Experiments[4–9] based, e.g., on light emission or probe currents, give higher average velocities which correspond to those found with shock waves.[10,11] These are to be expected with the large current pulses which have been superimposed on the auxiliary discharge. Such velocities are probably not representative of those in a steady discharge. The velocity spectrum in high-current arcs is still to be investigated. However, it appears that the emission of neutral vapor from low-current arcs cannot be described in terms of classical evaporation theory or expressed in terms of a temperature of the cathode spot.[12]

ACKNOWLEDGMENT

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[11] P. F. Little, Proceedings of the Conference on Ionization, Munich, 1961 (North Holland Publishing Company, Amsterdam, 1962), Vol. 2.

[12] A. von Engel and K. W. Arnold, Nature 187, 1101 (1960).

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Approach to Equilibrium of Electrons, Plasmons, and Phonons in Quantum and Classical Plasmas*

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The approach to equilibrium of electrons, plasmons, and phonons in finite-temperature plasmas is studied in the random phase approximation. It is first shown that for an electron plasma in equilibrium, the long-wavelength, well-defined, plasmons contribute a term to the free energy which is appropriate to a collection of independent bosons. In order to study nonequilibrium processes, an explicit plasmon distribution function is introduced. The matrix element for plasmon-electron coupling is calculated in the random phase approximation, and second-order perturbation theory is used to write down the equations which couple the electron and plasmon distribution functions. Equilibrium is shown to result from the competition between the spontaneous emission of plasma waves by single fast electrons and the Landau damping of the plasma waves due to the same group of particles. The equation for the time rate of change of the electron distribution function reduces, in the classical limit, to a Fokker-Planck equation in which there appear diffusion and friction terms associated with plasma waves, of the type first considered by Klimontovitch. When an initial arbitrary nonequilibrium electron distribution is considered, it is seen that a plasma wave instability corresponds to coherent excitation of plasma waves by the electrons in contrast to the incoherent excitation associated with spontaneous emission. The method is generalized to two-component electron-ion plasmas in which well defined acoustic plasma waves exist, by introducing an explicit distribution function for the phonons (the acoustic plasmons). The approach to equilibrium and the two-stream instability are derived and discussed for the coupled electron-phonon system; results similar to those obtained for the electron-plasmon system are found.

I

THE properties of an interacting electron gas in a uniform background of positive charge have been studied extensively in both the limit of low temperatures and high densities (the quantum plasma) and high temperatures and low densities (the classical plasma). In both these limits, the random phase approximation (RPA) is valid. In general there are two essentially distinct modes of excitation of the plasma: single-particle modes with energies appropriate to a gas of noninteracting electrons, and collective modes, the plasma oscillations, which possess a frequency near the plasma frequency, $\omega_p=(4\pi ne^2/m)^{\frac{1}{2}}$. At long wavelengths the coupling between the plasmons (the quantized plasma modes) and the individual electrons is weak, because there are few electrons which are capable of absorbing the plasmon energy and momentum, so that a long-wavelength plasmon constitutes a well-defined excitation mode. For short wavelengths there are many electrons available to absorb plasmons, so that a short-wavelength plasma mode is highly damped and cannot be usefully regarded as an elementary excitation of the plasma.

Despite the by-now extensive plasma literature,[1] the coupling between electrons and plasmons in a finite-temperature quantum plasma does not seem to have been studied in any great detail. In the present paper we carry out such a study with the aid of explicit plasmon and electron distribution functions and demonstrate its usefulness for an understanding of the way in which the equilibrium between plasmons and electrons comes about in both quantum and classical plasmas.

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[1] Some recent review articles dealing with both classical and quantum plasmas are: Y. Klimontovitch and V. P. Silin, Uspekhi Fiz. Nauk. 70, 247 [translation: Soviet Phys.—Uspekhi 3, 84 (1960)], which contains rather complete references to the Russian work in this field, and D. Pines, Physica 26, 103 (1960).

The use of an explicit plasmon distribution function represents a finite-temperature generalization of the collective coordinate method of Bohm and one of the authors.[2] The coupling between electrons and plasmons is obtained directly from the Bohm-Pines theory, although, for example, field-theoretic methods could also be utilized to this purpose. The method yields, in the classical limit for the electron distribution function, a Fokker-Planck equation in which there appear diffusion and friction terms associated with plasma waves.[3]

Equilibrium arises as a result of the competition between the spontaneous emission of plasma waves by single fast electrons and the Landau damping of the plasma waves due to this same group of single particles. The same method may be applied to an arbitrary nonequilibrium distribution and leads one directly to the concept of a plasma wave instability (e.g., the two-stream instability) as a coherent excitation of plasma waves (in contrast to the incoherent excitation associated with spontaneous emission).

The generalization of the method to a two-component plasma in which well-defined acoustic plasma modes exist is straightforward. Examples of such plasmas are the continuum model for the motion of electrons and ions in a metal, or an electron-ion classical plasma in which the electrons are at a temperature large compared to that of the ions. One obtains thereby the approach to equilibrium between the electrons and acoustic plasma waves, in addition to the electron high-frequency (optical) plasma wave equilibrium previously studied. For the classical electron-ion plasma, one finds a Fokker-Planck equation with terms describing acoustic plasmon diffusion and friction; the two-stream instability is seen to represent coherent excitation of acoustic plasma oscillations.

In Sec. II, we review briefly the RPA plasma properties. We show explicitly that at finite temperatures the long-wavelength, well-defined, plasmons contribute a term to the free energy which is appropriate to a collection of independent bosons. In Sec. III, we discuss the plasmon-electron coupling. Section IV is devoted to a brief survey of applications of the method to two-component plasmas, and Sec. V consists of a short discussion of some possible future applications of the methods developed in the preceding sections. Appendix A is devoted to the quantum-mechanical derivation of the plasma dispersion relation by solving the equations of motion of the system (within RPA) as an initial value problem.

II

We review briefly the properties of an electron plasma as calculated within the random phase approximation[1,4] (RPA). These properties may be determined from the retarded frequency and wave-vector-dependent longitudinal dielectric constant $\epsilon(\mathbf{q},\Omega)$ of the plasma. The plasmon dispersion relation is given by

$$\epsilon(\mathbf{q},\omega)=0. \tag{2.1}$$

The free energy of the plasma is given by[5]

$$F=F(0)-\sum_{\mathbf{q}\neq 0}\frac{2\pi Ne^2}{q^2}-\sum_{\mathbf{q}\neq 0}\frac{\hbar}{4\pi i}\int_{-\infty}^{\infty}d\Omega\coth\frac{\hbar\Omega}{2kT}\int_0^{e^2}\frac{dg}{g}\operatorname{Im}\frac{1}{\epsilon'(\mathbf{q},\Omega)}, \tag{2.2}$$

where $\epsilon'(q,\Omega)$ is the equilibrium retarded dielectric constant appropriate to an interaction between the electrons of strength g and N is the number of particles in the system, chosen to be of unit volume. In the RPA one has quite generally,

$$\epsilon(\mathbf{q},\Omega)=1+4\pi\alpha(\mathbf{q},\Omega), \tag{2.3}$$

where $4\pi\alpha(\mathbf{q},\Omega)$ is the free-electron polarizability, given by

$$4\pi\alpha(\mathbf{q},\Omega)=-\frac{4\pi e^2}{q^2}\sum_{\mathbf{K}}[f(\mathbf{K})-f(\mathbf{K}+\mathbf{q})]\left[\frac{1}{\hbar(\Omega-\nu_{\mathbf{K}\mathbf{q}})+i\eta}\right] \tag{2.4}$$

for Ω in the upper half of the complex Ω plane. Here η is an infinitesimal positive quantity, and $f(\mathbf{K})$ is the probability of finding an electron in state $\mathbf{K}$, energy, $E(\mathbf{K})=\hbar^2K^2/2m$; at thermal equilibrium one has the familiar Fermi-Dirac result

$$f(\mathbf{K})=f^0(\mathbf{K})=1/[e^{(E_\mathbf{K}-\mu)/kT}+1], \tag{2.5}$$

where μ is the free-electron chemical potential. We have also introduced the one-electron excitation frequency,

$$\nu_{\mathbf{K}\mathbf{Q}}=[E(\mathbf{K}+\mathbf{Q})-E(\mathbf{K})]/\hbar.$$

The expression (2.3) is valid both for a quantum plasma (high electron densities and low temperatures) and a classical plasma (low electron densities and high temperatures). In order to satisfy the plasmon dispersion relation (2.1) for damped oscillations, the definition of $\epsilon(q,\Omega)$ must be extended to the lower half of the complex Ω plane. The proper extension may be accomplished by analytic continuation of ϵ across the cut along the real axis, as discussed in the Appendix. For small damping, one finds for the dispersion relation:

$$1=\frac{4\pi e^2}{m}\sum_{\mathbf{K}}\frac{f(\mathbf{K})}{(\omega-\hbar\mathbf{q}\cdot\mathbf{K}/m)^2-\hbar^2q^4/4m^2}-\frac{i4\pi^2e^2}{q^2}\sum_{\mathbf{K}}[f(\mathbf{K})-f(\mathbf{K}+\mathbf{q})]\delta[\hbar(\omega-\nu_{\mathbf{K}\mathbf{q}})]. \tag{2.6}$$

[2] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953); hereafter referred to as BP.

[3] Such an equation was first obtained by Y. Klimontovitch, JETP **36**, 1405 (1959) [translation: Soviet Phys.—JETP **36**(9), 999 (1959)].

[4] For an explicit comparison of some of the many equivalent formulations of the RPA, see H. Ehrenreich and M. Cohen, Phys. Rev. **115**, 786 (1959) and D. Pines, J. Nuclear Energy **C2**, 5 (1960).

[5] F. Englert and R. Brout, Phys. Rev. **120**, 1085 (1960).

In the limit of $T=0$ the imaginary term in (2.6) vanishes for $q\leq q_0$, where q_0 is defined by

$$\hbar\omega(q_0)=\hbar q_0 V_F+\hbar q_0^2/2m;$$

q_0 is the wave vector at which it first becomes possible for a plasmon of momentum q to transfer its energy and momentum to an electron within the Fermi sphere of a radius k_F, and V_F is the velocity of an electron on the Fermi surface. For $q\leq q_0$ (and within the RPA), the plasmons thus represent a stable elementary excitation of the system; the dispersion relation takes the familiar form

$$\omega^2=\omega_p^2+\tfrac{3}{5}q^2V_F^2+\cdots. \tag{2.7}$$

At non-zero temperatures there is always at least a slight probability of finding an electron of momentum $\mathbf{K}$ such that it is capable of absorping a plasmon of wave vector $\mathbf{q}$ with conservation of both energy and momentum. For the quantum plasma, when $q\lesssim q_0$ (and for the classical plasma, when $q\lesssim k_D$, the Debye screening wave vector) the number of such electrons is quite small, so that the resulting damping of the plasmons is likewise relatively small. In the long-wavelength limit, this damping is obtained by substituting $\omega=\omega_{\mathbf{q}}+i\gamma_{\mathbf{q}}$ into (2.6) and solving for $\omega_{\mathbf{q}}$ and $\gamma_{\mathbf{q}}$ under the assumption that $|\gamma_{\mathbf{q}}|\ll\omega_{\mathbf{q}}$. One finds for the dispersion relation for $\omega_{\mathbf{q}}$,

$$\begin{aligned}1&=\frac{4\pi e^2}{m}\sum_{\mathbf{K}}\frac{f(\mathbf{K})}{(\omega_{\mathbf{q}}-\hbar\mathbf{q}\cdot\mathbf{K}/m)^2-(\hbar^2q^4/4m^2)}\\&=\frac{8\pi e^2}{\hbar q^2}\sum_{\mathbf{K}}{}'\frac{f(\mathbf{K})\nu_{\mathbf{Kq}}}{\omega_q^2-\nu_{\mathbf{Kq}}^2},\end{aligned} \tag{2.8}$$

where $\sum_{\mathbf{K}}'$ denotes that principal parts are to be taken when the indicated summation is transformed to an integration. For small q and low temperatures, (2.8) reduces to (2.7). The result for $\gamma_{\mathbf{q}}$ is

$$\frac{\gamma_{\mathbf{q}}}{\omega_{\mathbf{q}}}=\frac{2\pi^2e^2}{q^2}\sum_{\mathbf{K}}\{f(\mathbf{K})-f(\mathbf{K}+\mathbf{q})\}\delta[\hbar(\omega_{\mathbf{q}}-\nu_{\mathbf{Kq}})]. \tag{2.9}$$

In the classical limit, (2.9) yields the results of Landau[6] and Bohm and Gross,[7] which for a Maxwellian distribution reads

$$\frac{\gamma_{\mathbf{q}}^c}{\omega_{\mathbf{q}}^c}=-\frac{2\pi^2e^2}{mq^2}\int d^3v\;\mathbf{q}\cdot\nabla_v f_0(\mathbf{v})\delta(\omega_{\mathbf{q}}^c-\mathbf{q}\cdot\mathbf{v}), \tag{2.9a}$$

where $\omega_{\mathbf{q}}^c$ is the solution of (2.8) in the classical limit.

As long as $\gamma_{\mathbf{q}}\ll\omega_{\mathbf{q}}$, it would seem reasonable to regard the plasmon as a well-defined elementary excitation of the system. Thus one might expect that in thermal equilibrium the number of plasmons present would be given by the Einstein-Bose distribution function,

$$N_{\mathbf{q}}^0=1/(e^{\hbar\omega_{\mathbf{q}}/kT}-1); \tag{2.10}$$

these plasmons would then contribute a term to the free energy F, which is

$$F_{\text{coll}}=\sum_{q<q_c}[(\hbar\omega_{\mathbf{q}}/2)+kT\ln(1-e^{-\hbar\omega_{\mathbf{q}}/kT})], \tag{2.11}$$

appropriate to a collection of bosons of frequency $\omega_{\mathbf{q}}$. Here $\mathbf{q}_c$ is the maximum wave vector for which the assumption $[\gamma_{\mathbf{q}}/\omega_{\mathbf{q}}]\ll1$ is tenable: One finds

$$\begin{aligned}&q_c\cong\omega_p/V_F, &&\text{(quantum plasmas)}\\&q_c\cong k_D=\omega_p/(kT/m)^{\frac{1}{2}}, &&\text{(classical plasmas).}\end{aligned}$$

We proceed to the derivation of (2.11).

We are concerned with the contribution to the free energy (2.2) from those poles of $\epsilon'(\mathbf{q},\Omega)=\epsilon_1'(\mathbf{q},\Omega)+i\epsilon_2'(\mathbf{q},\Omega)$ which lie near the real axis. We can write, in the vicinity of such a pole,

$$\epsilon'(\mathbf{q},\Omega)=[\partial\epsilon_1'(\mathbf{q},\Omega)/\partial\Omega]_{\Omega=\omega(g)}[\Omega-\omega(g)]+i\epsilon_2'(\mathbf{q},\Omega), \tag{2.12}$$

provided $\epsilon_2'(\mathbf{q},\omega)\ll1$. The plasmon contribution to the free energy will then be

$$F_{\text{coll}}=-\sum_{q<q_c}\frac{\hbar}{4\pi i}\int_{-\infty}^{\infty}d\Omega\coth\frac{\hbar\Omega}{2kT}\int_{g_1}^{e^2}\frac{dg}{g}\times\text{Im}\frac{1}{(\partial\epsilon_1'/\partial\Omega)_{\Omega=\omega(g)}[\Omega-\omega(g)]+i\epsilon_2'(\mathbf{q},\Omega)]}, \tag{2.13}$$

where g_1 is the lowest coupling constant for which

$$\epsilon_1'(\mathbf{q},\Omega)=0\quad\text{and}\quad\epsilon_2'(\mathbf{q},\Omega)\ll1. \tag{2.14}$$

We can transform (2.13) into simpler form with the aid of the identity,

$$[\partial\epsilon_1'(\Omega)/\partial\Omega]_{\Omega=\omega(g)}=[\partial\omega(g)/\partial g]^{-1}, \tag{2.15}$$

which follows directly from the definitions:

$$\epsilon_1'(\Omega)=1-\frac{4\pi g}{m}\sum_{\mathbf{K}}\frac{f(\mathbf{K})}{(\Omega-\hbar\mathbf{q}\cdot\mathbf{K}/m)^2-\hbar^2q^4/4m^2},$$

and

$$1=\frac{4\pi g}{m}\sum_{\mathbf{K}}\frac{f(\mathbf{K})}{[\omega(g)-\hbar\mathbf{q}\cdot\mathbf{K}/m]^2-\hbar^2q^4/4m^2}.$$

We then have, on substituting (2.15) into (2.13), and changing variables to $\omega(g)$,

$$F_{\text{coll}}=-\sum_{q<q_c}\frac{\hbar}{4\pi i}\int_{-\infty}^{\infty}d\Omega\coth\frac{\hbar\Omega}{2kT}\int_{\omega_1}^{\omega}d\omega(g)\times\text{Im}\frac{1}{\Omega-\omega(g)+i\epsilon_2'(\Omega)}. \tag{2.16}$$

Where $\epsilon_2'(\Omega)\ll1$, we then find

$$F_{\text{coll}}=\sum_{q<q_c}\frac{\hbar}{2}\int_{\omega_1}^{\omega}d\Omega\coth\frac{\hbar\Omega}{2kT}, \tag{2.17}$$

[6] L. D. Landau, J. Phys. U.S.S.R. 10, 25 (1946).
[7] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 (1949).

with $\omega_1 \cong q\bar{V}$, where $\bar{V} \cong V_F$ for a nearly degenerate Fermi gas, and $\bar{V} \cong (kT/m)^{\frac{1}{2}}$ for a classical plasma. On carrying out the integration in (2.17), one finds the contribution from the upper limit is given by (2.11) and represents that part of the free energy which is explicitly associated with the plasmons. The contribution from the lower limit is to be associated with part of the individual particle contribution to the free energy and will not concern us here.

III

We now consider the way in which an equilibrium distribution of plasmons and electrons is established (again, within the RPA). We may do this by considering $dN_{\mathbf{q}}/dt$ and $\partial f_{\mathbf{K}}/\partial t$, the time rate of change of the plasmon and electron distribution functions, respectively, in consequence of the plasmon-electron interaction. We calculate $dN_{\mathbf{q}}/dt$ and $\partial f_{\mathbf{K}}/\partial t$ by using the collective description of Bohm and one of the authors,[2] in which collective coordinates for the plasmons are specified explicitly. The Hamiltonian which describes, in the RPA, the plasmons of momentum $q<q_c$, electrons, and their mutual interaction, is

$$H=\sum_{\mathbf{K}} E_{\mathbf{K}}C_{\mathbf{K}}^{\dagger}C_{\mathbf{K}}+\sum_{q<q_c}\hbar\omega_{\mathbf{q}}A_{\mathbf{q}}^{\dagger}A_{\mathbf{q}} +\sum_{\mathbf{K}}\sum_{q<q_c}\left(\frac{2\pi e^2\hbar^3}{q^2\omega_{\mathbf{q}}}\right)^{\frac{1}{2}}\left(\frac{\mathbf{K}\cdot\mathbf{q}}{m}+\frac{q^2}{2m}\right) \times(A_{\mathbf{q}}+A_{-\mathbf{q}}^{\dagger})C_{\mathbf{K}+\mathbf{q}}^{\dagger}C_{\mathbf{K}}+\sum_{q<q_c}\frac{\hbar}{4\omega_{\mathbf{q}}}[\omega_p^2-\omega_{\mathbf{q}}^2] \times(A_{\mathbf{q}}^{\dagger}A_{\mathbf{q}}+A_{\mathbf{q}}A_{\mathbf{q}}^{\dagger}-A_{\mathbf{q}}A_{-\mathbf{q}}-A_{-\mathbf{q}}^{\dagger}A_{\mathbf{q}}^{\dagger}), \quad (3.1)$$

where $C_{\mathbf{K}}$, $C_{\mathbf{K}}^{\dagger}$ are the second quantized operators for the electrons and $A_{\mathbf{q}}$ and $A_{\mathbf{q}}^{\dagger}$ are those for plasmons of frequency $\omega_{\mathbf{q}}$. There are in the BP description a set of subsidiary conditions on the system wave functions; however, they will not affect the present considerations.[8]

The plasmon-electron interaction in (3.1) acts to shift the plasmon frequency from ω_p to $\omega_{\mathbf{q}}$. The resulting dispersion relation for $\omega_{\mathbf{q}}$ may be obtained by a canonical transformation [which acts to cancel the last term in (3.1) and agrees with (2.8)]. The plasmon-electron interaction also gives rise to a shift in the one-electron energies (through an effective electron interaction), and makes possible a resonant transfer of energy and momentum between the electron and plasmon systems. It is this latter effect which will be of particular concern to us here. In the long-wave limit, as we have seen, the number of electrons which are resonantly coupled to the plasmons is small, and we may use perturbation theory to calculate the resulting changes in the electron and plasmon distribution functions. Before carrying out detailed calculations, it is convenient to rewrite the plasmon-electron interaction term in (3.1) as:

$$H_{\text{int}}=\sum_{\mathbf{K}}\sum_{q<q_c} M_{\mathbf{q}}(A_{\mathbf{q}}C_{\mathbf{K}+\mathbf{q}}^{\dagger}C_{\mathbf{K}}+A_{\mathbf{q}}^{\dagger}C_{\mathbf{K}}^{\dagger}C_{\mathbf{K}+\mathbf{q}}), \quad (3.2)$$

where

$$M_{\mathbf{q}}=\left(\frac{2\pi e^2}{q^2}\hbar\omega_{\mathbf{q}}\right)^{\frac{1}{2}}.$$

This simplification is made possible by the fact that we consider only resonant transitions.

We find, on the application of time-dependent perturbation theory, that the time-rate of change of the plasmon distribution function is given by[9]

$$\frac{\partial N_{\mathbf{q}}}{\partial t}=\sum_{\mathbf{K}}\frac{2\pi}{\hbar}|M_{\mathbf{q}}|^2\{(N_{\mathbf{q}}+1)f(\mathbf{K}+\mathbf{q})[1-f(\mathbf{K})] -N_{\mathbf{q}}f(\mathbf{K})[1-f(\mathbf{K}+\mathbf{q})]\}\delta(E_{\mathbf{K}}+\hbar\omega_{\mathbf{q}}-E_{\mathbf{K}+\mathbf{q}}), \quad (3.3)$$

where the first term in brackets arises from the emission of plasmons by the electrons, the second from plasmon absorption. It may readily be seen, upon substitution of (2.5) and (2.10), that in thermal equilibrium $\partial N_{\mathbf{q}}/\partial t=0$; this result is hardly surprising, since the use of time-dependent perturbation theory is equivalent to assuming detailed balancing, and necessarily leads to $\partial N_{\mathbf{q}}/\partial t=0$ in thermal equilibrium.

It is instructive to rewrite (3.3) as

$$\frac{\partial N_{\mathbf{q}}}{\partial t}=\sum_{\mathbf{K}}\frac{4\pi^2e^2}{q^2}\omega_{\mathbf{q}}[f(\mathbf{K}+\mathbf{q})-f(\mathbf{K})]\delta[\hbar(\omega_{\mathbf{q}}-\nu_{\mathbf{K}\mathbf{q}})]N_{\mathbf{q}} +\sum_{\mathbf{K}}\frac{4\pi^2e^2}{q^2}\omega_{\mathbf{q}}f(\mathbf{K}+\mathbf{q})[1-f(\mathbf{K})]\delta[\hbar(\omega_{\mathbf{q}}-\nu_{\mathbf{K}\mathbf{q}})]. \quad (3.4)$$

The two terms in (3.4) possess a simple physical interpretation. The first arises from induced emission and absorption of plasmons by the electrons; it may be written as $2\gamma_{\mathbf{q}}N_{\mathbf{q}}$, where $\gamma_{\mathbf{q}}$ is given by (2.9).[10] Usually $\gamma_{\mathbf{q}}$ is negative; corresponding to the damping of the plasma waves by the individual electrons. The second term is the rate of gain of plasmons due to spontaneous plasmon emission by the fast electrons in the plasma. This emission, which corresponds to the formation of a wake of plasma waves behind the electrons, is the longitudinal analog of Čerenkov radiation.[11] We see, therefore, that thermal equilibrium arises in consequence of a balance between two competing processes—

[8] The subsidiary conditions will not affect the system at zero temperature. At finite temperatures they will alter slightly the corrections (arising from electron interaction) to the one electron contribution to the free energy, specific heat, etc. For classical plasmas, such corrections are of order $e^2k_D/\kappa T$, and are negligible. For quantum plasmas, such corrections are of order r_s; they will be neglected in the present paper.

[9] At this stage the calculation is essentially identical to that required to determine the phonon-electron equilibrium in metal. See, for example, R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, New York, 1955), p. 127.

[10] The factor of 2 arises because $N_{\mathbf{q}}$ represents a probability and is therefore proportional to the square of the amplitude of the plasma waves of momentum $\mathbf{q}$. The latter changes at a rate $\gamma_{\mathbf{q}}$, so that $N_{\mathbf{q}}$ changes at a rate $2\gamma_{\mathbf{q}}$.

[11] D. Bohm and D. Pines, Phys. Rev. 85, 338 (1952).

plasmon damping and spontaneous plasmon emission—and that both effects are simply described in the above quantum mechanical description.

The time rate of change of the electron distribution function is given by

$$\frac{\partial f_{\mathbf{K}}}{\partial t}=\sum_{q<q_c}\frac{2\pi}{\hbar}|M_{\mathbf{q}}|^2(N_{\mathbf{q}}+1)\delta[E_{\mathbf{K}}+\hbar\omega_{\mathbf{q}}-E_{\mathbf{K}+\mathbf{q}}]f_{\mathbf{K}+\mathbf{q}}\times[1-f_{\mathbf{K}}]+N_{\mathbf{q}}\delta[E_{\mathbf{K}-\mathbf{q}}+\hbar\omega_{\mathbf{q}}-E_{\mathbf{K}}]f_{\mathbf{K}-\mathbf{q}}[1-f_{\mathbf{K}}]$$
$$-\sum_{q<q_c}\frac{2\pi}{\hbar}|M_{\mathbf{q}}|^2N_{\mathbf{q}}\delta[E_{\mathbf{K}}+\hbar\omega_{\mathbf{q}}-E_{\mathbf{K}+\mathbf{q}}]f_{\mathbf{K}}[1-f_{\mathbf{K}+\mathbf{q}}]+(N_{\mathbf{q}}+1)\delta[E_{\mathbf{K}}-\hbar\omega_{\mathbf{q}}-E_{\mathbf{K}-\mathbf{q}}]f_{\mathbf{K}}[1-f_{\mathbf{K}-\mathbf{q}}]. \quad (3.5)$$

The first two terms in (3.5) represent the gain of electrons in state **K** as a result of emission and absorption of plasmons; the last two, the loss of electrons from this state due to plasmon absorption and emission. Again, we can easily show upon substitution of (2.5) and (2.10) that $\partial f_{\mathbf{K}}/\partial t=0$ in thermal equilibrium.

It is convenient to group the terms in (3.5), so that one obtains

$$\frac{\partial f_{\mathbf{K}}}{\partial t}=\sum_{q<q_c}\frac{4\pi e^2}{q^2}\frac{\omega_{\mathbf{q}}}{\hbar}N_{\mathbf{q}}[f(\mathbf{K}+\mathbf{q})-f(\mathbf{K})]\times\delta[\omega_{\mathbf{q}}-\hbar(\mathbf{K}\cdot\mathbf{q}/m+q^2/2m)]+[f(\mathbf{K}-\mathbf{q})-f(\mathbf{K})]\delta[\omega_{\mathbf{q}}-\hbar(\mathbf{K}\cdot\mathbf{q}/m-q^2/2m)]$$
$$+\sum_{q<q_c}\frac{4\pi e^2}{q^2}\frac{\omega_{\mathbf{q}}}{\hbar}\{f_{\mathbf{K}+\mathbf{q}}(1-f_{\mathbf{K}})\delta[\omega_{\mathbf{q}}-\hbar(\mathbf{K}\cdot\mathbf{q}/m+q^2/2m)]+f_{\mathbf{K}}(1-f_{\mathbf{K}-\mathbf{q}})\delta[\omega_{\mathbf{q}}-\hbar(\mathbf{K}\cdot\mathbf{q}/m-q^2/2m)]\}. \quad (3.6)$$

Again, we remark on the balance in thermal equilibrium between the rate of gain of electrons in state **K** due to plasmon damping and the rate of loss of electrons from this state due to spontaneous plasmon emission.

It is interesting to go to the classical limit of (3.6). This may be done with the aid of the following relations:

$$\hbar\mathbf{K}/m\rightarrow\mathbf{v}:\quad f(\mathbf{K})\rightarrow f(\mathbf{v}),$$
$$f(\mathbf{K}+\mathbf{q})\rightarrow f(\mathbf{v}+\hbar\mathbf{q}/m)=f(\mathbf{v})+(\hbar/m)\mathbf{q}\cdot\nabla_v f(\mathbf{v})+\cdots, \quad (3.7)$$
$$\delta[\omega_{\mathbf{q}}-\hbar(\mathbf{K}\cdot\mathbf{q}/m+q^2/2m)]\rightarrow\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{V})+(\hbar/2m)\mathbf{q}\cdot\nabla_v\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})+\cdots.$$

One then finds, after a certain amount of algebra,

$$\frac{\partial f(\mathbf{v})}{\partial t}=\sum_{q<q_c}\frac{4\pi e^2}{mq^2}\left[\frac{\mathcal{E}_{\mathbf{q}}}{m}\mathbf{q}\cdot\nabla_v\{\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})\mathbf{q}\cdot\nabla_v f\}+\omega_{\mathbf{q}}\mathbf{q}\cdot\nabla_v\{f(\mathbf{v})\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})\}\right], \quad (3.8)$$

where we have introduced the energy in the qth plasma mode, $\mathcal{E}_{\mathbf{q}}$, and

$$\mathcal{E}_{\mathbf{q}}=N_{\mathbf{q}}\hbar\omega_{\mathbf{q}}.$$

Equation (3.8) is in agreement with the kinetic equation result of Klimontovitch.[3] The first of the terms in (3.8) may be calculated directly from the nonlinear terms in the classical collisionless Boltzmann equation[12]; the second term associated with spontaneous plasmon emission appears in a classical treatment only when one goes to one higher order in the equations of motion of the coupled distribution functions, the Fokker-Planck equation. We see that both terms appear on the same basis in our quantum description utilizing collective coordinates.

The classical limit of (3.4) is obtained by considering the time rate of change of $\mathcal{E}_{\mathbf{q}}$, the energy in the qth plasmon mode. One finds then

$$\frac{\partial\mathcal{E}_{\mathbf{q}}}{\partial t}=2\gamma_{\mathbf{q}}\mathcal{E}_{\mathbf{q}}+\int d\mathbf{v}\,\frac{4\pi e^2}{q^2}\omega_{\mathbf{q}}^2 f(\mathbf{v})\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v}), \quad (3.9)$$

where the first term represents, usually, the damping of plasma waves, while the second gives the appropriate classical rate of plasma wave excitation by the electrons due to spontaneous plasmon emission.

We should like to emphasize that (3.4) and (3.6) or (3.8) and (3.9), do not possess an equilibrium solution for an arbitrary distribution of electron momenta. As is well known, when the momentum distribution of the electrons possesses two humps, there is the possibility of a solution for (2.9) in which the sign of $\gamma_{\mathbf{q}}$ is positive, corresponding to a growing plasma wave. The growing-wave solution corresponds to a coherent excitation of plasma oscillations by the electrons, in contrast to the incoherent excitation represented by, say, the second term on the right-hand side of (3.9). Under these circumstances, both terms in (3.4) and (3.9) or (3.6) and (3.8) will possess the same sign, and there is, of course, no possibility of equilibrium if one considers only these equations. What will happen is that the plasma waves will grow in amplitude until limited by nonlinear effects in the set of coupled equations under consideration, combined with a nonlinear coupling between the plasmons of different wave vector, a coupling which lies outside the framework of the RPA.[13]

IV

It is straightforward to extend the discussion given above to a plasma formed by two sets of particles having masses m and M, and charges $-e$ and e. In addition to the high-frequency "optical" plasma oscillations in which the charges move out of phase there may exist, under certain conditions, low-frequency acoustic plasma oscillations in which the electrons

[12] W. E. Drummond and D. Pines, Proceedings of the Salzburg Conference on Plasma Physics, 1961 (to be published).

[13] For a discussion of this problem for classical plasmas, see reference 12.

move in phase with the positive (heavy) charges in such a two-component plasma. The complex frequency, $\Omega_{\mathbf{q}}$ of these collective oscillations may be determined as a function of the wave vector q by requiring that the total dielectric constant vanish. Thus one has, in the RPA,[14]

$$\epsilon(\mathbf{q},\Omega_{\mathbf{q}})=1+4\pi\alpha(\mathbf{q},\Omega_{\mathbf{q}})+4\pi\alpha_{+}(\mathbf{q},\Omega_{\mathbf{q}})=0, \quad (4.1)$$

where $4\pi\alpha_{+}$ is the polarizability associated with the free positive charges, and is given by an expression similar to (2.4);

$$4\pi\alpha_{+}(\mathbf{q},\mathbf{\Omega})=-\frac{4\pi e^2}{q^2}\sum_{\mathbf{K}}[f_{+}(\mathbf{K})-f_{+}(\mathbf{K}+\mathbf{q})]\times\left[\frac{1}{\hbar\Omega-E_{+}(\mathbf{K}+\mathbf{q})+E_{+}(\mathbf{K})+i\eta}\right]. \quad (4.2)$$

Here $f_{+}(\mathbf{K})$ is probability of finding the positive charge in state $\mathbf{K}$, with energy, $E_{+}(\mathbf{K})=\hbar^2K^2/2M$; for positive charges at a temperature T_{+}, it is simply

$$f_{+}(\mathbf{K})=1/\{\exp[(E_{+}(\mathbf{K})-\mu_{+})/kT_{+}]+1\},$$

where μ_{+} is the chemical potential appropriate to the positive charges.

For both classical and quantum two-component plasmas, the positive charge polarizability, $4\pi\alpha_{+}(\mathbf{q},\omega)$, reduces in the high-frequency limit to

$$4\pi\alpha_{+}(\mathbf{q},\omega)\cong-\Omega_p^2/\omega^2+O(q^2\langle v^2\rangle_{\text{av}}/\omega^2), \quad (4.3)$$

where

$$\Omega_p=(4\pi Ne^2/M)^{\frac{1}{2}}$$

is the positive charge plasma frequency, and $\langle v^2\rangle_{\text{av}}$ is their average squared velocity. Closer investigation shows that under the circumstances the acoustic sound waves are well-defined elementary excitations the expansion (4.3) is justified in the determination of their frequency and it suffices to keep the leading term. For example, in the continuum model of a metal, for which the ions are treated as a continuous fluid, the ions possess a polarizability $-\Omega_p^2/\omega^2$. For a classical plasma of electrons and ions, or a quantum plasma of electrons and holes in a semi-metal, the sound wave is well defined only when the effective positive charge temperature, T_{+}, is small compared to that for the electrons, T; in this case the expansion (4.3) is likewise justified.[15] We shall accordingly use (4.3) in this section.

The dispersion relation for the sound waves is then, according to (4.1) and (4.3),

$$\Omega_q=\omega_a(\mathbf{q})+i\gamma_a(\mathbf{q}), \quad (4.4)$$

where

$$\omega_a^2(\mathbf{q})=\frac{\Omega_p^2}{1+4\pi\alpha(\mathbf{q},\omega_a)}\cong\frac{\Omega_p^2}{\epsilon(\mathbf{q},0)}, \quad (4.5)$$

and

$$\frac{\gamma_a}{\omega_a}=\frac{1}{\epsilon(\mathbf{q},0)}\frac{2\pi^2e^2}{q^2}\sum_{\mathbf{K}}\{f(\mathbf{K})-f(\mathbf{K}+\mathbf{q})\}\times\delta[\hbar\omega_a(\mathbf{q})-E(\mathbf{K}+\mathbf{q})+E(\mathbf{K})]. \quad (4.6)$$

In (4.5), $\epsilon(\mathbf{q},0)$ is the static electronic dielectric constant, which takes on the following values in the quantum and classical domains:

Quantum domain:

$$\epsilon(\mathbf{q},0)\cong1+k_{\text{FT}}^2/q^2=1+3\omega_p^2/q^2V_F^2. \quad (4.7\text{a})$$

$(T\rightarrow 0)$.

Classical domain:

$$\epsilon(\mathbf{q},0)\cong1+k_D^2/q^2=1+4\pi ne^2/q^2kT. \quad (4.7\text{b})$$

With the aid of these expressions, and (4.6), one obtains the following long-wavelength ($q\ll q_D$ or q_{FT}) results for ω_a and γ_a [15]:

Quantum domain:

$$\omega_a=qV_F(m/3M)^{\frac{1}{2}}, \quad (\gamma_a/\omega_a)=-(\pi/4)(m/3M)^{\frac{1}{2}}. \quad (4.8\text{a})$$

Classical domain:

$$\omega_a=q(kT/M)^{\frac{1}{2}}, \quad \gamma_a/\omega_a=-(\pi m/8M)^{\frac{1}{2}}. \quad (4.8\text{b})$$

The reader may remark that the results (4.8a) and (4.8b) show that the use of $\epsilon(\mathbf{q},0)$ in (4.5) and (4.6) is justified; he should also bear in mind the fact that we have assumed the Landau damping of the sound waves by the positive charges is negligible, a condition tantamount to asserting that $T\gg T_{+}$.

We now write the Boltzmann equations for the coupled electron-phonon (acoustic plasmon) system.[16] As we have done for the plasmons, we introduce an explicit set of collective coordinates, and a corresponding distribution function for the phonons. The necessary additional terms which describe the phonons and their interaction with the electrons may be obtained directly from the work of Bardeen and Pines,[17] and are

$$H'=\sum_{q<q_a}b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}\hbar\omega_a(\mathbf{q})+\sum_{q<q_a}m_{\mathbf{q}}(b_{\mathbf{q}}C_{\mathbf{K}+\mathbf{q}}^{\dagger}C_{\mathbf{K}}+b_{\mathbf{q}}^{\dagger}C_{\mathbf{K}}^{\dagger}C_{\mathbf{K}+\mathbf{q}}), \quad (4.9)$$

where $b_{\mathbf{q}}$ and $b_{\mathbf{q}}^{\dagger}$ are the phonon annihilation and

[14] For a discussion of this dispersion relation as applied to sound wave frequencies in metals, see D. Pines in *The Many-Body Problem* (John Wiley & Sons, Inc., New York, 1959), p. 517; as applied to electrons and holes in semiconductors or semimetals, see D. Pines and J. R. Schrieffer, Phys. Rev. **124**, 1387 (1961); as applied to electrons and ions in a classical plasma, see Klimontovitch and Silin[1] and I. Bernstein, Phys. Rev. **109**, 10 (1958).

[15] It is meaningful to consider a meta-equilibrium condition in which $T\gg T_{+}$ provided the time required for the positive charges to come to equilibrium with the electrons is long compared to any of the times of physical interest. We assume that to be the case in this section.

[16] We do not consider here the phonon-plasmon coupling, which is negligible, on the ion-phonon coupling, which is small for $T_{+}\ll T$. The latter may simply be included by writing the analogous Boltzmann equation for the ions.

[17] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).

creation operators. The matrix element for scattering of a phonon by an electron is

$$m_{\mathbf{q}}=\left[\frac{2\pi e^2}{q^2}\frac{\hbar\omega_a(\mathbf{q})}{\epsilon(\mathbf{q},0)}\right]^{\frac{1}{2}}, \tag{4.10}$$

where $\omega_a(\mathbf{q})$ is the phonon frequency defined by (4.5). In (4.9), $\mathbf{q}_a$ denotes the maximum wave vector for which the phonons are a well-defined elementary excitation. For example, for a classical plasma of electrons and ions for which $T\gg T_+$, one finds

$$q_a\cong(4\pi ne^2/kT_+)^{\frac{1}{2}}.$$

The rate of change of the phonon distribution function, $n_{\mathbf{q}}$, is then

$$dn_{\mathbf{q}}/dt=2\gamma_a(\mathbf{q})n_{\mathbf{q}}+\sum_{\mathbf{K}}\frac{4\pi e^2}{q^2\epsilon(\mathbf{q},0)}\omega_a(\mathbf{q})f(\mathbf{K}+\mathbf{q})$$
$$\times[1-f(\mathbf{K})]\delta[\hbar\omega_a(\mathbf{q})-(\hbar^2/m)(\mathbf{K}\cdot\mathbf{q}+q^2/2)], \tag{4.11}$$

where we have made use of (4.6). It is easy to show that $f_{\mathbf{K}}{}^0$ and

$$n_{\mathbf{q}}{}^0=1/[\exp(\hbar\omega_a(\mathbf{q})/kT)-1] \tag{4.12}$$

provide the equilibrium solution of (4.11). Thus the acoustic plasmons do not reflect the positive charge temperature T_+ under the present physical conditions, and, instead, come to thermal equilibrium with the electron temperature T.

There will also be an additional term in the Boltzmann equation for the electron distribution function due to the electron-phonon interaction. This is

$$\left(\frac{\partial f_{\mathbf{K}}}{\partial t}\right)_{\text{phonon}}=\sum_{q<q_a}\frac{4\pi e^2}{q^2\epsilon(\mathbf{q},0)}\frac{\omega_a(\mathbf{q})n_{\mathbf{q}}}{\hbar}$$
$$\times\{[f(\mathbf{K}+\mathbf{q})-f(\mathbf{K})]\delta[\omega_a-\hbar(\mathbf{K}\cdot\mathbf{q}/m+q^2/2m)]$$
$$+[f(\mathbf{K}-\mathbf{q})-f(\mathbf{K})]\delta[\omega_a-\hbar(\mathbf{K}\cdot\mathbf{q}/m-q^2/2m)]\}$$
$$+\sum_{q<q_a}\frac{4\pi e^2}{q^2\epsilon(\mathbf{q},0)}\frac{\omega_a(\mathbf{q})}{\hbar}$$
$$\times\{f_{\mathbf{K}+\mathbf{q}}(1-f_{\mathbf{K}})\delta[\omega_a-\hbar(\mathbf{K}\cdot\mathbf{q}/m+q^2/2m)]$$
$$+f_{\mathbf{K}}(1-f_{\mathbf{K}-\mathbf{q}})\delta[\omega_a-\hbar(\mathbf{K}\cdot\mathbf{q}/m-q^2/2m)]\}. \tag{4.13}$$

The distribution functions (4.12) and (2.5) may likewise be shown to provide an equilibrium solution of (4.13). We see that for the phonons equilibrium arises as a balance between phonon damping and spontaneous phonon emission by the electrons; for the electrons it comes as a balance between spontaneous phonon emission and the induced emission and absorption of phonons.

For the two-component plasma one may likewise encounter a two-stream instability, corresponding to a positive $\gamma_a(\mathbf{q})$. Under present circumstances, the instability may be attributed directly to coherent excitation of phonons. Thus if one considers the electrons to move with a drift velocity, v_d, corresponding to a distribution function,

$$f^0(\mathbf{K},\mathbf{v}_d)=\left[\exp\left(\frac{[(\hbar\mathbf{K}-m\mathbf{v}_d)/2m]^2-\mu}{kT}\right)+1\right]^{-1},$$

the threshold for the instability is reached when $\mathbf{q}\cdot\mathbf{v}_d=\omega_a(\mathbf{q})$; for drift velocities in excess of this value, coherent phonon excitation occurs, and the mechanism by which equilibrium comes about is very much altered. Such an effect is probably of principal importance for classical plasmas.[18] We therefore quote the relevant expressions for the equations which couple $f(\mathbf{v})$ and $\mathcal{E}_a(\mathbf{q})$, the energy in the $\mathbf{q}$th phonon mode, in the classical limit. They are:

$$\left[\frac{\partial f(\mathbf{v})}{\partial t}\right]_{\text{phonon}}=\sum_{q<q_a}\frac{4\pi e^2}{mq^2\epsilon(\mathbf{q},0)}$$
$$\times\left[\frac{\mathcal{E}_a(\mathbf{q})}{m}\mathbf{q}\cdot\nabla_v\{\mathbf{q}\cdot\nabla_v f(v)\delta(\omega_a-\mathbf{q}\cdot\mathbf{v})\}\right.$$
$$\left.+\omega_a\mathbf{q}\cdot\nabla_v\{f(\mathbf{v})\delta(\omega_a-\mathbf{q}\cdot\mathbf{v})\}\right], \tag{4.14}$$

and

$$\frac{\partial\mathcal{E}_a(\mathbf{q})}{\partial t}=2\gamma_a(\mathbf{q})\mathcal{E}_a(\mathbf{q})$$
$$+\int d^3v\frac{4\pi e^2\omega_a{}^2}{q^2\epsilon(\mathbf{q},0)}f(\mathbf{v})\delta(\omega_a-\mathbf{q}\cdot\mathbf{v}). \tag{4.15}$$

V

The coupled equations for the electron and plasmon (or electron and phonon) distribution functions provide a simple physical picture of the way equilibrium is established between the single-particle modes and the collective excitations in finite temperature plasmas. They also serve to demonstrate clearly the relationship between growing waves in plasmas and the spontaneous Čerenkov excitation of plasmons; the former correspond to coherent excitation of plasmons (or phonons) by the electrons, the latter to incoherent plasmon or phonon excitation.

The use of collective coordinates plus time-dependent perturbation theory and the random phase approximation has led to a particularly simple derivation of the relevant classical equation: the collision terms in the Fokker-Planck equation which explicitly derive from the collective modes, together with the equations which describe the time dependence of the energy stored in the collective modes. These equations may be useful in studying the conductivity of a classical plasma for which $T\gg T_+$ (the ion temperature) under the action

[18] It may, for example, explain the phenomena of "pump-out" observed in the Princeton stellerator. See I. Bernstein, E. Frieman, R. Kulsrud, and M. Rosenbluth, Phys. Fluids 3, 136 (1960).

of weak external fields[19]; they may also be of use in determining the relaxation rates and effect of strong electric fields under circumstances that growing plasma waves play a role.[20] In addition to these collision terms, there will also be terms associated with collisions between individual electrons; in fact, this latter class of terms comprise those usually considered in calculations of plasma relaxation times and conductivity.[21] It is likely that for many problems both kinds of collision terms will be of importance.

We may mention some solid-state applications of the coupled equations for electrons, plasmons, and phonons. The calculation of the resistance of a metal, in the continuum model, follows directly from (4.13); in this model, sound wave attenuation is determined from (4.8a), while the approach to equilibrium between phonons and electrons is determined by (4.11) and (4.13).[22] These equations may likewise prove useful for the determination of electron-electron or electron-hole scattering in semiconductors and semi-metals.

Finally, we should like to emphasize that all of the foregoing considerations are based on the use of the random phase approximation. For classical plasmas near thermal equilibrium, the approximation is justified; its validity depends on $e^2k_D/kT\ll 1$, a condition that is well satisfied. For states well away from equilibrium, it is difficult to assess the region of its validity; thus far, studies of problems in this area have not fully exploited the RPA, much less assessed its accuracy. For quantum plasmas, the RPA is valid for the long-wavelength phenomena here considered, but cannot be relied on in a description of short-range electron-electron collisions.[23]

APPENDIX A. QUANTUM MECHANICAL DISPERSION RELATION

The dispersion relation giving the complex frequency of the collective modes is best obtained within RPA by solving as an initial value problem the equation of motion for the one-particle density matrix in the absence of external fields. This approach, which follows that used by Landau[6] to treat the dynamics of a classical plasma has the advantage over the more conventional dielectric constant approach that damped as well as growing oscillations may be treated. Strictly speaking the dielectric constant method can only describe the driven response of the system to an external field; the free decay of the plasma oscillations follows only if one makes the assumption that $\epsilon(\mathbf{q},\omega)$ should be analytically continued into the lower half of the ω plane. We give below an extension of the initial value problem approach to a one-component quantum plasma, and thereby justify the results obtained by analytic continuation.

The one-particle density matrix is given in terms of creation and annihilation operators $C_{\mathbf{K}}^\dagger$ and $C_{\mathbf{K}}$ for particles of momentum K by

$$\rho_{\mathbf{K}}(\mathbf{q})=C_{\mathbf{K}}^\dagger C_{\mathbf{K}+\mathbf{q}}. \tag{A.1}$$

Within the random phase approximation it satisfies

$$-\frac{i\partial}{\partial t}\rho_{\mathbf{K}}(\mathbf{q})=[H,\rho_{\mathbf{K}}(\mathbf{q})]/\hbar=\nu_{\mathbf{K}\mathbf{q}}\rho_{\mathbf{K}}(\mathbf{q}) +V_q[N_{\mathbf{K}+\mathbf{q}}-N_{\mathbf{K}}/\hbar]\sum_{\mathbf{K}'}\rho_{\mathbf{K}'}(\mathbf{q}), \tag{A.2}$$

where $N_{\mathbf{K}}=C_{\mathbf{K}}^\dagger C_{\mathbf{K}}$ and $V_q=4\pi e^2/q^2$. Taking matrix elements with respect to RPA eigenstates $|\alpha\rangle$ and $|\beta\rangle$ of H, (A.2) becomes

$$\left[-i\frac{\partial}{\partial t}+\nu_{\mathbf{K}}(\mathbf{q})\right]R_{\mathbf{K}}(\mathbf{q},t) =V_q[f_{\mathbf{K}+\mathbf{q}}-f_{\mathbf{K}}/\hbar]\sum_{\mathbf{K}'}R_{\mathbf{K}'}(\mathbf{q},t), \tag{A.3}$$

where

$$R_{\mathbf{K}}(\mathbf{q},t)=\langle\beta|\rho_{\mathbf{K}}(\mathbf{q},t)|\alpha\rangle, \tag{A.4}$$

and

$$f_{\mathbf{K}}=\langle\beta|N_{\mathbf{K}}|\beta\rangle. \tag{A.5}$$

To treat the field-free motion of the system given the boundary values $R_{\mathbf{K}}(\mathbf{q},0)$ at $t=0$, we take the one-sided Fourier transform of (A.3) with respect to time:

$$\bar{R}_{\mathbf{K}}(\mathbf{q},\omega)[\nu_{\mathbf{K}\mathbf{q}}-\omega] =V_q(f_{\mathbf{K}+\mathbf{q}}-f_{\mathbf{K}})\sum_{\mathbf{K}'}\bar{R}_{\mathbf{K}'}(\mathbf{q},\omega)+iR_{\mathbf{K}}(\mathbf{q},0), \tag{A.6}$$

$$\bar{R}_{\mathbf{K}}(\mathbf{q},\omega)=\int_0^\alpha e^{i\omega t}R_{\mathbf{K}}(\mathbf{q},t)dt, \tag{A.7}$$

where ω is defined to be in the upper half-plane. Solving for $\bar{R}_{\mathbf{K}}(\mathbf{q},\omega)$ and summing on $\mathbf{K}$, one obtains

$$\sum_{\mathbf{K}'}\bar{R}_{\mathbf{K}'}(\mathbf{q},\omega)=i\sum_{\mathbf{K}'}\frac{R_{\mathbf{K}'}(\mathbf{q},0)}{\nu_{\mathbf{K}'\mathbf{q}}-\omega}H(\mathbf{q},\omega). \tag{A.8}$$

The response kernal $H(q,\omega)$ is given by

$$H(\mathbf{q},\omega)=1+V_q\sum_{\mathbf{K}}\frac{(f_{\mathbf{K}}-f_{\mathbf{K}+\mathbf{q}})}{\nu_{\mathbf{K}\mathbf{q}}-\omega}. \tag{A.9}$$

[19] H. W. Wyld (private communication) has independently derived (4.14) and has used it to determine the plasma conductivity.

[20] Studies of relaxation rates based on (3.8) and (3.9) have been carried out in reference 12.

[21] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956). For a derivation of these terms within an approximation equivalent to the RPA, see M. Rosenbluth and N. Rostoker, Phys. Fluids **3**, 1 (1960) and R. Balescu, *ibid.*, 52 (1960). A simple quantum derivation based explicitly on RPA has recently been shown to lead directly to the Rosenbluth, Rostoker, and Balescu result (D. Pines and H. W. Wyld, to be published).

[22] D. Bohm and T. Staver, Phys. Rev. **84**, 836 (1952); T. Staver, Ph.D. thesis, Princeton University, 1952 (unpublished).

[23] P. Nozières and D. Pines, Phys. Rev. **111**, 442 (1958).

From Poisson's equation it follows that the Laplace transform of the electric field $E(\mathbf{q},\omega)$ is given by

$$i\mathbf{q}E(\mathbf{q},\omega)=4\pi e\sum_{\mathbf{K}'}\bar{R}_{\mathbf{K}'}(\mathbf{q},\omega), \tag{A.10}$$

so that the electric field is given as a function of time by

$$E(\mathbf{q},t)=\frac{4\pi eq}{q^2}\int_C\left\{\sum_{\mathbf{K}'}\frac{R_{\mathbf{K}'}(\mathbf{q},0)}{\nu_{\mathbf{K}'\mathbf{q}}-\omega}\bigg/H(\mathbf{q},\omega)\right\}e^{-i\omega t}\frac{d\omega}{2\pi}. \tag{A.11}$$

It is well known from Fourier transform theory that the contour C is to be taken along a line from $-\infty$ to $+\infty$ above any singularities in the integrand.[24] For $t>0$, it is possible to close the contour by a semi-circle of infinite radius in the lower half-plane. The essential point in the derivation is the analytic continuation of the integrand into the lower half plane so that the Cauchy theorem may be used to evaluate the integral. In general, poles of the integrand in (A.11) come from both the numerator and denominator. However, the poles which are characteristic of the collective modes of the system rather than the specific form of the initial disturbance are given by the zeros of $H(\mathbf{q},\omega)$. The zeros of the analytic continuation of $H(\mathbf{q},\omega)$ into the lower half plane, give the frequency and damping rate of the damped plasma oscillations. If $H(\mathbf{q},\omega)$ has zeros in the upper half plane, the plasma oscillations undergo unstable growth. As Landau has pointed out, the analytic continuation of $H(\mathbf{q},\omega)$ may be accomplished by replacing the sum in (A.9) by an integral and suitably deforming the resulting contour. For small damping rate, this result is equivalent to the familiar prescription of replacing the resonance denominator by

$$\frac{1}{\nu_{\mathbf{K}\mathbf{q}}-\omega}=\frac{P}{\nu_{\mathbf{K}\mathbf{q}}-\omega}+i\pi\delta(\nu_{\mathbf{K}\mathbf{q}}-\omega).$$

[24] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1958), p. 468.

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Magnetoacoustic Measurements in Silver at 230 Mc/sec and 4.2°K*

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Measurements of acoustic attenuation in silver have been made with 150–233 Mc/sec longitudinal sound waves in magnetic fields up to 15 000 oersteds and at a temperature of 4.2°K. Plots of the ultrasonic pulse height as a function of the reciprocal of the magnetic field strength show from ten to fifteen maxima and minima for several orientations. Numerical data are presented and discussed in some detail. The Pippard type theoretical model of the silver Fermi surface is compared with our measurements.

INTRODUCTION

In recent years, the oscillatory behavior of ultrasonic attenuation as a function of applied magnetic field has been reported for several metals.[1] Recent theoretical discussions[2–4] of the interaction between sound waves and electrons in very pure metals subject to a magnetic field, indicate that the observed periods of oscillation are directly related to the Fermi surface dimensions. Some preliminary magnetoacoustic data taken on silver were reported by one of us at the International Conference on the Fermi Surfaces in Metals held at Cooperstown, New York, in August, 1960.[5] Since that time, our data accuracy has been improved through development of more refined experimental techniques and the construction of higher frequency equipment. The presentation and discussion of these more recent data is the purpose of this paper.

EXPERIMENTAL TECHNIQUES

Ultrasonic pulse techniques similar to those described by Morse[6] were used. For the frequency range up to 190 Mc/sec, a commercially available[7] combination rf transmitter, receiver, and cathode-ray oscilloscope unit was used. For the frequency range from 200 to 250 Mc/sec, a new transmitter-receiver system has been constructed in our laboratory. In order to obtain more efficient transmission of power to the sample, we have found it useful to incorporate a tuned circuit into the sample mount (see Fig. 1). The sample mount, including the tuning elements, is suspended in a glass Dewar system and immersed in liquid helium. Final tuning is

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[1] Extensive discussion of recent work as well as a complete set of references may be found in *The Fermi Surface*, edited by W. A. Harrison and M. B. Webb (John Wiley & Sons, Inc., New York, 1960).

[2] A. B. Pippard, Proc. Roy. Soc. (London) **A257**, 165 (1960).

[3] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

[4] T. Kjeldaas and T. Holstein, Phys. Rev. Letters **2**, 340 (1959).

[5] H. V. Bohm, reference 1, p. 245.

[6] R. W. Morse, *Progress in Cryogenics*, edited by K. Mendelssohn (Haywood and Company, Ltd., London, 1959), Vol. I.

[7] "Ultrasonic Attenuation Comparator" manufactured by Sperry Products, Inc., Danbury, Connecticut.