma}_2$ are the self-energy parts which involve no net variation of the number of particles, the net creation and the net annihilation of one pair, respectively. We shall see later that each F operator contributes a factor of the order of the ratio of the average energy-gap Δ to the Fermi energy ϵ_F. In the weak coupling limit, we want to neglect the second- and higher-order terms, which result from diagrams including two or more operators F. We shall then restrict the summation for Σ_1 to the class of all self-energy diagrams involving no operator F or $\bar{F}$, i.e., the diagrams which make up Σ_1 in the normal state. Similarly, we shall only include in Σ_2 or $\bar{\Sigma}_2$ those self-energy diagrams which involve only one operator F or $\bar{F}$, respectively[11]:

$$\begin{aligned}\bar{\Sigma}_2(x)&=iV(x)\bar{F}(x),\\ \Sigma_2(x)&=-iV(x)F(x),\end{aligned} \quad (7)$$

where $V(x)$ is the renormalized vertex part representing the quasi-particle interaction in the normal fluid (V includes no operator F). In the first order of the interaction, $V(x)$ is simply the (instantaneous) potential between two bare helium-3 atoms. In higher orders, $V(x)$ includes noninstantaneous polarization terms but we shall assume that these corrections can be neglected.

[16] We have been careful to take into account the spin-inversion properties of the system and particularly, we have introduced a minus sign on the right-hand side of (5) because the self-energy part of the last diagram of Fig. 1(a) has the opposite spin combination to that of definition (7).

In other words, we replace the true energy dependent potential $V(\mathbf{k},\omega)$ by the limiting value $V(\mathbf{k},0)$. In this approximation, the self-energy parts Σ_2 and $\bar{\Sigma}_2$ are instantaneous so that we need only to know the relation between F and $\bar{F}$ for equal times, in order to solve the Eqs. (5) and (6).[17] We can easily derive this relation from the commutation rules:

$$\bar{F}_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}', +0)=i[\langle\psi_{\sigma'}(\mathbf{r}',t)\psi_\sigma(\mathbf{r},t)\rangle]^* = [F_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}', +0)]^*. \quad (8)$$

Introducing the gap function $C(\mathbf{k})$ defined by Anderson and Morel, we obtain, therefore,

$$\Sigma_2(\mathbf{k},\omega)\simeq C(\mathbf{k}), \quad \bar{\Sigma}_2(\mathbf{k},\omega)\simeq C^*(\mathbf{k}). \quad (9)$$

Carrying this into (5) and (6), we find then

$$F(\mathbf{k},\omega)=C(\mathbf{k})/ [(\epsilon_k-\omega-\Sigma_1(\mathbf{k},\omega)][\epsilon_k+\omega-\Sigma_1(\mathbf{k}, -\omega)]+|C(\mathbf{k})|^2.$$

Since Σ_1 is identical to the self-energy in the normal fluid (within second order corrections), $[\epsilon_{\mathbf{k}}-\omega-\Sigma_1(\mathbf{k},\omega)]^{-1}$ is just the renormalized propagator $G_n(\mathbf{k},\omega)$ in the normal fluid; consequently, we may rewrite the expression for F in a more elegant form:

$$F(\mathbf{k},\omega)=C(\mathbf{k})/[G_n(\mathbf{k},\omega)]^{-1}[G_n(\mathbf{k}, -\omega)]^{-1}+|C(\mathbf{k})|^2, \quad (10)$$

and finally [from (7) and (10)]:

$$C(\mathbf{k})=-\frac{i}{(2\pi)^4}\int\frac{V(\mathbf{k}-\mathbf{k}')C(\mathbf{k}')}{[G_n(\mathbf{k}',\omega')]^{-1}[G_n(\mathbf{k}', -\omega')]^{-1}+|C(\mathbf{k}')|^2}d^3k'd\omega'. \quad (11)$$

Hartree Approximation

We shall conveniently represent the renormalized propagator G_n by its Lehmann expansion in terms of spectral densities:

$$G_n(\mathbf{k},\omega)=\int_0^\infty\left[\frac{A_+(\mathbf{k},\omega')}{\omega'-\omega-i\eta}-\frac{A_-(\mathbf{k},\omega')}{\omega'+\omega-i\eta}\right]_{\eta\to+0}d\omega'. \quad (12)$$

The simplest approximation consists in neglecting all dynamical correlations (Hartree approximation) and associating a definite energy with each momentum. Then,

$$A_+=\delta(\omega-\epsilon_{\mathbf{k}}), \quad \epsilon_{\mathbf{k}}>0$$
$$A_-=\delta(\omega+\epsilon_{\mathbf{k}}), \quad \epsilon_{\mathbf{k}}<0, \quad (13)$$

and zero otherwise. The corresponding propagator is essentially the unperturbed propagator G_0 (possibly including a correction of the particle energy in the form of an effective mass different from the true mass):

$$G_0(\mathbf{k},\omega)=1/(\epsilon_{\mathbf{k}}-\omega+i\eta) \quad \text{if} \quad \epsilon_{\mathbf{k}}<0$$
$$=1/(\epsilon_{\mathbf{k}}-\omega-i\eta) \quad \text{if} \quad \epsilon_{\mathbf{k}}>0. \quad (14)$$

Carrying this into (11) and performing the frequency integration, we obtain the familiar BCS equation corresponding to the case of infinite lifetimes

$$C(\mathbf{k})=\int V(\mathbf{k}-\mathbf{k}')\frac{C(\mathbf{k}')}{2E_{\mathbf{k}'}}\frac{d^3k'}{(2\pi)^3}, \quad (15)$$

$$E_{\mathbf{k}}=[\epsilon_{\mathbf{k}}^2+|C(\mathbf{k})|^2]^{\frac{1}{2}}. \quad (16)$$

[17] This is quite fortunate because the relation between F and $\bar{F}$ for different times is not simple, in general. These operators are, indeed, related by simple symmetry requirements when the ground state is invariant under time reversal like the superconducting state ($l=0$) [see P. Nozières, Lecture Notes, University of Paris, 1960 (unpublished)]. However, the condensed state of liquid helium-3 has a d-type configuration, involving a net correlation current, and, therefore, is not time reversal invariant.

The actual resolution of Eq. (15) has been carried out in detail by Anderson and Morel.[5] They showed in particular how to eliminate the energy dependence of the potential $V(\mathbf{k}-\mathbf{k}')$ by a suitable transformation of this equation. We shall not be concerned here with such refinements and simply take V as independent of the moduli of $\mathbf{k}$ and $\mathbf{k}'$. This approximation has, however, the disadvantage of introducing an artificial divergence into the initially convergent Eq. (15). In order to remove this divergence, we shall cut off the integral at a fixed energy of the order of the Fermi energy. If we are allowed to make this approximation, $V(\mathbf{k}-\mathbf{k}')$ can be considered as function of the angle between $\mathbf{k}$ and $\mathbf{k}'$ which we expand in series of spherical harmonics. In the case of helium-3 we retain only the term $l=2$. Finally, we replace the integration over $\mathbf{k}'$ by an integration over the corresponding quasi-particle energy:

$$\epsilon_{\mathbf{k}}=\hbar v_0(k-k_0). \quad (17)$$

Then,

$$C(\hat{k})=N_0V\sum_{m=-l}^{m=+l}Y_{lm}(\hat{k}) \times\int d\Omega'\,Y_{lm}^*(\hat{k}')C(\hat{k}')\int_0^{\epsilon_F}\frac{d\epsilon}{E}, \quad (18)$$

where $Y_{lm}(\hat{k})$ is a shorthand notation for $Y_{lm}(\theta,\varphi)$; $\hat{k}$ is the unit vector parallel to $\mathbf{k}$. Using the solution derived in reference (5), we estimate from (18) the value of the angular average Δ of the gap $C(\hat{k})$:

$$\Delta=2\epsilon_F D\exp(-1/N_0V). \quad (19)$$

D is a form factor which depends upon the particular configuration considered. For the ground state configuration of helium-3, Anderson and Morel find $D=3.17$.

BMN Model

Actually, the approximation (13) neglects two major effects. Firstly, in addition to the coherent quasi-particle peak, $A_\pm$ has a continuous background corresponding to the simultaneous excitation of several quasi-particles. The strength of the peak is therefore reduced from one to some smaller value $z_\mathbf{k}$. Secondly, the quasi-particle peak has a finite width $2\Gamma_\mathbf{k}$ corresponding to a pole at a finite distance $i\Gamma_\mathbf{k}$ from the real axis; this broadening clearly describes the damping of the quasi-particle. We should, therefore, use instead of (13):

$$A_\pm = A_\pm^{\text{inc}} + \frac{z_\mathbf{k}}{\pi}\frac{\Gamma_\mathbf{k}}{(\omega \mp \epsilon_\mathbf{k})^2 + \Gamma_\mathbf{k}^2}, \tag{20}$$

the first term on the right-hand side corresponding to the incoherent background and the second to the quasi-particle peak (the latter occurs in A_+ if $\epsilon_\mathbf{k} > 0$ or in A_- if $\epsilon_\mathbf{k} < 0$). Since we are mainly interested in the lifetime effect, we shall use a model spectral density in which we neglect the incoherent part and, therefore, take $z_\mathbf{k} = 1$:

$$A_\pm = \frac{1}{\pi}\frac{\Gamma_\mathbf{k}}{(\omega \mp \epsilon_\mathbf{k})^2 + \Gamma_\mathbf{k}^2}. \tag{21}$$

Hence [with the help of (12)]

$$\begin{aligned} G_n(\mathbf{k},\omega) &= 1/(\epsilon_\mathbf{k} - \omega - i\Gamma_\mathbf{k}) \quad \text{if} \quad \omega > 0 \\ &= 1/(\epsilon_\mathbf{k} - \omega + i\Gamma_\mathbf{k}) \quad \text{if} \quad \omega < 0. \end{aligned} \tag{22}$$

Equation (15) is then replaced by

$$C(\mathbf{k}) = \int V(\mathbf{k}-\mathbf{k}')\frac{C(\mathbf{k}')}{2E_{\mathbf{k}'}}\left[\frac{2}{\pi}\tan^{-1}\left(\frac{E_{\mathbf{k}'}}{\Gamma_{\mathbf{k}'}}\right)\right]\frac{d^3k'}{(2\pi)^3}. \tag{23}$$

The effect of the bracketed factor which appears in the integrand is, quite properly, to cut off the integral when the imaginary part $\Gamma_\mathbf{k}$ of the frequency associated with the quasi-particle $\mathbf{k}$ reaches the same magnitude as the real part $E_\mathbf{k}$. It is well known that $\Gamma_\mathbf{k}$ is a quadratic function of $\epsilon_\mathbf{k}$ (at $T = 0°\text{K}$):

$$\Gamma_\mathbf{k} = \epsilon_\mathbf{k}^2/\Lambda, \tag{24}$$

where we introduce a parameter Λ which measures the amount of scattering. We can solve (23) in the same fashion as (15), the only difference appearing in the energy integral. A numerical computation shows however that

$$\int_0^\infty \frac{2}{\pi}\tan^{-1}\left(\frac{\Lambda}{\epsilon}\right)\frac{d\epsilon}{[\epsilon^2 + |C|^2]^{\frac{1}{2}}} \simeq \ln\left(\frac{2\Lambda}{|C|} + \frac{4}{\pi}\right) + O(C^3/\Lambda^3).$$

Thus, we finally obtain the expression for the average gap:

$$\Delta = 2\Lambda D[1 + 0.18\Delta/\Lambda]\exp(-1/N_0V). \tag{25}$$

Apart from the bracketed factor (which is practically equal to one), the effect of the finite lifetime of the quasi-particles is simply to reduce the magnitude of the energy gap by the factor Λ/ϵ_F. We suspect that the same effect obtains for the transition temperature although one cannot conclude positively from the present theory valid only at absolute zero temperature.

III. CONDENSED STATE OF A NONIDEAL FERMI FLUID AT FINITE TEMPERATURE

We shall follow here the diagrammatic approach developed by Luttinger and Ward[12] and use the same notation as much as possible. The gist of the finite temperature "propagator" formalism lies in the possibility of generalizing Wick's theorem for a "temperature variable" u allowed to vary from $-\beta$ to $+\beta$ ($\beta = 1/kT$) and playing the role of an imaginary time variable. Accordingly, all (geometrical) relations derived by mean of diagram analysis at absolute zero temperature obtain also in this new formalism provided that it is replaced by u and the continuous frequency ω by the discrete variable ζ which takes the values

$$\zeta_l = (2l+1)(\pi i/\beta). \tag{26}$$

The required ensemble average is then carried out automatically by summing over all ζ_l. Since the relations (5) to (10) of the previous section have been derived by straightforward diagram counting, they obtain also in the present formalism, and particularly,

$$F(\mathbf{k},\zeta_l) = C(\mathbf{k})/[G_n(\mathbf{k},\zeta_l)]^{-1}[G_n(\mathbf{k}, -\zeta_l)]^{-1} + |C(\mathbf{k})|^2. \tag{27}$$

Now

$$C(\mathbf{k}) = \frac{1}{\beta(2\pi)^3}\sum_{l'=-\infty}^{l'=+\infty}\int d^3k'\, V(\mathbf{k}-\mathbf{k}')F(\mathbf{k}',\zeta_{l'}). \tag{28}$$

Luttinger has shown[12] that the "temperature propagator" $G_n(\zeta)$ in the normal fluid can be expressed in terms of a Lehmann expansion, just like the usual time propagator $G_n(\omega)$:

$$G_n(\mathbf{k},\zeta) = \int_{-\infty}^{+\infty}\frac{A(\mathbf{k},x)}{x-\zeta}dx, \tag{29}$$

where A is the spectral density. At zero temperature, A reduces to A_+, if $x > 0$ and A_-, if $x < 0$. As in the previous section, we shall use a model spectral density instead of the exact density $A(\mathbf{k},x)$. The simplest approximation (Hartree approximation) is to take the spectral density as a delta function, thereby obtaining

$$G_0(\mathbf{k},\zeta) = 1/(\epsilon_\mathbf{k} - \zeta). \tag{30}$$

This approximation leads of course to the usual BCS equation, as we shall briefly demonstrate below. Carrying (30) into (27) and (28), we obtain indeed:

$$C(\mathbf{k}) = \int \frac{d^3k'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\frac{C(\mathbf{k}')}{2E_{\mathbf{k}'}} \times \frac{1}{\beta}\sum_{-\infty}^{+\infty}\left(\frac{e^{\eta\zeta_l}}{E_{\mathbf{k}'} - \zeta_l} + \frac{e^{\eta\zeta_l}}{E_{\mathbf{k}'} + \zeta_l}\right), \tag{31}$$

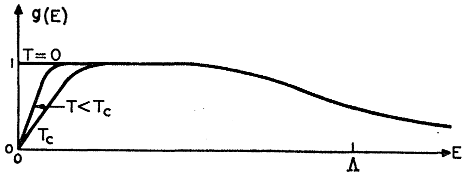

FIG. 2. Plot of the cutoff function g vs energy and temperature [see Eq. (36) of the text].

where $e^{\eta\zeta}$ is a convergence factor which we shall make equal to one at the end of the derivation. Using a result quoted in reference 12 (Appendix), we see that the sum over all l is just

$$f^+(E_{k'})-f^-(E_{k'})=\tanh(\beta E_{k'}/2), \tag{32}$$

as expected.

Following now BMN's method, let us introduce the model spectral density,

$$A(\mathbf{k},x)=\frac{1}{\pi}\frac{\Gamma_k(T)}{[x-\epsilon_k]^2+\Gamma_k^2(T)}. \tag{33}$$

Here, Γ_k is proportional to the decay probability of the quasi-particle $\mathbf{k}$ at the finite temperature T (different from the decay probability at $T=0°$K on account of the different statistical distribution). From Appendix A, we have

$$\Gamma_k(T)=\frac{1}{\Lambda\beta^2}\frac{\pi^2+(\beta\epsilon_k)^2}{1+e^{-\beta\epsilon_k}}. \tag{34}$$

Note that this expression correctly reduces to (24) in the low-temperature limit. From (29) and (33), we derive the expression for the "temperature propagator" in the normal fluid:

$$\begin{aligned} G_n(\mathbf{k},\zeta)&=1/(\epsilon_k-\zeta-i\Gamma_k) \quad \text{if} \quad \mathrm{Im}\zeta>0 \\ &=1/(\epsilon_k-\zeta+i\Gamma_k) \quad \text{if} \quad \mathrm{Im}\zeta<0. \end{aligned} \tag{35}$$

Finally, we derive from (28) and (35) the generalized gap equation at finite temperature:

$$C(\mathbf{k})=\int\frac{d^3k'}{(2\pi)^3}V(\mathbf{k}-\mathbf{k}')\frac{C(\mathbf{k}')}{2E_{k'}}g(E_{k'},T), \tag{36}$$

$$g(E_k,T)=\frac{2}{\beta}\sum_{l=0}^{\infty}\left(\frac{1}{E_k-\zeta_l-i\Gamma_k}+\frac{1}{E_k+\zeta_l+i\Gamma_k}\right). \tag{37}$$

The latter expression cannot be computed exactly; we can, however, consider limiting cases and firstly the limit $E_k\ll kT$. We see from (34) that Γ_k is much smaller than π/β and *a fortiori* all ζ_l. In other words, the lifetime is long enough in the low-energy limit to permit the use of the Hartree approximation: $g(E_k,T)$, therefore, reduces to (32). In the high-energy limit, on the other hand, an approximate summation method can be used to find (see Appendix B)

$$g(E_k,T)\approx(2/\pi)\tan^{-1}(E_k/\Gamma_k)+O(C^2/\Lambda^2), \quad (E_k\gtrsim kT). \tag{38}$$

It is reasonable that we should find the same cutoff factor as for $T=0°$K since the effect of the finite lifetime is significant only for particle energies much larger than kT_c. We have plotted the variation of this function g vs energy and temperature in Fig. 2; note that the effects of the thermodynamic average and of the finite lifetime are well separated, at least in the weak-coupling limit. The equation for the transition temperature T_c is a special case of (36):

$$1=N_0V\int g(\epsilon,T_c)\frac{d\epsilon}{\epsilon}, \tag{39}$$

the explicit solution of which is approximately (weak-coupling limit):

$$kT_c\simeq 1.105\Lambda\exp(-1/N_0V). \tag{40}$$

This last result together with (25) finally demonstrates that the effect of the finite lifetime of the quasi-particles can be adequately described by a cutoff Λ limiting the domain of the useful transitions. One problem remains, however: how to relate this parameter pertaining to the microscopic theory, to the experimental data obtained from macroscopic measurements.

IV. EVALUATION OF THE SCATTERING PROBABILITY

We have found in the previous sections that all physical properties of a condensed Fermi fluid can be expressed in terms of only two parameters: the strength N_0V of the interaction which binds the pairs and the cutoff Λ which measures the amount of scattering. However, estimating these parameters from the known experimental data is not straightforward and consequently, their accepted values are subject to some uncertainty. Much attention and sizable computational efforts have been devoted to the calculation of N_0V. We shall use here the value given by Emery and Sessler,

$$\exp(-1/N_0V)\simeq 1/40. \tag{41}$$

On the other hand, little is known about the scattering probability and the corresponding decay rate of the quasi-particle excitations of liquid helium-3. We shall presently, analyze this decay probability in the framework of Landau's theory of Fermi liquids.[13]

There are many processes which may produce the decay of a single quasi-particle excitation into several excitations of lower energies but, on account of the phase space available, the most probable process in the low-energy limit is the creation of one particle-hole pair (Fig. 3). This scattering process has been studied in great detail by Abrikosov and Khalatnikov[14]; we shall follow their analysis and use the same notation as much as possible. The decay probability of a quasi-particle of momentum $\mathbf{p}_1$ ($\mathbf{p}_1=\hbar k_1$) and spin σ_1 by creating one particle-hole pair is also the collision probability with a second particle $\mathbf{p}_2$, σ_2, the two particles ending up in $\mathbf{p}_1'$, σ_1 and $\mathbf{p}_2'$, σ_2. Consequently,

$$\frac{\partial n_1}{\partial t}=-\frac{1}{(2\pi\hbar)^6}\sum_{\sigma_2}\int W_{\sigma_1\sigma_2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2')\times f_2(1-f_1')(1-f_2')\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')\times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')d^3p_2d^3p_1'd^3p_2', \quad (42)$$

where W is the scattering probability, f_i is the Fermi distribution $f(\epsilon_i)$, and the two delta-functions enforce the conservation of energy and momentum. When σ_2 is antiparallel to σ_1, the summation over $\mathbf{p}_2$, $\mathbf{p}_1'$, and $\mathbf{p}_2'$ must be extended over the whole momentum space. If σ_2 is parallel to σ_1, on the other hand, the final state is unchanged by permutation of $\mathbf{p}_1'$ and $\mathbf{p}_2'$ as the particles are undistinguishable. The integration over $\mathbf{p}_2'$ should therefore, be restricted to one half of the momentum space. We shall find it more convenient to integrate over the whole momentum space in both cases and write the suitable factor $\frac{1}{2}$ into the spin sum. Let us then define

$$2w=W_{\uparrow\downarrow}+\tfrac{1}{2}W_{\uparrow\uparrow}. \quad (43)$$

We have introduced a factor 2 on the left-hand side, in order to fit our definition of the function w with the notation of AK. The relation (42) becomes

$$\frac{\partial n_1}{\partial t}=-\frac{2}{(2\pi\hbar)^6}\int wf_2(1-f_2')(1-f_2')\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')\times\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')d^3p_2d^3p_1'd^3p_2', \quad (44)$$

i.e., identical to Abrikosov and Khalatnikov's equation (AK 7.8). Here, w is now a function of the angular variables only: angle χ between the momenta of the incident particles and angle φ of the planes $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_1'\mathbf{p}_2'$. This scattering probability or at least the forward scattering probability ($\varphi=0$) are closely related to the quantities which appear in the Landau's theory and can, in principle, be derived from the thermodynamic properties of the fluid. We shall, however, postpone the discussion of this matter to Appendix C and state immediately the conclusion reached there: the available experimental data are not sufficient to give any precise information about the function $w(\chi,\varphi)$ beyond the zeroth order approximation, i.e., the isotropic w approximation. We can estimate this constant w from first principles (i.e., molar density, effective mass, and speed of sound data) using Landau's theory (see Appendix C). Proceeding then as AK, we do the angular integration and obtain, after a slight change of notation involving the renaming of the energy variable ($\epsilon_2\to-\epsilon_1$; $\epsilon_1'\to\epsilon_2$; $\epsilon_2'\to\epsilon_3$),

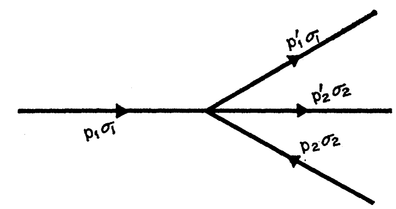

FIG. 3. Diagram of the most probable collision process producing the decay of a single quasi-particle excitation of the system. The lines representing the initial and final particles bear no resemblance to the actual disposition of the momenta in the reciprocal space.

$$\frac{\partial n}{\partial t}=-\frac{m^{*3}}{8\pi^4\hbar^6}\left[\frac{w}{\cos(\chi/2)}\right]_{\text{av}}\int(1-f_1)(1-f_2)(1-f_3)\times\delta(\epsilon-\epsilon_1-\epsilon_2-\epsilon_3)d\epsilon_1d\epsilon_2d\epsilon_3. \quad (45)$$

The bracketed expression is to be averaged over the whole sphere; in the constant w approximation, this average is simply $2w$. The energy integration can be done as we see from Appendix A. Comparing this relation (45) with (A1), we see that

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=-\frac{m^{*3}\Lambda}{16\pi^4\hbar^6}\left[\frac{w}{\cos(\chi/2)}\right]_{\text{av}}\Gamma_{\mathbf{k}}. \quad (46)$$

On the other hand, we can also compute the initial decay rate of the quasi-particle $\mathbf{k}$ in the Green's function formalism. Let us add this quasi-particle to the ground state at the time $t=0$ and compute the probability of finding it again at a later time t:

$$\bar{n}_{\mathbf{k}}(t)=\langle\varphi_0|c_{\mathbf{k}}(0)c_{\mathbf{k}}^\dagger(t)c_{\mathbf{k}}(t)c_{\mathbf{k}}^\dagger(0)|\varphi_0\rangle=|G(\mathbf{k},t)|^2. \quad (47)$$

We can easily derive the expression of $G(\mathbf{k},t)$ from the spectral density (21), using the usual representation for positive time:

$$G(\mathbf{k},t)=i\int_0^\infty A_+(\mathbf{k},\omega)e^{-i\omega t}d\omega\simeq e^{-i(\epsilon_{\mathbf{k}}-i\Gamma_{\mathbf{k}})t}. \quad (48)$$

The initial decay rate is, therefore, also given by

$$\partial n_{\mathbf{k}}/\partial t=-2\Gamma_{\mathbf{k}}/\hbar. \quad (49)$$

The relations (46) and (49) provide us with a convenient bridge between the Green's function formalism and Landau's theory. Using the expression (C11) for the collision probability, we obtain finally,

$$\frac{1}{\Lambda}=\frac{m^{*3}}{32\pi^4\hbar^5}\left[\frac{w}{\cos(\chi/2)}\right]_{\text{av}}=\frac{12.1\pi m^*}{16p_0^2}. \quad (50)$$

Using the most recent data on the specific heat[18] and the molar volume[18,19] we compute a value of the cutoff of the order of 1.5°K, i.e., about one-half of the Fermi energy.

We could also estimate w from physical properties which are intimately dependent upon scattering such as thermal conductivity, viscosity, self-diffusion. Indeed, the expressions of these quantities[20] involve various angular averages of the scattering probability with different weights. Since the angular average which appears in the expression for the thermal conductivity K is reasonably similar to that of Eq. (50), we feel that the actual dependence of w upon the angular variables χ and φ does not matter much and that a reliable estimate of Λ can be derived from the experimental value of K even though w has to be approximated by a constant. It is very gratifying to find $\Lambda \simeq 1.6$°K, in good agreement with the value computed from first principles.

It is reasonable that Λ should be of the same order of magnitude as the Fermi energy, for a much smaller cutoff would indicate a very strong interaction which would be very unlikely in the case of a liquid because the equilibrium density is usually such as to balance the kinetic and potential energies. Thus, the Fermi energy ϵ_F is essentially the only characteristic energy of the problem. We should also remark that, in view of Λ being a sizable fraction of ϵ_F, we were not allowed to neglect the energy dependence of the potential $V(\mathbf{k}-\mathbf{k}')$. This approximation had the effect of overestimating V for large excitation energies. This error can be corrected by reducing somewhat the estimated cutoff, possibly to a value of the order of 1°K instead of 1.5°K. Considering this difficulty, we think that our result would account for a transition temperature T_c in the range 0.02 to 0.03°K. However, the effect of the finite lifetimes of the quasi-particles cannot, in our opinion, account for the much lower T_c suggested by the absence of an observed condensation above 0.008°K; any discrepancy between the predicted T_c and the experimental data should then be ascribed to the estimated value of N_0V.

ACKNOWLEDGMENTS

We want to thank Dr. P. W. Anderson for many enlightening discussions and Professor Joachim Luttinger for introducing us to his "temperature propagator" method. One of us (PN) would also like to express his gratitude to the Bell Telephone Laboratories, Murray Hill, New Jersey, for their hospitality during the period when the work reported here was performed.

[18] A. C. Anderson, G. L. Salinger, W. A. Steyert, and J. C. Wheatley, Phys. Rev. Letters **7**, 295 (1961).

[19] L. Goldstein, Phys. Rev. **117**, 375 (1960).

[20] For viscosity and thermal conductivity, see AK, Eqs. (7.22) and (8.10). The expression for the coefficient of self-diffusion is given by D. Hone, Phys. Rev. **121**, 669 (1961).

APPENDIX A

The object of this Appendix is to compute the following integral:

$$\Gamma(\epsilon)=\frac{2}{\Lambda}\int[1-f(\epsilon_1)][1-f(\epsilon_2)][1-f(\epsilon_3)] \times\delta(\epsilon-\epsilon_1-\epsilon_2-\epsilon_3)d\epsilon_1 d\epsilon_2 d\epsilon_3, \quad \text{(A1)}$$

where f is the Fermi distribution. In order not to spoil the symmetry of this expression, we shall replace the delta-function by its Fourier transform

$$\delta(\epsilon)=\int_{-\infty}^{+\infty} e^{-i\epsilon t}\frac{dt}{2\pi}. \quad \text{(A2)}$$

The right-hand side of (A.1) is then the product of three identical factors:

$$\int_{-\infty}^{+\infty}\frac{e^{(\beta+it)x}}{1+e^{\beta x}}dx=\frac{i\pi}{\beta\sinh(\pi t/\beta)}. \quad \text{(A3)}$$

This last result has been obtained by a straightforward integration in the complex plane. Now

$$\Gamma(\epsilon)=\frac{2}{\Lambda}\int_{-\infty}^{+\infty} e^{-i\epsilon t}\left(\frac{\pi i}{\beta\sinh(\pi t/\beta)}\right)^3\frac{dt}{2\pi}. \quad \text{(A4)}$$

The integrand has an infinite series of triple poles

$$t_n=in\beta. \quad \text{(A5)}$$

Some care must be exercised to find the corresponding residues

$$Z_n=\frac{i}{2}\left[\frac{\pi^2}{\beta^2}+\epsilon^2\right](-1)^n e^{n\beta\epsilon}. \quad \text{(A6)}$$

A final integration in the complex t plane gives then:

$$\Gamma(\epsilon)=-\frac{2i}{\Lambda}\sum_{n=0}^{-\infty} Z_n=\frac{1}{\Lambda}\left(\frac{\pi^2}{\beta^2}+\epsilon^2\right)\Big/(1+e^{-\beta\epsilon}). \quad \text{(A7)}$$

Note that this expression properly reduces to ϵ^2/Λ in the low-temperature limit ($\epsilon \gg kT$), as can be seen directly from (A1).

APPENDIX B

The object of this Appendix is to evaluate, in the limit $E\gg kT$, the function $g(E)$ introduced in Sec. III:

$$g(E)=\frac{4E}{\beta}\sum_{l=0}^{\infty}\frac{1}{E^2+[(2l+1)\pi/\beta+\Gamma]^2}. \quad \text{(B1)}$$

Let us first prove the following lemma.

Lemma. If $f(x)$ is a continuous function decreasing monotonically from a finite value $f(0)$ for $x=0$ to zero at infinity, then,

$$\sum_{l=0}^{\infty} f(l+\tfrac{1}{2})=\int_0^{\infty} f(x)dx+\frac{1}{4\times 3!}f'(0)-\frac{7}{48\times 5!}f'''(0)+\cdots. \quad \text{(B2)}$$

Let us expand $f(x)$ in Taylor's series near $x=\frac{1}{2}$:

$$\int_0^1 f(x)dx=f(\tfrac{1}{2})+\frac{1}{4\times 3!}f''(\tfrac{1}{2})+\frac{1}{16\times 5!}f^{\text{iv}}(\tfrac{1}{2})+\cdots. \quad \text{(B3)}$$

Adding the corresponding equations for the intervals 1–2, 2–3, etc., we obtain, therefore,

$$\sum_{l=0}^{\infty} f(l+\tfrac{1}{2})=\int_0^{\infty} f(x)dx-\frac{1}{4\times 3!}\sum_{l=0}^{\infty} f''(l+\tfrac{1}{2})-\frac{1}{16\times 5!}\sum_{l=0}^{\infty} f^{\text{iv}}(l+\tfrac{1}{2}). \quad \text{(B4)}$$

Repeating the same process for the sum of the second derivatives, and then for the sum of the fourth derivatives, etc., one does indeed obtain (B.2). Moreover, if the function $f(x)$ is smooth, i.e., decreases gently and regularly, one sees that the second and higher order terms of (B.2) are quite small.

Now, the function

$$g(E)=\frac{\beta E}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(\beta^2E^2/4\pi^2)+[(\beta\Gamma/2\pi)+l+\frac{1}{2}]^2}, \quad \text{(B5)}$$

fulfills exactly these conditions in the limit we are considering. Then,

$$g(E)=\frac{2}{\pi}\tan^{-1}\left(\frac{E}{\Gamma}\right)-\frac{2\pi}{3\beta^2}\frac{E\Gamma}{(E^2+\Gamma^2)^2}+\frac{14\pi^3}{15\beta^4}\frac{E\Gamma(E^2-\Gamma^2)}{(E^2+\Gamma^2)^4}+\cdots. \quad \text{(B6)}$$

For a typical value $\beta\Lambda=100$ (weak coupling), we find that the combined correction due to the second and third terms of (B6) does not exceed 0.002 in a range where the first term varies from 0.98 to zero. It is, therefore, permissible to retain only the first term.

APPENDIX C

We wish here to analyze the collision probability w in further detail and relate it to the Landau theory of Fermi liquids. The collision probabilities $w_{\uparrow\uparrow}$ or $w_{\uparrow\downarrow}$ for two particles with parallel or antiparallel spins can be expressed in terms of the corresponding diffusion amplitudes:

$$w_{\sigma\sigma'}=(2\pi/\hbar)|A_{\sigma\sigma'}(\chi,\varphi)|^2. \quad \text{(C1)}$$

Restricting our attention to the special case of forward scattering without spin exchange ($\varphi=0$), we see that these collision amplitudes are related to the interaction energy of the quasi-particles $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$ introduced by Landau (see AK, Appendix A). Both $A_{\sigma\sigma'}(\chi,0)$ and $f_{\sigma\sigma'}(\chi)$ can be decomposed into symmetrical and antisymmetrical parts according to

$$A_{\uparrow\uparrow}(\chi,0)=A^s(\chi,0)+A^a(\chi,0),\quad A_{\uparrow\downarrow}(\chi,0)=A^s(\chi,0)-A^a(\chi,0), \quad \text{(C2)}$$

and

$$f_{\uparrow\uparrow}(\chi)=f^s(\chi)+f^a(\chi),\quad f_{\uparrow\downarrow}(\chi)=f^s(\chi)-f^a(\chi). \quad \text{(C3)}$$

Let us further expand these four functions A^s, A^a, f^s, and f^a in series of Legendre polynomials:

$$A^s(\chi,0)=\sum_{l=0}^{\infty} A_l{}^sP_l(\cos\chi),\text{ etc.} \quad \text{(C4)}$$

and introduce instead of the coefficients A^s, etc., the dimensionless quantities

$$\alpha_l{}^s=A_l{}^s(m^*p_0/\pi^2\hbar^3),\quad \varphi_l{}^s=f_l{}^s(m^*p_0/\pi^2\hbar^3),\text{ etc.} \quad \text{(C5)}$$

Then, the relation between A and f can be expressed in the following fashion [see AK, Eqs. (A2.14)]:

$$\alpha_l{}^s=\frac{\varphi_l{}^s}{1+\varphi_l{}^s/(2l+1)},\quad \alpha_l{}^a=\frac{\varphi_l{}^a}{1+\varphi_l{}^a/(2l+1)}, \quad \text{(C6)}$$

These relations give us the means to compute the forward collision amplitudes, provided we are able to estimate the coefficients $f_l{}^s$ and $f_l{}^a$ from the experimental data. However, only three among these coefficients can be reached experimentally. From AK's equations (5.8) and (9.9) we see that $\varphi_0{}^s$, $\varphi_1{}^s$, and $\varphi_0{}^a$ can be expressed in terms of the effective mass m^*, the velocity of sound u, and the magnetic susceptiblity χ_M:

$$\varphi_0{}^s=(3mm^*u^2/p_0{}^2)-1,\quad \varphi_1{}^s=(m^*/m)-1,\quad \varphi_0{}^a=(e\hbar/mc)^2(m^*p_0/4\pi^2\hbar^3\chi_M)-1. \quad \text{(C7)}$$

Using the most recent data,[18,21,22,23] we find then,

$$\varphi_0{}^s=9,\quad \varphi_1{}^s=5.4,\quad \varphi_0{}^a=-0.71. \quad \text{(C8)}$$

[21] W. R. Abel, A. C. Anderson, and J. C. Wheatley, Phys. Rev. Letters **7**, 299 (1961).

[22] H. L. Laquer, S. G. Sydoriak, and T. R. Roberts, Phys. Rev. **113**, 417 (1959).

[23] A. C. Anderson, W. Reese, R. J. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters **7**, 220 (1961).

Hence,

$$\alpha_0{}^s=0.9,$$
$$\alpha_1{}^s=1.92, \qquad (C9)$$
$$\alpha_0{}^a=-2.41.$$

Actually, the important quantities are $A_{\uparrow\uparrow}$ and $A_{\uparrow\downarrow}$, rather than A^s and A^a. Their mean values averaged on the sphere are, in reduced units:

$$\alpha_{0\uparrow\uparrow}=-1.51,$$
$$\alpha_{0\uparrow\downarrow}=3.31. \qquad (C10)$$

Note that the collision amplitude is larger for antiparallel spins than for parallel spins; this is quite reasonable since the exchange correlation is expected to reduce the effective interaction in the latter case. In view of the importance of $\alpha_0{}^a$, it does not seem reasonable to include the first-order term $\alpha_1{}^s$, when we do not know $\alpha_1{}^a$.[24] We are, thus, led to assume a constant forward diffusion amplitude $A(\chi,0)$. At this stage, we may as well proceed and assume that $A(\chi,\varphi)$ is constant and equal to the average value for $\varphi=0$.

[24] A slightly different view has been taken by D. Hone, Phys. Rev. **125**, 1494 (1962). This author estimates $\alpha_1{}^a$ from the known values of $\alpha_0{}^a$, $\alpha_0{}^s$ and $\alpha_1{}^s$ by requiring that the scattering amplitude vanish for parallel spins and $\theta=0$. However, the φ dependence is still completely unknown.

With the help of Eqs. (43), (C1), and (C5) and the experimental values (C10), we obtain then,

$$w\approx 12.1(\pi^5\hbar^5/m^{*2}p_0{}^2). \qquad (C11)$$

Inserting this result into the expressions for the thermal conductivity, K, the viscosity η and the coefficient of self-diffusion D,[20] we get

$$K(\text{erg/cm sec°K})=48/T,$$
$$\eta(\text{poise})=1.2\times10^{-6}/T^2, \qquad (C12)$$
$$D(\text{cm}^2/\text{sec})=5.9\times10^{-6}/T^2.$$

These theoretical values compare reasonably well with the experimental data[21,23,25]

$$K=51/T,$$
$$\eta=2.8\times10^{-6}/T^2, \qquad (C13)$$
$$D=1.54\times10^{-6}/T^2.$$

The theoretical value of the viscosity is too small and the value of the self-diffusion coefficient too large, but the discrepancies are well within the range of error we may expect from our crude approximation (in fact, one should even try to infer from these results a qualitative behavior of the function $w(\chi,\varphi)$). The most reliable result seems to be the thermal conductivity which we use in the body of the paper.

[25] A. C. Anderson, G. L. Salinger, and J. C. Wheatley, Phys. Rev. Letters **6**, 443 (1961).