Singularities in Partial-Wave Amplitudes for Two Ingoing and Two Outgoing Particles*

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The singularities which appear in the energy plane of a partial-wave amplitude are investigated for the general process with two ingoing and two outgoing particles. These lie either on the real axis or on curves symmetrical about the real axis. The equations of these curves and the conditions under which they occur are obtained; also the ranges in the energy spectra of the crossed channels to which they correspond. These general results are applied to the particular cases of pion-nucleon scattering and pion photoproduction from nucleons.

1. INTRODUCTION

It is frequently of interest when studying an interaction process involving two ingoing and two outgoing particles to isolate that part of the interaction which takes place in a particular state of orbital angular momentum. This is represented by a partial wave amplitude. It is possible to derive approximate relations between these partial-wave amplitudes and amplitudes for processes occurring with fixed momentum transfer; through these the dispersion relations for amplitudes at fixed momentum transfer may be used to give information about partial-wave amplitudes.[1] In many cases, however, the physical content of the situation may be seen more clearly by considering the analyticity properties of the partial wave amplitudes directly and writing dispersion relations for these amplitudes explicitly. Particular cases have been treated by several authors.[2] MacDowell has discussed K-N scattering, π-$\pi\rightarrow N$-$\bar{N}$ has been analyzed by Frazer and Fulco. π-N scattering has been treated by Frazer and Fulco; Hamilton and Spearman; Frautschi, and Walecka; Bowcock, Cottingham, and Lurié. The photoproduction of strange particles has been looked at by Fayyazuddin. Amati, Leader, and Vitale have examined nucleon-nucleon scattering.

A process with two ingoing and two outgoing particles may be considered in terms of three channels, corresponding to the possible combinations of pairs of ingoing and outgoing particles. Following the hypothesis of Mandelstam,[3] we suppose that the complete amplitude for scattering in one of these channels may be continued as an analytic function in the complex energy plane with the exception of certain regions of singularities determined by the energy spectra in all three channels. The singularities arising from the energy spectra in the three channels correspond in each case to a range on the positive real axis of the appropriate energy variable. Consequently, the distribution of these singularities in the energy plane for channel 3 is determined by the relationships between the energy variables for the three channels.

In the case of pion-nucleon scattering[4] these relationships are given by

$$s_3=[(M^2+q^2)^{\frac{1}{2}}+(\mu^2+q^2)^{\frac{1}{2}}]^2,$$
$$s_2=-2q^2(1-\cos\theta),$$
$$s_1=2M^2+2\mu^2+2q^2(1-\cos\theta)-[(M^2+q^2)^{\frac{1}{2}}+(\mu^2+q^2)^{\frac{1}{2}}]^2,$$

where s_1, s_2, s_3 are the squares of the total energies in channels 1, 2, 3 respectively; q and θ are the magnitude of the pion momentum and the scattering angle, respectively, in the center-of-mass (c.m.) system for channel 3; and M and μ are the nucleon and pion masses. It is easy to see that for any real scattering angle θ, the parts of the s_3, s_2, s_1 real axes defined, respectively, by $s_3=M^2$, $s_3\geqslant(M+\mu)^2$; $s_2\geqslant 4\mu^2$; $s_1=M^2$, $s_1\geqslant(M+\mu)^2$ all correspond to *real* values of q^2. Thus the singularities in the pion-nucleon scattering amplitude arising from channels 1, 2, 3 for any real scattering angle all lie on the real real axis in the q^2 plane. This maps on to the real axis in the s_3 plane together with the circle $|s_3|=M^2-\mu^2$.

However, in the case of pion photoproduction from nucleons[5] the relationships between the energy variables are given by

$$s_3=[(k^2+M^2)^{\frac{1}{2}}+k]^2=[(q^2+M^2)^{\frac{1}{2}}+(q^2+\mu^2)^{\frac{1}{2}}]^2,$$
$$s_2=\mu^2-2(q^2+\mu^2)^{\frac{1}{2}}k+2qk\cos\theta,$$
$$s_1=M^2-2(q^2+M^2)^{\frac{1}{2}}k-2qk\cos\theta,$$

where s_1, s_2, s_3, q, θ, M, and μ are defined as for pion-nucleon scattering and k is the magnitude of the photon's momentum in the center-of-momentum system. Here the singular points on the positive s_1 and s_2 real axes

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[1] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[2] S. W. MacDowell, Phys. Rev. **116**, 774 (1959); W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960); **119**, 1420 (1960); J. Hamilton and T. D. Spearman, Ann. Phys. (New York) **12**, 172 (1961); S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960); J. Bowcock, N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960); Fayyazuddin, Phys. Rev. **123**, 1882 1961); D. Amati, E. Leader, and B. Vitale, Nuovo cimento **18**, 459 (1960).

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 and 1952 (1959).

[4] See Appendix I.

[5] See Appendix II.

do not map on to the real axis in the q^2 or k^2 plane, and for different values of θ these map on to different curves in the s_3 plane. Thus, due to the range of physical angles, $0 \leqslant \theta \leqslant \pi$, these points from the s_1 and s_2 real axis map into an area in the s_3 plane formed by the aggregate of curves, each of which is associated with an individual value of θ.[6]

The purpose of the present paper is to examine the location of the singularities arising from the energy spectra in the three channels in the s_3 plane of the partial wave amplitudes for the general mass case. In Sec. 2 we describe the kinematics of the general mass process with two ingoing and two outgoing particles. In Sec. 3 we examine the form of the partial wave amplitudes and find that these have branch point singularities at the points in the s_3 plane corresponding to the angles $\theta=0$, $\theta=\pi$ but that the areas of singularities arising from the range of intermediate angles do not appear. In Sec. 4 we find the location of these branch points in the s_3 plane and describe the cuts made necessary by these partial wave amplitudes. These are summarized in Sec. 5. Finally, in Appendices I and II we apply these general results obtained in Sec. 4 to the particular cases of pion-nucleon scattering and pion photoproduction from nucleons.

2. KINEMATICS OF THE GENERAL MASS CASE

$$
\begin{aligned}
s_3 &= -(q_1+q_2)^2 = -(q_3+q_4)^2, \\
s_2 &= -(q_1+q_3)^2 = -(q_2+q_4)^2, \\
s_1 &= -(q_2+q_3)^2 = -(q_1+q_4)^2,
\end{aligned}
\tag{1}
$$

where q_i denote the 4-momenta of the particles, all considered as ingoing. (See Fig. 1.)

We shall look at the process occurring in channel 3 $(1+2 \rightarrow 3+4)$. In the c.m. system for channel 3 we may write

$$
\begin{aligned}
q_1 &= (\omega_1, \mathbf{q}), \\
q_2 &= (\omega_2, -\mathbf{q}), \\
q_3 &= (-\omega_3, \mathbf{q}'), \\
q_4 &= (-\omega_4, -\mathbf{q}'),
\end{aligned}
$$

where

$$
\omega_{1,2} = (m_{1,2}^2+q^2)^{\frac{1}{2}}, \quad \omega_{3,4} = (m_{3,4}^2+q'^2)^{\frac{1}{2}},
$$

ω_i, m_i being the energies and masses, respectively, of the four particles, $\mathbf{q}$ and $\mathbf{q}'$ the ingoing and outgoing momenta, and $q=|\mathbf{q}|$, $q'=|\mathbf{q}'|$.

Then if θ is the channel-3 scattering angle,[7]

$$
\begin{aligned}
s_3 &= (\omega_1+\omega_2)^2 = (\omega_3+\omega_4)^2, \\
s_2 &= (\omega_1-\omega_3)^2 - q^2 - q'^2 - 2qq' \cos\theta, \\
s_1 &= (\omega_1-\omega_4)^2 - q^2 - q'^2 + 2qq' \cos\theta.
\end{aligned}
\tag{2}
$$

[6] See A. Minguzzi, Nuovo cimento **20**, 599 (1961).
[7] We define $p\cdot q = \mathbf{p}\cdot\mathbf{q} - p_0 q_0$.

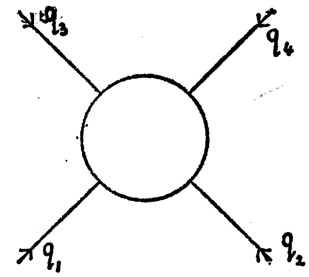


FIG. 1. General four-particle process.

From Eq. (2) we find

$$
\begin{aligned}
q^2 &= [s_3-(m_1+m_2)^2][s_3-(m_1-m_2)^2]/4s_3, \\
q'^2 &= [s_3-(m_3+m_4)^2][s_3-(m_3-m_4)^2]/4s_3.
\end{aligned}
\tag{3}
$$

Further

$$
\begin{aligned}
\omega_1 &= (m_1^2+q^2)^{\frac{1}{2}} = (s_3+m_1^2-m_2^2)/(4s_3)^{\frac{1}{2}}, \\
\omega_3 &= (m_3^2+q'^2)^{\frac{1}{2}} = (s_3+m_3^2-m_4^2)/(4s_3)^{\frac{1}{2}}, \\
\omega_4 &= (m_4^2+q'^2)^{\frac{1}{2}} = (s_3+m_4^2-m_3^2)/(4s_3)^{\frac{1}{2}},
\end{aligned}
\tag{4}
$$

The signs of ω_1, ω_2, ω_3, ω_4 have been chosen so that when s_3 takes physical values for channel 3, ω_1, ω_2, ω_3, ω_4 are to be positive.

This gives in terms of s_3 and $\cos\theta$,

$$
\begin{aligned}
s_2 = {} & m_1^2+m_3^2-[s_3+m_1^2-m_2^2][s_3+m_3^2-m_4^2]/(2s_3) \\
& -\{[s_3-(m_1+m_2)^2][s_3-(m_1-m_2)^2] \\
& \times[s_3-(m_3+m_4)^2][s_3-(m_3-m_4)^2]\}^{\frac{1}{2}} \cos\theta/(2s_3), \\
s_1 = {} & m_1^2+m_4^2-[s_3+m_1^2-m_2^2][s_3+m_4^2-m_3^2]/(2s_3) \\
& +\{[s_3-(m_1+m_2)^2][s_3-(m_1-m_2)^2] \\
& \times[s_3-(m_3+m_4)^2][s_3-(m_3-m_4)^2]\}^{\frac{1}{2}} \cos\theta/(2s_3).
\end{aligned}
\tag{5}
$$

We observe the crossing relation, that $s_2 \leftrightarrow s_1$ when we interchange m_3 and m_4 and replace $\cos\theta$ by $-\cos\theta$. For s_2 and s_1 to be defined as single-valued functions of s_3 it is necessary to introduce cuts in the s_3 plane at the branch points $(m_1 \pm m_2)^2$, $(m_3 \pm m_4)^2$. It is normally convenient to take these cuts along the real axis from $\min\{(m_1+m_2)^2, (m_3+m_4)^2\}$ to ∞ and from $-\infty$ to $\max\{(m_1-m_2)^2, (m_3-m_4)^2\}$.

3. SINGULARITIES INTRODUCED BY THE ENERGY SPECTRA IN THE THREE CHANNELS

We denote by σ_1, σ_2, σ_3 the squares of the lowest masses of strongly interacting intermediate states in channels 1, 2, 3, respectively. In some cases σ_1, σ_2, σ_3 may denote discrete terms corresponding to single-particle intermediate states and the continuous spectra will begin at ρ_1, ρ_2, ρ_3 corresponding to two-particle intermediate states. Where there is no discrete term $\sigma_i = \rho_i$.

The spectrum of intermediate states in channel 3 introduces, in the s_3 plane, a possible pole at σ_3 and a cut from ρ_3 to ∞ along the real axis.

Let us now investigate the singularities in the s_3 plane introduced in the partial wave amplitudes for channel-3 processes by the spectrum of intermediate states in channel 2. These singularities appear in the lth partial wave amplitude in the form[8]:

$$A_l(s_3)=\int_{-1}^{+1} d(\cos\theta)P_l(\cos\theta)\int_{\sigma_2}^{\infty} ds_2' \frac{\mathfrak{F}(s_2's_3)}{s_2'-s_2}, \quad (6)$$

where $P_l(x)$ is the lth Legendre polynomial and $\mathfrak{F}(s_2',s_3)$ depends only on s_3 and the parameter s_2'.

From Eq. (5) we see that the denominator $s_2'-s_2$ may be written as

$$\alpha(s_2',s_3)+\beta(s_3)\cos\theta, \quad (7)$$

where

$$\begin{aligned}\alpha(s_2',s_3)&=s_2'-m_1^2-m_3^2+(s_3+m_1^2-m_2^2)\\&\quad\times(s_3+m_3^2-m_4^2)/(2s_3),\\ \beta(s_3)&=\{[s_3-(m_1+m_2)^2][s_3-(m_1-m_2)^2]\\&\quad\times[s_3-(m_3+m_4)^2][s_3-(m_3-m_4)^2]\}^{\frac{1}{2}}/(2s_3).\end{aligned} \quad (8)$$

Assuming suitable convergence of the integral in Eq. (6) we may interchange the order of integration and obtain

$$A_l(s_3)=\int_{\sigma_2}^{\infty} ds_2'\, \mathfrak{F}(s_2',s_3)\int_{-1}^{1} dx \frac{P_l(x)}{\alpha(s_2',s_3)+\beta(s_3)x}. \quad (9)$$

We must discuss the three cases:

a. $\beta(s_3)=0, \quad \alpha(s_2',s_3)=0.$

$\beta(s_3)=0$ implies that s_3 has one of the values $(m_1\pm m_2)^2$, $(m_3\pm m_4)^2$. We may write $\alpha(s_2',s_3)$ as

$$\alpha(s_2',s_3)=(s_3-\gamma_1)(s_3-\gamma_2)/(2s_3),$$

where γ_1 and γ_2 are functions of s_2'. Since $\beta(s_3)=0$, $\alpha(s_2',s_3)$ can only vanish when γ_1 or γ_2 takes one of the values $(m_1\pm m_2)^2$, $(m_3\pm m_4)^2$; say γ_1. Then for the appropriate values of s_2' and s_3 we may write

$$\beta(s_3)=\bar{\beta}(s_3)(s_3-\gamma_1)^{\frac{1}{2}},$$
$$\alpha(s_2',s_3)=\bar{\alpha}(s_2',s_3)(s_3-\gamma_1),$$

where $\bar{\alpha}(s_2',s_3)$, $\bar{\beta}(s_3)$ are both nonvanishing and finite.

The integral then becomes

$$\frac{1}{(s_3-\gamma_1)^{\frac{1}{2}}}\int_{-1}^{+1}\frac{P_l(x)dx}{\bar{\alpha}(s_2',s_3)(s_3-\gamma_1)^{\frac{1}{2}}+\bar{\beta}(s_3)x}. \quad (10)$$

Since $\bar{\alpha}(s_2',s_3)(s_3-\gamma_1)^{\frac{1}{2}}=0$ and $\bar{\beta}(s_3)\neq 0$ it follows that the only singularity introduced by this term comes from the term $(s_3-\gamma_1)^{\frac{1}{2}}$. This gives a branch point singularity at γ_1. γ_1 can have any of the four values $(m_1\pm m_2)^2$, $(m_3\pm m_4)^2$ but we have already introduced cuts at these points to define s_1 and s_2 so no new singularities are introduced by this case.

b. $\beta(s_3)=0, \quad \alpha(s_2',s_3)\neq 0.$

[8] A singularity of the form $1/(s_2'-s_2)(s_1'-s_1)$ can be reduced to partial fractions and gives a term as above together with one containing $1/(s_1'-s_1)$.

The integral (10) can be written in this case as

$$\frac{1}{\alpha(s_2',s_3)}\int_{-1}^{1} P_l(x)dx, \quad (11)$$

which gives no singularities.

c. $\beta(s_3)\neq 0, \quad \alpha(s_2',s_3)\neq 0.$

As $\beta(s_3)\neq 0$ we can write $P_l(x)$ as a polynomial in $[\alpha(s_2',s_3)+\beta(s_3)x]$. Then the integral (9) may be written as

$$\int_{-1}^{1} F(s_2',s_3,x)dx+\int_{-1}^{1}\frac{C(s_2',s_3)dx}{a(s_2',s_3)+\beta(s_3)x}, \quad (12)$$

where $F(s_2',s_3,x)$ and $C(s_2',s_3)$ are nonsingular functions of s_3. The first integral in (12) gives no singularities; performing the second integral we obtain

$$[C(s_3)/\beta(s_3)]\{\ln[\alpha(s_2',s_3)+\beta(s_3)] - \ln[\alpha(s_2',s_3)-\beta(s_3)]\}. \quad (13)$$

This function has branch point singularities at points s_3 such that

$$\alpha(s_2',s_3)=\pm\beta(s_3) \quad (14)$$

for any s_2' lying between σ_2 and ∞. The problem of locating the remaining singularities of the partial wave amplitudes in the s_3 plane reduces to the mathematical one of determining the solutions of Eq. (14).

4. LOCATION OF SINGULARITIES IN THE s_3 PLANE

From Eq. (8) we see that $s_3=0$ satisfies the equation,

$$\alpha(s_2',s_3)=+\beta(s_3),$$

for all values of s_2'. Thus for all values of s_2' we must introduce a cut extending from the origin to infinity. It is convenient to take this along the negative real axis. To find the other roots we can replace Eq. (14) by

$$as_3^2+2bs_3+c=0, \quad (15)$$

where

$$\begin{aligned}a&=s_2'\\ 2b&=s_2'^2-\sum_{i=1}^{4} m_i^2 s_2'+(m_1^2-m_3^2)(m_2^2-m_4^2),\\ c&=(m_1^2-m_2^2)(m_3^2-m_4^2)s_2'\\&\quad+(m_1^2m_4^2-m_2^2m_3^2)(m_1^2-m_1^2-m_3^2+m_4^2).\end{aligned} \quad (16)$$

To find the roots of $as_3^2+2bs_3+c=0$, denote the real and imaginary parts of s_3 by x and y. Equation (15) then gives

$$a(x^2-y^2)+2bx+c=0, \quad (17)$$

$$axy+by=0. \quad (18)$$

The singularities in the partial wave amplitudes are given by the points of intersection of the curves given by Eq. (17) and (18). Equation (18) represents the pair of straight lines $y=0$, $x=-b/a$. The hyperbola represented by Eq. (17) only intersects the line $y=0$ if $b^2 \geqslant ac$ and only intersects the line $x=-b/a$ if $b^2 \leqslant ac$. The former case gives rise to singularities lying on the real axis; the latter to singularities off the real axis. This is shown in Fig. 2. The discriminant b^2-ac may be written in the form

$$\tfrac{1}{4}[s_2'-(m_1+m_3)^2][s_2'-(m_1-m_3)^2] \times[s_2'-(m_2+m_4)^2][s_2'-(m_2-m_4)^2]. \quad (19)$$

This is represented graphically in Fig. 3.

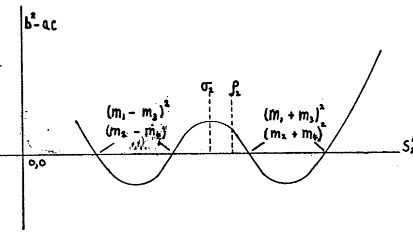


FIG. 3. b^2-ac in the general mass case.

It is clear that no channel-2 intermediate state (under strong interactions) can be of mass less than $|m_1-m_3|$ or $|m_2-m_4|$ for if this were so, the heavier of m_1, m_3 or m_2, m_4 would be unstable under strong interactions. So $\rho_2 \geqslant \sigma_2 \geqslant \max\{(m_1-m_3)^2, (m_2-m_4)^2\}$. If all four particles are strongly interacting $\sigma_2 \leqslant \rho_2 \leqslant \min\{(m_1+m_3)^2, (m_2+m_4)^2\}$ and so σ_2 and ρ_2 must lie as shown in Fig. 3. In the event of some of the particles not being strongly interacting ρ_2 or σ_2 may be greater than m_1+m_3 or m_2+m_4. An example of this is seen in the case of pion-photoproduction (cf. Appendix II). Here the channel-2 process is $\gamma+\pi\rightarrow N+\bar{N}$ and so $m_1=0$, $m_3=\mu$. The matrix element for $\gamma\pi$ scattering is of the second order in the em coupling constant e: if we consider only first-order processes in e the spectrum of intermediate states consists of one pion, two pion and higher mass states. Thus $\sigma_2=\mu^2$ corresponding to the one-pion pole, and $\rho_2=4\mu^2$. So in this case $\sigma_2=(m_1+m_3)^2<\rho_2$. As will be seen later this means that the singularities off the real axis lie on a curve which is open at the right-hand side.

From Fig. 3 we see that singularities lying off the real axis can arise only from values of s_2' lying between the values $(m_1+m_3)^2$ and $(m_2+m_4)^2$. The singularities on the real axis can come from

$$s_2' \leqslant \min\{(m_1+m_3)^2, (m_2+m_4)^2\}$$

and also from

$$s_2' \geqslant \max\{(m_1+m_3)^2, (m_2+m_4)^2\}.$$

We shall now investigate the location of these singularities.

(a) Singularities Lying Off the Real Axis

The positions of these singularities are defined by the locus of intersection of the hyperbolas given by Eq. (17) with the lines $x=-b/a$: this is obtained by eliminating the parameter s_2'. The locus may be written in the form

$$\{-(x-\tfrac{1}{2}\Sigma)\pm[(x-\tfrac{1}{2}\Sigma)^2-\lambda]^{\frac{1}{2}}\}[x^2+y^2-\kappa]=\nu, \quad \text{for} \quad \lambda\neq 0 \quad (20)$$

or

$$-2(x-\tfrac{1}{2}\Sigma)(x^2+y^2-\kappa)=\nu, \quad \text{for} \quad \lambda=0,$$

where

$$\begin{aligned} \Sigma &= \sum_{i=1}^{4} m_i^2, \\ \lambda &= (m_1^2-m_3^2)(m_2^2-m_4^2), \\ \kappa &= (m_1^2-m_2^2)(m_3^2-m_4^2), \\ \nu &= (m_1^2m_4^2-m_2^2m_3^2)(m_1^2-m_2^2-m_3^2+m_4^2). \end{aligned} \quad (21)$$

In this notation Eq. (16) become

$$\begin{aligned} a &= s_2', \\ 2b &= s_2'^2-\Sigma s_2'+\lambda, \\ c &= \kappa s_2'+\nu. \end{aligned} \quad (16')$$

From Eq. (16′) it follows that as s_2' ranges from ρ_2 to ∞ so $-b/a$ ranges from $-\infty$ to L, where

$$\begin{aligned} L &= \Sigma/2-\lambda^{\frac{1}{2}} && \text{for} \quad \lambda>\rho_2^2, \\ &= \Sigma/2-\tfrac{1}{2}(\rho_2+\lambda/\rho_2) && \text{for} \quad \lambda<\rho_2^2. \end{aligned} \quad (21)$$

When there is a pole term $s_2'=\sigma_2$ in the case of

$$\max\{(m_1+m_3)^2, (m_2+m_4)^2\} \geqslant \sigma_2 \geqslant \min\{(m_1+m_3)^2, (m_2+m_4)^2\}$$

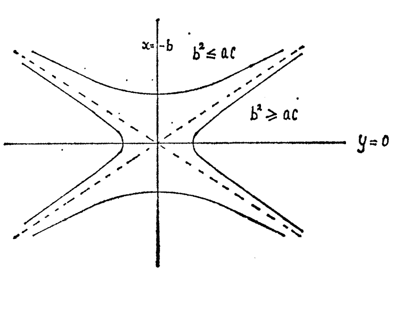
the branch arising from this pole will lie on the line

$$x=\frac{-1}{2\sigma_2}(\sigma_2^2-\sigma_2\Sigma+\lambda);$$

otherwise the branch points will lie on the real axis. To take account of this term it is sufficient to introduce a cut joining the two branch points.

So we see that apart from the pole term contribution the singularities due to the channel-2 energy spectrum which lie off the real axis are defined by that part of the curve given by Eq. (20) which lies to the left of the line $x=L$.

FIG. 2. Representation of the equations: $a(x^2-y)^2+2bx+c=0$, $axy+by=0$.



(b) Singularities Lying On the Real Axis

The positions of these singularities are defined by the locus of intersection of the hyperbolas given by Eq. (17) with the line $y=0$. On eliminating the parameter s_2' we get the equation

$$ax^2+2bx+c=0, \tag{22}$$

which gives the values of x

$$ax=-b\pm(b^2+ac)^{\frac{1}{2}}, \tag{23}$$

where a, b, c are given as functions of s_2' by Eq. (16). As we have seen the values of s_2' that may contribute are a possible pole term $s_2'=\sigma_2$ provided that

$$\sigma_2\leqslant\min\{(m_1+m_3)^2,\ (m_2+m_4)^2\};$$

values of s_2' (if any) such that

$$\rho_2\leqslant s_2'\leqslant\min\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$$

and values

$$s_2'\geqslant\max\{(m_1+m_3)^2,\ (m_2+m_4)^2,\ \rho_2\}.$$

A similar analysis in which m_3 and m_4 are everywhere interchanged gives the singularities arising from the channel-1 energy spectrum. We see that either crossed channel will lead to singularities lying off the real axis if the thresholds for ingoing and outgoing states in that channel are unequal and provided that the threshold for strongly interacting intermediate states is less than the greater of these.

Corresponding to the continuum part of the energy spectrum in one of the crossed channels there will be a continuum of pairs of branch points. To allow the partial wave amplitude to be defined in the s_3 plane it is necessary to introduce cuts which prevent any one of these branch points being encircled without its partner also being encircled. This is achieved by a continuous cut, on which all these pairs of branch points lie, given by the part of the curve defined by the Eq. (20) lying to the left of the line $x=L$; the real axis from the origin to $-\infty$ and the part of the real axis which is determined by Eq. (23). A discrete term in the energy spectrum will give rise normally to two branch points which may not overlap with the region containing singular points due to the continuum part: in this case a cut joining these two branch points together with a cut from the origin to $-\infty$ (cf. Appendix I) are sufficient.

5. SUMMARY

So finally we may summarize the cuts and poles which are necessary to define a partial wave amplitude in the s_3 plane. These are:

(i) Cuts from $\max\{(m_1-m_2)^2,\ (m_3-m_4)^2\}$ to $-\infty$, and from $\min\{(m_1+m_2)^2,\ (m_3+m_4)^2\}$ to ∞, are required to define s_1 and s_2 in the s_3 plane.

(ii) A possible pole at σ_3 and a cut from ρ_3 to ∞ on the real axis, arising from the channel-3 energy spectrum.

(iii) A cut on the real axis from the origin to $-\infty$ corresponding to the branch point $s_3=0$ for any value of s_2' in the channel-2 energy spectrum and a cut on the real axis defined by Eq. (23) for s_2' lying in the ranges ρ_2 to $\min\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$ and $\max\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$ to ∞. A similar cut for the continuum part of the channel-1 energy spectrum.

(iv) A cut, on the real axis if $\sigma_2\leqslant\min\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$ (possibly detached from any of the above cuts) due to a possible discrete term in the channel-2 energy spectrum, whose end points are determined by Eq. (23) and a similar cut due to a discrete term in the channel-1 energy spectrum.

(v) A curve lying off and symmetrical about the real axis, determined by Eq. (20), which is closed at its left-hand end but may be open at the right-hand end if some of the particles are not strongly interacting. This arises from values of s_2' lying in the range $\min\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$ to $\max\{(m_1+m_3)^2,\ (m_2+m_4)^2\}$ and only occurs if $m_1+m_3\neq m_2+m_4$. A similar curve may also arise from the channel-1 energy spectrum.

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APPENDIX I

Pion-Nucleon Scattering

In the particular case of pion-nucleon scattering $m_1=\mu$, $m_2=M$, $m_3=\mu$, $m_4=M$, where μ and M are the pion and nucleon masses, respectively. For the channel-2 process $\pi\pi\rightarrow N\bar{N}$, $\sigma_2=\rho_2=4\mu^2$. For the channel-1 process $\pi\bar{N}\rightarrow\pi\bar{N}$, $\sigma_1=M^2$, $\rho_1=(M+\mu)^2$.

Then the singularities of the partial wave amplitudes in the s_3 plane, as given by Sec. (5), are:

(i) Cuts from $-\infty$ to $(M-\mu)^2$ and $(M+\mu)^2$ to ∞ to define s_1 and s_2 in the s_3 plane.

(ii) A pole at M^2 and a cut from $(M+\mu)^2$ to ∞. The physical region for channel 3.

(iii) A cut on the real axis from the origin to $-\infty$.

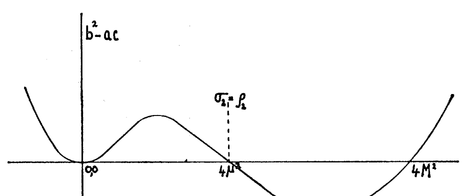

FIG. 4. b^2-ac for channel 2 of π-N scattering.

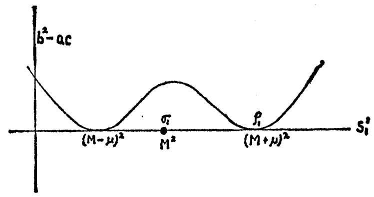

FIG. 5. b^2-ac for channel 1 of $\pi-N$ scattering.

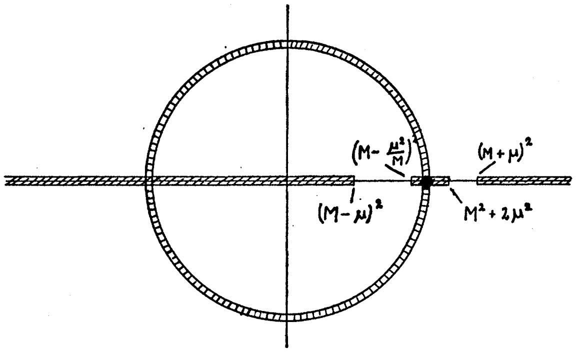

FIG. 6. Singularities in the partial-wave amplitudes for $\pi-N$ scattering.

From Fig. 4, we see that $b^2-ac\geqslant 0$ for $4M^2\leqslant s_2'<\infty$ giving a cut on the real axis. This extends from $-\infty$ to 0. From Fig. 5, we see that $b^2-ac\geqslant 0$ for $(M+\mu)^2\leqslant s_1'<\infty$. This gives a cut on the real axis, from $-\infty$ to $(M-\mu)^2$.

(iv) There is no discrete term in channel 2.

The discrete term at M^2 in channel 1 gives a cut on the real axis from $(M-\mu^2/M)^2$ to $M^2+2\mu^2$.

(v) From Fig. 4 we see that channel 2 gives rise to a cut off the real axis for $4\mu^2\leqslant s_2'\leqslant 4M^2$. For channel 2

$$\Sigma=2(M^2+\mu^2),\quad \lambda=0,\quad \kappa=(M^2-\mu^2)^2,\quad \nu=0.$$

By Eq. (16) this gives

$$x^2+y^2=(M^2-\mu^2)^2,$$

a circle of radius $(M^2-\mu^2)$.

As seen in Fig. 5, b^2-ac is nowhere negative and thus we conclude that channel 1 does not give rise to cuts off the real axis.

The cuts for π-N scattering are shown in Fig. 6.

APPENDIX II

Photoproduction of Pions or Nucleons

In the case of photoproduction we have more asymmetry in the masses involved and the situation is more complex. We take $m_1=0$, $m_2=M$, $m_3=\mu$, $m_4=M$. For the channel-1 process $\gamma\bar{N}\rightarrow\pi\bar{N}$, $\sigma_1=M^2$ and $\rho_1=(M+\mu)^2$. For the channel-2 process $\gamma\pi\rightarrow N\bar{N}$, $\sigma_2=\mu^2$ and $\rho_2=4\mu^2$.

In channel 2 we take as the beginning of the continuous spectrum of intermediate states, $\rho_2=4\mu^2$ because the $\gamma\pi$ intermediate state in $\gamma\pi\rightarrow\gamma\pi\rightarrow N\bar{N}$ leads to a matrix element which is of second order in the electromagnetic coupling constant. For a similar reason we take $\rho_1=\rho_3=(M+\mu)^2$.

Following Sec. 5 we find as singularities in the s_3 plane:

(i) Cuts from $-\infty$ to $(M-\mu)^2$ and $(M+\mu)^2$ to ∞ to define s_1 and s_2 in the s_3 plane.

(ii) A pole at M^2 and a cut from $(M+\mu)^2$ to ∞. The physical region for channel 3.

(iii) A cut on the real axis from the origin to $-\infty$ on account of the branch point at $s_3=0$.

From Fig. 7 we see that for $4M^2\leqslant s_2'<\infty$ we have a cut on the real axis. This runs from $-\infty$ to 0.

From Fig. 8 we see that for $(M+\mu)^2\leqslant s_1'<\infty$ we have a cut on the real axis. This is from $-\infty$ to $[M/(M+\mu)](M^2-M\mu-\mu^2)$.

(iv) The discrete term in channel 2 at $s_2'=\sigma_2=\mu^2$ gives a pole at $s_3=M^2$. b^2-ac is zero at this value of s_2' (Fig. 7).

The discrete term in channel 1 at $s_1'=\sigma_1=M^2$ also gives a pole at $s_3=M^2$. Again b^2-ac is zero at this value s_1' (Fig. 8).

(v) From Fig. 7 we see that channel 2 gives rise to singularities off the axis for $\rho_2=4\mu^2\leqslant s_2'\leqslant 4M^2$. The cut off the axis is given by Eq. (18). In channel 2,

$$\Sigma=2M^2+\mu^2,\quad \lambda=0,\quad \kappa=M^2(M^2-\mu^2),\quad \nu=\mu^2+M^2.$$

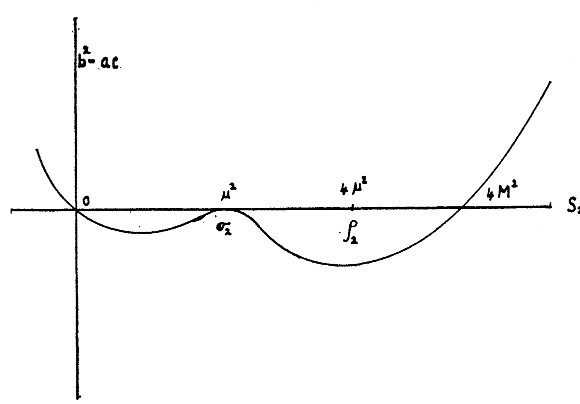

FIG. 7. b^2-ac for channel 2 of $\gamma N\rightarrow\pi N$.

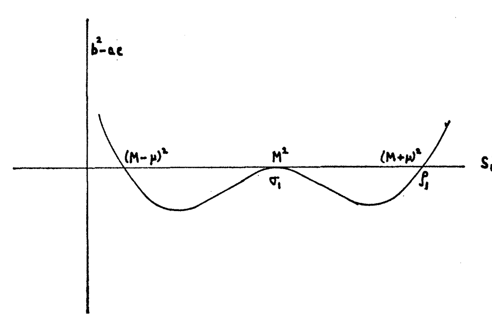

FIG. 8. b^2-ac for channel 1 of $\gamma N\rightarrow\pi N$.

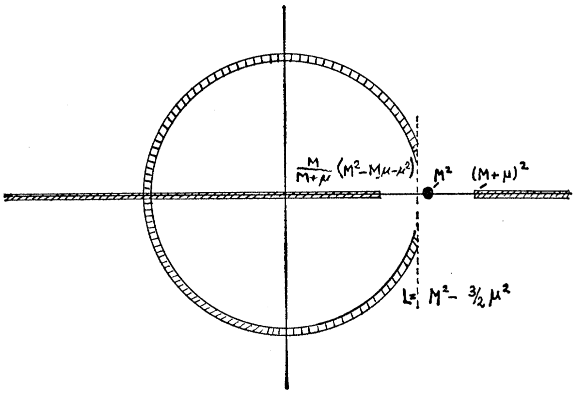

FIG. 9. Singularities in the partial-wave amplitudes for photoproduction.

We can write the curve as

$$x^2+y^2=M^2(M^2-\mu^2)-\mu^4M^2/(2x-2M^2-2\mu^2).$$

Since $x<M^2$, $\mu^4M^2/(2x-2M^2-\mu^2)<M^2\mu^2$, which is much less than $M^2(M^2-\mu^2)$. Thus the curve off the axis deviates very little from a circle of radius $M(M^2-\mu^2)^{\frac{1}{2}}$. This curve lies between two circles one of radius 45.2 μ^2 and the other of radius 44.6 μ^2.

The curve off the real axis is not closed on the right-hand side as x is limited by Eq. (20). Here

$$L=\tfrac{1}{2}\Sigma-\tfrac{1}{2}(\sigma_2+\lambda/\sigma_2)$$
$$=M^2-\tfrac{3}{2}\mu^2.$$

As can be seen from Fig. 7 the fact that the curve is open on the right-hand side comes about because there is no contribution from values of s_2' such that $\mu^2<s_2'<4\mu^2$ due to the neglect of the $\gamma\pi$ intermediate state.

For channel 1 we see from Fig. 8 that the neglect of the γN intermediate state means that there is no contribution to the singularities from values of s_1' which make b^2-ac negative. Thus channel 1 does not give rise to singularities off the real axis. (See Fig. 9).