Isoscalar Nucleon Structure*

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The three-pion contribution to the isoscalar nucleon structure is examined in dispersion theory. We assume that the amplitudes (photon → three pions) and (three pions → nucleon pair) are characterized by the pions interacting in pairs. This interaction is taken to be the low energy $J=T=1$ resonance, which consequently dominates the picture. The effects of a three-pion interaction are included so as to satisfy an extended unitarity condition. The lowest mass singularities, including the complex ones, are discussed, interpreted, and included. If the π-π resonance is at $10\mu^2$, then a reasonable radius can be easily obtained. On the other hand, if the resonance is at $20\mu^2$ or higher, then a strong intrinsic three-pion resonance or a bound state seems to be needed for agreement with experiment.

I. INTRODUCTION

RECENT advances, both experimental and theoretical, have brought the problem of the electromagnetic structure of the nucleon into sharp focus. The continuing work of Hofstadter *et al.*[1] and the recent work of Wilson *et al.*[2] have clarified the charge and magnetic moment form factors of both the proton and the neutron. The first attempts to apply dispersion theory to certain aspects of this problem were carried out by Chew, Karplus, Gasiorowicz, and Zachariasen,[3] and by Federbush, Goldberger, and Treiman[4] in their classic paper. A qualitative, if not quantitative, understanding of the isotopic vector nucleon properties has been achieved in the work of Frazer and Fulco[5] and improved by the normalization procedure of Ball and Wong.[6] These works may even prove to be valid if the much heralded but elusive low energy pion-pion resonance should be found experimentally to possess reasonable parameters.[7] We may add that this resonance is also significant for interpreting other experimental facts. For instance, Bowcock, Cottingham, and Lurié[8] have shown that the small phase shifts in pion-nucleon elastic scattering can be better understood if such a resonance is present.

Our understanding of the isotopic scalar structure of the nucleon has been much less satisfactory. The lowest mass state contributing to the isoscalar structure consists of three pions. There has been much discussion of this state[9] and of its experimental ramifications.[10] A perturbation calculation of the isoscalar structure was carried out by Hiida, Nakanishi, and collaborators,[11] and Bosco and De Alfaro[12] carried out a calculation based on a static model with a cutoff. However, the significance of these results is very much in doubt if a low energy pion-pion resonance is present.

A different dispersion theoretic treatment of nucleon structure was suggested by Bincer,[13] who used a nucleon mass rather than the photon mass for the dispersion variable. In this case the lowest mass state is the pion-nucleon system, and the small pion-nucleon phase shifts contribute to the isoscalar structure. The effects of the pion-pion resonance are therefore indirectly included.[8] However, this approach has two potential disadvantages. The first lies in the fact that these dispersion relations are significant only as long as subtractions are not introduced. In particular, one must assume that the amplitudes vanish at infinity. The second disadvantage is related to the customary neglect of higher mass intermediate states. These states involve the (3-3) resonance as well as the 2π and the 3π resonances, and therefore may be both prohibitive to calculate and crucial. Numerical calculations based on this approach were attempted by Kawarabayashi and Machida.[14]

We would like to present a treatment of the three-pion contribution to these rather mysterious isotopic scalar properties (i.e., with reference to the usual dispersion relations). Our treatment will include the

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[1] R. Hofstadter, C. deVries, and R. Herman, Phys. Rev. Letters **6**, 290 (1961); R. Hofstadter and R. Herman, *ibid.* **6**, 293 (1961).
[2] D. N. Olson, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters **6**, 286 (1961).
[3] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958).
[4] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).
[5] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).
[6] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **6**, 29 (1961).
[7] J. A. Anderson, V. X. Bang, P. G. Burke, D. D. Carmony, and N. Schmitz, Phys. Rev. Letters **6**, 365 (1961); A. R. Erwin, R. March, W. D. Walker, and E. West, *ibid.* **6**, 628 (1961).
[8] J. Bowcock, W. N. Cottingham, and D. Lurié, Phys. Rev. Letters **5**, 386 (1960); Nuovo cimento **16**, 918 (1960) and **19**, 142 (1961).
[9] See for example G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).
[10] Y. Nambu, Phys. Rev. **106**, 1366 (1957).
[11] K. Hiida and N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **22**, 863 (1959); further references are given in this paper.
[12] B. Bosco and V. De Alfaro, Phys. Rev. **115**, 215 (1959).
[13] A. M. Bincer, Phys. Rev. **118**, 855 (1960). We thank Professor Bincer for bringing this paper and the paper of reference 14 to our attention.
[14] K. Kawarabayashi and S. Machida, Progr. Theoret. Phys. (Kyoto) **25**, 17 (1961).

effects of the pion-pion resonance directly and in a natural way. However, we will be forced to make a number of uncontrollable approximations.

Our objectives are very modest. We will try to understand the structure of the form factors in terms of a low-energy pion-pion scattering resonance, a direct three-pion force, and the singularities of the annihilation amplitude. Now although the three-pion force is at least in part a manifestation of the two-pion interaction, we are unable to estimate its magnitude in a reliable manner, and the effect of singularities is likewise difficult to estimate. We will therefore introduce parameters to describe the strength of these effects. Because of our approximations and the arbitrary parameters we are forced to introduce, a subtraction will be performed so that the correct value of the static charge and the static magnetic moment is assured.

For a treatment of the three-pion contribution to the form factors, one needs the amplitudes (photon $\to$ three pions) and (three pions $\to$ nucleon pair). In discussing these, the approach to the problem of constructing unitary inelastic amplitudes which was proposed recently by one of us[15] will be used. It must be stressed that we cannot at the present time give a systematic discussion of a three-particle state. This does not prohibit us from making a model of this state and from forming physically reasonable approximations based on this picture. Such models are in fact constructed and discussed in A.

Our procedure has been to start with one of these models, and to modify it by including the effect of the interaction $3\pi\to 3\pi$ and the effect of singularities[16] of the annihilation amplitude. Of course, both of these modifications likewise depend on simple models. The interaction $3\pi\to 3\pi$ has been constructed according to A, and the treatment of singularities is based in part on perturbation theory.

With these warnings, let us proceed by stating the problem in more definite terms. We are interested in the isotopic scalar part of the nucleon electromagnetic current:

$$J_\mu{}^s=\left(\frac{\bar{P}_0P_0}{M^2}\right)^{\frac{1}{2}}\langle 0|\,j_\mu{}^s|\,\bar{P},P;\,\text{in}\rangle \tag{1.1}$$

$$=-\bar{v}(\bar{P})[G_1{}^s(t)i\gamma_\mu+G_2{}^s(t)(\bar{P}-P)_\mu]u(P), \tag{1.2}$$

where $t=-(\bar{P}+P)^2$ is the momentum transfer. The superscript s (for scalar) will be dropped from now on. The usual dispersion representation will be assumed for G_1 and G_2 as functions of t, and their absorptive parts are to be determined from

$$A_\mu=-\pi\left(\frac{P_0}{M}\right)^{\frac{1}{2}}\sum_s\langle 0|\,j_\mu|\,s\rangle\bar{v}(\bar{P})\langle s|\,f|\,P\rangle \times\delta(P_s-\bar{P}-P). \tag{1.3}$$

Our calculation will be restricted to the three-pion intermediate states. If there is a true bound state[10] of the three-pion system, then there will be a delta function contribution to A_μ. We will assume that no such state exists for the rest of this calculation. If such a state is found experimentally, it can be constructed formally by an analytic continuation in the three-pion interaction parameters.[17]

The two production amplitudes which must be constructed are then

$$F_\mu{}^{\alpha\beta\gamma}=\langle 0|\,j_\mu|\,\alpha k_1\beta k_2\gamma k_3;\,\text{in}\rangle(8\omega_1\omega_2\omega_3)^{\frac{1}{2}}, \tag{1.4}$$

and

$$M_{\alpha\beta\gamma}=(8\omega_1\omega_2\omega_3)^{\frac{1}{2}}\bar{v}(\bar{P})\langle\alpha k_1\beta k_2\gamma k_3;\,\text{out}|\,f|\,P\rangle \times(P_0/M)^{\frac{1}{2}}. \tag{1.5}$$

Instead of explicitly constructing these amplitudes by the methods of A, we will simply write down our solution for these functions and will try to make them reasonable. The function F will be discussed in Sec. II together with its physical interpretation in terms of graphs and unitarity. The production amplitude M is constructed in Sec. III by using F as a guide, and a brief mathematical and physical analysis of its analytic properties is given. In constructing production amplitudes by dispersion methods, the problem of treating the necessarily present complex singularities[16] must be faced. We will present an argument, based on perturbation theory, that these singularities can be interpreted in simple physical terms. This is turn leads to a natural mode of (approximate) treatment. In Sec. IV the rather involved phase space integral is carried out in detail. The numerical results and conclusions are presented in Sec. V for the pion-pion resonance at a total center of mass energy squared of 10 and 20 measured in units of the pion mass. Let us now turn to the physics of the problem.

II. ELECTROPRODUCTION OF THREE PIONS

Electroproduction of three pions is illustrated in Fig. 1. This process has been discussed in some detail in A,

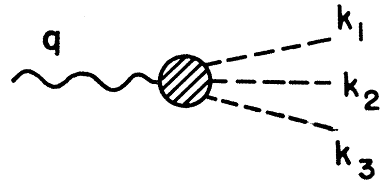

FIG. 1. Electroproduction of three pions.

[15] R. Blankenbecler, Phys. Rev. **122**, 983 (1961). Hereafter to be called A.

[16] L. F. Cook, Jr., and J. Tarski, J. Math. Phys. (to be published); Y. S. Kim, Phys. Rev. Letters **6**, 313 (1961); P. V. Landshoff and S. B. Treiman, Nuovo cimento **19**, 1249 (1961).

[17] Such a construction would be a natural generalization of the method described by R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

so that only a short review will be given here. We will, however, include the effect of a direct $3\pi \to 3\pi$ scattering, which was assumed negligible in the treatment of A. The variables are defined by

$$t = -q^2,\quad s_{ij} = -(k_i + k_j)^2, \tag{2.1}$$

and they satisfy the condition

$$s_{12} + s_{23} + s_{31} = 3 + t, \tag{2.2}$$

where the pion mass has been taken to be unity, and t is the mass of the virtual photon. The general structure of the matrix element is[4]

$$F_\mu{}^{\alpha\beta\gamma} = i\epsilon_{\mu\nu\lambda\sigma} k_1{}^\nu k_2{}^\lambda k_3{}^\sigma \epsilon^{\alpha\beta\gamma} F(s_{12}, s_{23}, s_{31}; t), \tag{2.3}$$

where α, β, and γ are the isotopic labels of the pions, and F is a symmetric function of the three variables s_{ij}.

Following the procedures of A, the function F is taken to be

$$F(s_{12}, s_{23}, s_{31}; t) = F_0 D^{-1}(t) \exp[\Delta_{12} + \Delta_{23} + \Delta_{31}], \tag{2.4}$$

where

$$D(t) = 1 - \frac{t}{\pi} \int_9^\infty \frac{dt'}{t'(t'-t-i\epsilon)} \times \sum\nolimits_{3'} |\exp[\Delta_{12}' + \Delta_{23}' + \Delta_{31}']|^2 \times \sigma(s_{12}', s_{23}', s_{31}'; t')$$
$$\Delta_{ij} = \frac{1}{\pi} \int_4^\infty ds' \delta(s')(s' - s_{ij} - i\epsilon)^{-1}, \tag{2.5}$$

and F_0 is a constant. The phase shift $\delta(s)$ is the $T = J = 1$ phase shift for pion-pion scattering. The function $D(t)$ serves to sum the connected three-pion diagrams, and will be discussed fully below.

The form $F_0 \exp(\sum \Delta_{ij})$ was given in A, and was compared there with the works of Gourdin and Martin[18] and of Wong[19] for the case of photoproduction. The factors $\exp(\Delta_{ij})$ sum in an approximate manner the disconnected two-particle graphs.[20] A further interpretation of the product $\exp(\sum \Delta_{ij})$ comes from potential scattering, where such an expression would be interpreted as a product wave function for the three-pion system. Such product wave functions are in common use in nuclear physics[21] and in the statistical model.[22]

The form (2.5) of the function $D(t)$ is suggested by the formalism and by the examples of A. This form can be interpreted with the help of the unitarity condition. As we shall see below, Eqs. (2.4)–(2.5) lead to a reasonable approximation for the amplitude $3\pi \to 3\pi$.

For simplicity we shall not attempt to construct here any field-theoretic models which might yield the form (2.4), but there do exist solvable models in which the relevant amplitudes have a structure resembling (2.4). In these models, however, not all of the possible interactions are allowed, so that some of the factors $\exp(\Delta_{ij})$ are missing. One such model is the Lee model, for which Amado[20] has calculated the relevant production amplitude. Another example, which includes recoil effects, has been presented in A.

The form (2.4) neglects the effect of singularities other than the two-pion and three-pion normal thresholds. This may seem unjustified, since it follows from the work of Barton and Kacser[23] that there are other singularities near the relevant region (i.e., where the pions can have real momenta). These authors argue that the effect of these singularities is negligible for the process $K \to 3\pi$, and we shall make an analogous assumption. This might be justified by the fact that the amplitude is finite rather than infinite at the singularities in question. On the other hand, we will see in the next section how the complex vertex singularities of the amplitude $N\bar{N} \to 3\pi$ can be interpreted and their effects included.

Let us return to the function $D(t)$ in (2.5). The symbol $\sum_{3'}$ stands for an integral over the phase space available for the three intermediate pions, with momenta k_1', k_2', k_3', at a fixed center of mass momentum t' (see A). The function σ is the "potential" and includes the effects of the detailed structure of the interaction $3\pi \to 3\pi$. We will be led to a candidate for σ, and to an expression for the amplitude $3\pi \to 3\pi$, when we examine the interpretation of our functions in terms of unitarity.

These remarks about D may not be superfluous: First, D has been subtracted, so that $D(0) = 1$, and therefore F_0 measures the strength of the photo-process. Second, D sums the connected three-pion diagrams, as we stated previously. Therefore, if there is a strong three-pion interaction, it will manifest itself in the t dependence of D. Third, if there is a three-pion bound state of mass t_0, then presumably $D(t_0)$ is zero. We shall return to this possibility in the last section.

Let us now examine the interpretation of Eqs. (2.4)–(2.5) in more definite terms. For the independent variables of F we may select t, s_{23}, and $z = \cos\theta$, where θ is the angle between the relative momentum of the two-three pair ($2Q$) and the momentum of pion one (k_1), in the center of mass of the two-three pair. Many other sets of independent variables could be used just as well in the following discussion. The invariants s_{12} and s_{31} can then be expressed as follows:

$$s_{12} = \tfrac{1}{2}(t - s_{23} + 3) + 2|\mathbf{k}_1||\mathbf{Q}|z,\quad s_{31} = \tfrac{1}{2}(t - s_{23} + 3) - 2|\mathbf{k}_1||\mathbf{Q}|z, \tag{2.6}$$

[18] M. Gourdin and A. Martin, Nuovo cimento **16**, 78 (1960).
[19] H. S. Wong, Phys. Rev. Letters **5**, 548 (1960).
[20] R. D. Amado, Phys. Rev. **122**, 696 (1961). It is not difficult to show that Amado's expression for the amplitude $\langle N\theta\theta/V\theta\rangle$ can be put into a form analogous to (2.4).
[21] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. V.
[22] See, for instance, L. F. Cook, Jr., and J. V. Lepore, Phys. Rev. **120**, 1028 (1960).
[23] G. Barton and C. Kacser, Nuovo cimento (to be published).

where

$$Q^2=\tfrac{1}{4}s_{23}-1,$$
$$4s_{23}\mathbf{k}_1^2=[t-(s_{23}^{\frac{1}{2}}+1)^2][t-(s_{23}^{\frac{1}{2}}-1)^2].$$

The unitarity relation which would be naturally expected in the physical region is

$$F_\mu(t+i\epsilon,\, s_{23}+i\epsilon',\, z)-F_\mu(t-i\epsilon,\, s_{23}+i\epsilon',\, z) =2iA_\mu(t,\, s_{23}+i\epsilon',\, z), \quad (2.7)$$

where

$$A_\mu=\sum_{3'}\langle 0|\,j_\mu|3\pi';\,\mathrm{out}\rangle\langle 3\pi';\,\mathrm{out}|J_1|k_2k_3;\,\mathrm{in}\rangle \quad (2.8)$$
$$=\sum_{3'} F_\mu^*(t,\, s_{23}+i\epsilon',\, z')M_{33}(s_{23}'+i\epsilon',\, z',\,\cdots s_{23}+i\epsilon,\, z,\, t). \quad (2.9)$$

Unitarity relations of this kind can be inferred by analytic continuation from space-like values of all the variables except t. Such relations were used in the development of A.

The next step is to carry out the explicit calculation of A_μ from the expression (2.4) and to cast this result into the form (2.9). We will, however, ignore the spin factors and consider only scalar functions. We will denote the limits $\mathrm{Im}t\rightarrow 0\pm$ by the superscripts $\pm$, respectively. The limit $\mathrm{Im}s_{23}\rightarrow 0+$ is to be understood.

We first assert that

$$\exp(\Delta_{jk})^-=\exp(\Delta_{jk})^*,\quad (jk)=(12)\quad\text{or}\quad(31).$$

This is not immediately obvious from (2.6), but can be easily established as follows. In continuing from $t+i\epsilon$ to $t-i\epsilon$ with a fixed s_{23}, we must decrease t until it is away from its cut; in particular, we must avoid the cut which corresponds to $s_{jk}\geqq 4$. Then the continuation to $t-i\epsilon$ will lead us to the other side of this cut. But this is the only cut for the function $\exp(\Delta_{jk})$, and our assertion follows.

Next, let us consider the expression $F^+D^+-F^-D^-$. We note that $D^-=D^*$ and that $FD=F_0\exp(\sum\Delta_{ij})$, and we obtain

$$F^+D^+-F^-D^-=2i[F^+\,\mathrm{Im}D-AD^-]$$
$$=F_0\exp[\Delta_{12}+\Delta_{23}+\Delta_{31}]^*\exp[2i\delta_{23}]\times\{\exp[2i(\delta_{12}+\delta_{31})]-1\}.$$

These equations yield

$$A=(2i)^{-1}F^*(t,s_{23},z)\exp[2i\delta_{23}]\{\exp[2i(\delta_{12}+\delta_{31})]-1\}$$
$$+F^+(D^-)^{-1}\sum_{3'}|\exp[\Delta_{12}'+\Delta_{23}'+\Delta_{31}']|^2\times\sigma(s_{12}',s_{23}',s_{31}';\,t')$$
$$\equiv A^{(0)}+A^{(1)}$$
$$\equiv\sum_{3'}F^*M_{33}^{(0)}+\sum_{3'}F^*M_{33}^{(1)}.$$

We have now, in effect, integral equations for $M_{33}^{(0)}$ and $M_{33}^{(1)}$. These can be easily solved if, in the case of $M_{33}^{(0)}$, one approximates certain angular integrations in such a way as to be consistent with our neglect of rescattering in higher angular momentum states. We obtain

$$M_{33}^{(0)}=(2\pi)^3\exp[2i\delta_{23}]\{f_{12}2\omega_3\delta(\mathbf{k}_3'-\mathbf{k}_3)+\exp[2i\delta_{12}]f_{31}2\omega_2\delta(\mathbf{k}_2'-\mathbf{k}_2)\},$$

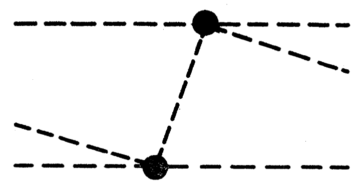

FIG. 2. One-pion exchange graph for the process $3\pi\rightarrow 3\pi$.

where

$$f_{ij}=e^{i\delta_{ij}}\sin\delta_{ij}/\rho_{ij},$$
$$\rho_{ij}=(16\pi^2)^{-1}(s_{ij}-4)^{\frac{1}{2}}(s_{ij})^{-\frac{1}{2}}.$$

Further,

$$M_{33}^{(1)}=\exp[\Delta_{12}'+\Delta_{23}'+\Delta_{31}']\sigma(s_{12}',s_{23}',s_{31}';\,t)D^{-1}(t)\times\exp[\Delta_{12}+\Delta_{23}+\Delta_{31}],$$

and

$$M_{33}=M_{33}^{(0)}+M_{33}^{(1)}.$$

This form for M_{33} has not been properly symmetrized according to the statistics of the pions, but this can be readily done. The physical interpretation of M_{33} is straightforward. The part $M_{33}^{(0)}$ has two terms which correspond to disconnected graphs involving π-π scattering. The S-matrix factors which are present, for example $\exp(2i\delta_{23})$, can be understood by comparison with the solution to the Lee model.[20] They are also just what one expects by analogy with a "two potential" situation. We find there that disconnected graphs lead to such a phase factor. One notes that there are no graphs in M_{33} that involve pion number one in a completely disconnected manner. This is, of course, required by the asymptotic condition and by the definition of M_{33}, which is implied.

The last term corresponds to connected interactions of all three pions which can, however, interact by pairs as they enter or leave. The function σ is to describe the detailed structure of the interaction. One of the simplest graphs which contributes to σ is given in Fig. 2. We may note that in the case of the Lee model, σ is in fact given in terms of graphs which are analogous to this one. This diagram describes a one-pion exchange force and thus has a rather long range. This diagram, however, cannot be directly included, since it would lead to a dependence of D on the invariants s_{ij}. We will therefore replace σ with a pole approximation so chosen that it has roughly the range of the one-pion exchange diagram. This point certainly deserves further study. We will choose

$$\sigma(t)=\Gamma s_0(t-s_0)^{-1},$$

where s_0 is expected to lie between zero and nine. The strength Γ is to be chosen to fit the experimental nucleon form factor.

III. NUCLEON PAIR ANNIHILATION

The form that the function M can take is restricted by the fact that the absorptive part of the form factors

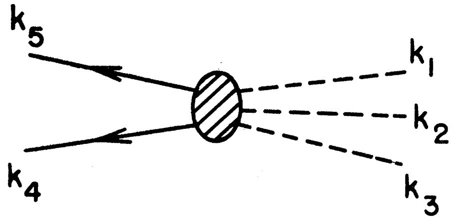

FIG. 3. Nucleon pair annihilation.

must be real. Since we are committed to the solution for F given by Eq. (2.4), it will prove both necessary and convenient to use it as a model for the construction of M. The momentum variables for the annihilation process are chosen as in Fig. 3. The scalar variables can be classified as energy-like and momentum transfer-like. The latter will also be called the crossed invariants. The energy-like invariants are s_{12}, s_{23}, s_{31}, and $s_{45}=t$. These variables enter into the problem at hand in much the same way as they enter into the electroproduction amplitude discussed in Section II.

A few words about the dependence of M on the invariants may be appropriate at this point. As is well known, five of the variables s_{ij} are independent. The invariants s_{12}, s_{23}, s_{31}, and t are linearly related by Eq. (2.2). Therefore, only three of these are independent, and two crossed invariants may be chosen for the other independent variables. We may choose, for example, s_{15} and s_{34}. Of course, the complete annihilation amplitude must be symmetric in the pions.

The matrix element that contributes to the form factor is taken to be of the form

$$M_{\alpha\beta\gamma}=i\epsilon_{\mu\nu\lambda\sigma}k_1{}^{\nu}k_2{}^{\lambda}k_3{}^{\sigma}\epsilon_{\alpha\beta\gamma}M^{\mu}, \qquad (3.1)$$

where

$$M^{\mu}=\bar{v}(k_5)[C_1 i\gamma^{\mu}+C_2(k_5-k_4)^{\mu}]u(k_4). \qquad (3.2)$$

The invariant functions C_1 and C_2 depend upon the scalar variables s_{ij}. The functions C_l are chosen to be consistent with our previous considerations, and also with the fact that the process should be dominated by the low energy pion-pion interaction:

$$C_l=d_l(s_{ij})D^{-1}(t)\exp[\Delta_{12}+\Delta_{23}+\Delta_{31}], \qquad (3.3)$$

where d_l is a real function of the scalar invariants when they are in the physical region for creation of three pions. The other functions were introduced earlier. It would seem as if we were neglecting the complex singularities[16] which must be present in any physical five-point function. This is *not* the case, and let us examine this point by discussing the analytic properties of M in more detail.

For simplicity we will assume that M can be written as a sum of terms, each of which depends on only one of the crossed invariants, but perhaps on all of the energy-like invariants. This is certainly incomplete, since even the simplest Feynman diagram for the process leads to an amplitude with the factors $(s_{15}-M^2)^{-1}(s_{34}-M^2)^{-1}$. However, this procedure can be justified to some extent. One would expect that the dependence of M on these invariants is governed primarily by the nearest singularities. We will find that, with reference to one particular term, taking into account the singularities in s_{15} affects only slightly the calculated nucleon structure. It would then be surprising if terms with both factors $(s_{15}-M^2)^{-1}(s_{34}-M^2)^{-1}$ would have a great effect. On the other hand, our assumption greatly simplifies the evaluation of the phase space integral.

Symmetry in the pion variables allows us to consider analyticity in only one of the crossed invariants, s_{15} for example. The lowest mass state which contributes is the one nucleon state. This, of course, contributes a pole at $s_{15}=M^2$. The residue of this pole can be related to the pion-nucleon coupling constant and in the present approximation on the allowed angular momentum states, to the matrix element for the process $N+\bar{N}\rightarrow 2\pi$ in the $T=J=1$ state. This is exactly the same matrix element that occurs in the isotopic vector nucleon structure and has been discussed in references (5) and (6).

The next state which contributes, and the last we will consider, is the pion-nucleon state depicted in Fig. 4. The analyticity of this diagram can most easily be discussed in perturbation theory. In doing so, all the vertices are replaced by point interactions and the two outgoing lines, s_{23}, behave like a single particle of that mass. In order to ensure that we are discussing this function on the physical sheet, it is convenient to start with the "mass" $(s_{23})^{\frac{1}{2}}$ of the order of unity and then to perform an anlytic continuation to larger masses.[24] In the stable case, we find for the contribution of this graph,

$$I=\frac{1}{\pi}\int_{(M+1)^2}^{\infty} ds'A(s',s_{23})(s'-s_{15})^{-1}.$$

As the "mass" $(s_{23}+i\epsilon)^{\frac{1}{2}}$ is increased, this function acquires an anomalous threshold which reaches the point (M^2+2) as s_{23} reaches four. Above this point, instability sets in and the threshold passes into the

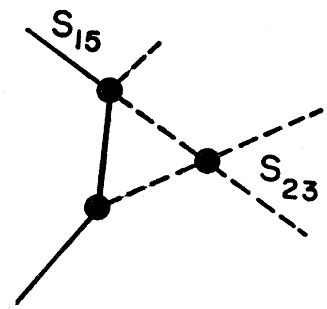

FIG. 4. Vertex graph in annihilation process.

[24] R. Blankenbecler and Y. Nambu, Nuovo cimento 18, 595 (1960); S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

complex plane to the point

$$s^*=M^2+\tfrac{1}{2}s_{23}-\tfrac{1}{2}i[s_{23}(s_{23}-4)(1-1/4M^2)]^{\frac{1}{2}}. \quad (3.4)$$

The path followed by the threshold s^* is shown in Fig. 5. The discontinuity of $I(s_{15})$ across the anomalous and complex cut is found by continuation in the standard manner.[24] It is essentially the inverse of the product of the center of mass energy and the absolute magnitude of the momentum of the intermediate nucleon:

$$\{[s-(M-s_{23}^{\frac{1}{2}})^2][(M+s_{23}^{\frac{1}{2}})^2-s]\}^{-\frac{1}{2}}. \quad (3.5)$$

The function C_l is then a pole plus a line integral from s^* to infinity. Let us evaluate this function for s_{15} off its cut, say less than (M^2-2), which is the physical region for the annihilation process. Then C_l is still complex because of the presence of the complex spike. The question is now to interpret the resultant imaginary part of C_l in physical terms. In order to do so, it is convenient to backtrack for a moment.

Consider C_l as a function of the variable s_{23}, with s_{15} real and less than (M^2-2). An ordinary dispersion relation holds, and the complex part in this case is due to the possible physical state of two pions which simply gives the function the phase of π-π scattering. The imaginary part must be identical with that due to the complex spike in s_{15} because these are just two different procedures for constructing the same function. This leads to a natural physical interpretation of the complex singularity in s_{15}. It is simply a manifestation of the fact that there is a real process possible on another leg of the vertex which, in turn, must give the function the phase of that scattering process even if the original variable, s_{15}, is real and not on its cut. It should be stressed that this argument has been carried out only to lowest order in perturbation theory.

It is also possible to interpret the complex singularity in terms of wave functions. The analogous problem in the anomalous threshold case has been fully discussed.[25] The essential point is that the wave function has an exponential fall-off which is given by the anomalous threshold or, equivalently, the binding energy, instead of the Compton wavelengths of the particles involved. The case of the complex spike may be interpreted in terms of a complex fall-off distance which in turn implies a finite lifetime for the state in question. These properties have become familiar in discussions of nuclear decay problems.[26]

We may point out that the complex spike such as in Fig. 5 is one of the two common ways by which the spectrum of the imaginary part of an amplitude arises. A well-known example of the other kind of spectrum for the imaginary part is given by the Mandelstam spectral functions. In this case the spectrum is on the real axis (or on the real plane).

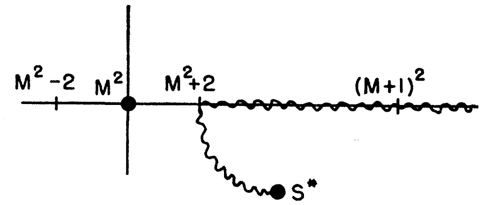

FIG. 5. Vertex singularity in annihilation amplitude.

In constructing our approximate solution for C_l, Eq. (3.3), we are taking into account the complex singularities insofar as they affect the phase of the function. For d_l, we shall use the following expression:

$$d_l=g_l(s_{15}-\bar{M}^2)^{-1}. \quad (3.6)$$

The quantity $\bar{M}^2$ is to be chosen so as to simulate the effect of the pole at $s_{15}=M^2$, and also of the real contribution of the line singularities shown in Fig. 5. We choose

$$\bar{M}^2=M^2+\tfrac{3}{2}. \quad (3.7)$$

Next, the g_l are constants which are to ensure the correct values of the static form factors. It seems difficult to relate g_1 and g_2 to more fundamental constants, since both the form (3.6) and the value (3.7) are somewhat arbitrary.

It is worthwhile remarking that if we were to choose d_l to be a constant and also $D=1$, then (3.3) achieves the form that one would expect on the basis of the statistical model.[22] The phase factors of $\exp(\Delta_{ij})$ are roughly speaking the values of the pion wave functions at the origin compared to what they would be if there were no final state interaction.

The final unsymmetrized form for the annihilation amplitude is therefore taken to be

$$M^\mu=b^\mu(s_{15}-\bar{M}^2)^{-1}D^{-1}(t)\exp[\Delta_{12}+\Delta_{23}+\Delta_{31}], \quad (3.8)$$

where

$$b^\mu=\bar{v}(k_5)[g_1 i\gamma^\mu+g_2(k_5-k_4)^\mu]u(k_4). \quad (3.9)$$

It is not necessary to symmetrize this function since it is to be inserted into a completely symmetric phase space integral. Let us now turn to this aspect of the problem.

IV. PHASE SPACE INTEGRAL

In order to evaluate the absorptive part of the form factor, the three-pion phase space integral must be evaluated. The integral to be calculated is

$$(2\pi)^6A_\mu=\pi\int d^4k_1d^4k_2d^4k_3\delta(k_1^2+1)\delta(k_2^2+1)\delta(k_3^2+1)$$
$$\times\theta(k_1^0)\theta(k_2^0)\theta(k_3^0)\delta^4(k_1+k_2+k_3-q)\epsilon_{\mu\nu\lambda\sigma}k_1^\nu k_2^\lambda k_3^\sigma$$
$$\times\epsilon_{\alpha\beta\gamma\delta}k_1^\beta k_2^\gamma k_3^\delta F^*M^\alpha(s_{12},s_{23},s_{31};t;s_{15},s_{34}). \quad (4.1)$$

[25] See, for instance, R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960), where references to earlier works are given.

[26] See, for example, E. C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), p. 192.

Here q is the total available four-momentum:

$$q=k_4+k_5,\quad -q^2=t.$$

For completeness we note that M may depend on s_{34}. This dependence will be ignored from now on, in accordance with Eq. (3.8). Now, there are several ways to proceed. For instance, Lardner[27] transformed an integral such as in (4.1) but without the factors k_j^{μ} into an integral over the invariants. Such a procedure cannot be easily adapted to our problem, and we shall proceed to carry out the successive integrations directly.

We first carry out the integrations over k_2 and k_3 by transforming to their center-of-mass system:

$$K=k_2+k_3,\quad Q=\tfrac{1}{2}(k_2-k_3).$$

We introduce an arbitrary vector a_μ, and in this system the spin factors take the form

$$a^{\mu}\epsilon_{\mu\nu\lambda\sigma}k_1{}^{\nu}k_2{}^{\lambda}k_3{}^{\sigma}=s_{23}{}^{\frac{1}{2}}(\mathbf{a}\times\mathbf{k}_1)\cdot\mathbf{Q}.$$

The integrations over K and Q now lead to a result which in the center of mass system of s_{23} can be written as follows:

$$(2\pi)^6\mathbf{A}=\frac{\pi^2}{3}\int d^4k_1\theta(k_1{}^0)\delta(k_1{}^2+1)s_{23}{}^{\frac{1}{2}}Q^3\mathbf{k}_1\times(\mathbf{B}\times\mathbf{k}_1).\quad(4.2)$$

The time component A_0 equals zero. Here

$$\mathbf{B}(s_{23},s_{15};t)=\frac{3}{4}\int_{-1}^{1}dz(1-z^2)F^*\mathbf{M}.\qquad(4.3)$$

The expressions for s_{12}, s_{31}, $\mathbf{Q}^2$, and $\mathbf{k}_1{}^2$ in terms of s_{23}, t, and z are given in Eqs. (2.6), and

$$Q^2\equiv\mathbf{Q}^2=\tfrac{1}{4}s_{23}-1.$$

Before going on, let us comment on changing frames of reference. If we do the integration over the angles of Q in a frame where $\mathbf{k}_1=(0,0,|\mathbf{k}_1|)$, then we get a result which is proportional to $\mathbf{k}_1{}^2(a_xB_x+a_yB_y)$. We can write this result in the following form, which has a manifest three-dimensional invariance:

$$\mathbf{a}\cdot\mathbf{B}(\mathbf{k}_1{}^2)-(\mathbf{a}\cdot\mathbf{k}_1)(\mathbf{B}\cdot\mathbf{k}_1).$$

This expression gives the triple cross product occurring in (4.2). Next, it is convenient to carry out the integration over k_1 in the center of mass system of q, and for this reason we write the triple cross product in the Lorentz invariant way

$$s_{23}{}^{-1}\epsilon_{\mu\nu\lambda\sigma}(q-k_1)^{\nu}k_1{}^{\lambda}\epsilon^{\sigma\kappa\rho\eta}(q-k_1)_{\kappa}B_{\rho}(k_1)_{\eta}.$$

We see at once that

$$[\mathbf{k}_1\times(\mathbf{B}\times\mathbf{k}_1)]_{\text{c.m.}(23)}=ts_{23}{}^{-1}[\mathbf{k}_1\times(\mathbf{B}\times\mathbf{k}_1)]_{\text{c.m.}(q)}.$$

Let us also list some useful relations for the center of mass system of q:

$$\mathbf{k}_4{}^2=\mathbf{k}_5{}^2=\tfrac{1}{4}t-M^2.$$

$$\begin{aligned}\kappa^2=\mathbf{k}_1{}^2&=[t-(s_{23}{}^{\frac{1}{2}}+1)^2][t-(s_{23}{}^{\frac{1}{2}}-1)^2]/4t\\&=(\mathbf{k}_1{}^2)_{\text{c.m.}(23)}(s_{23}/t).\qquad(4.4)\\ \bar{M}^2-s_{15}&=M^2+\tfrac{3}{2}+(k_1-k_5)^2\\&=\tfrac{1}{2}(t-s_{23}+2)-2|\mathbf{k}_5|\,|\mathbf{k}_1|x,\end{aligned}$$

where x is the cosine of the angle between $\mathbf{k}_5$ and $\mathbf{k}_1$, in c.m.(q).

To complete our work, we insert M as given by Eqs. (3.6)–(3.7), and we change the integral over $k_1{}^0$ to one over s_{23}. The absorptive part A_μ can be brought easily into the form given by Eq. (1.2). The imaginary parts of the form factors can now be read off; the results, with the over-all constant factors omitted, are as follows:

$$\text{Im}G_1(t)=g_1\int ds_{23}\,\kappa_1{}^3Q^3(t/s_{23})^{\frac{1}{2}}H_1(s_{23},t)\frac{\bar{F}(s_{23},t)}{|D(t)|^2}\qquad(4.5)$$

$$\text{Im}G_2(t)=\int ds_{23}\,\kappa_1{}^3Q^3(t/s_{23})^{\frac{1}{2}}\bar{F}(s_{23},t)\,|D(t)|^{-2}\times\left[g_2(H_1-H_2)-g_1H_2\frac{M}{2}(\tfrac{1}{4}t-M^2)^{-1}\right],\quad(4.6)$$

where

$$H_1=\frac{3}{8}\int_{-1}^{1}dx\,(1+x^2)(s_{15}-\bar{M}^2)^{-1},$$

$$H_2=\frac{3}{8}\int_{-1}^{1}dx\,(3x^2-1)(s_{15}-\bar{M}^2)^{-1},$$

$$\bar{F}=\frac{3}{4}\int_{-1}^{1}dz(1-z^2)\,|\exp[\Delta_{12}+\Delta_{23}+\Delta_{31}]|^2.$$

The integrals over s_{23} are from 4 to $(t^{\frac{1}{2}}-1)^2$. In evaluating the integral over $s_{12}(z)$ and $s_{31}(z)$ in $\bar{F}$, it is convenient to use the fact that the integrand is a symmetric function of z, and hence can be rewritten as an integral from zero to one. Finally, in order to avoid a double numerical integral, we have simply evaluated the integrand at the midpoint, $z=\frac{1}{2}$. This does not introduce a very large error.

It may be worthwhile to comment on the integrands in Eqs. (4.5)–(4.6). These integrands vanish at the endpoints, due to phase space factors. For small t, each of these integrands has one maximum, which represents the effect of one excited resonance. For large t, there are two maxima, one of which is the effect of the resonance in s_{23}, and the other, of the resonance in s_{12} or s_{31}.

In order to simplify the presentation of the numerical evaluation of the form factors, let us write Eqs. (4.5)-

[27] R. W. Lardner, Nuovo cimento **19**, 77 (1961).

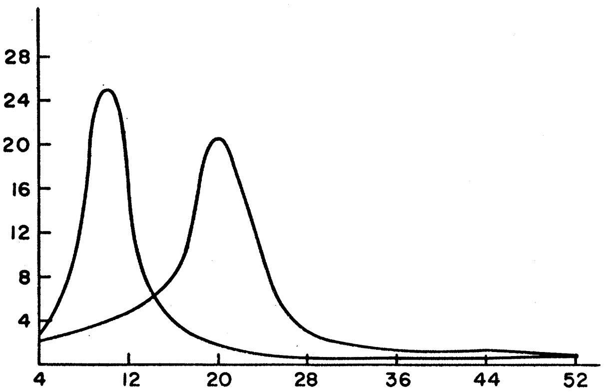

FIG. 6. Pion-pion phase functions, $|e^{\Delta}|^2$.

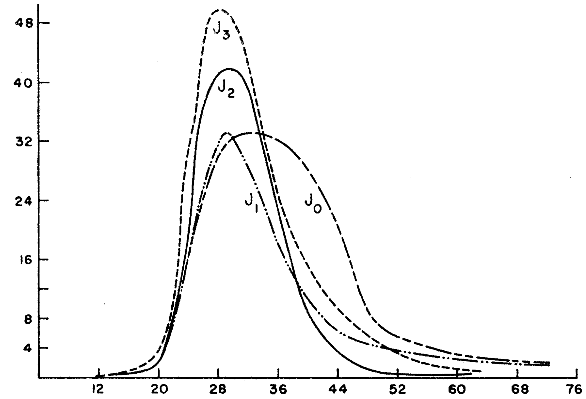

FIG. 7. Phase space integrals for $t_r=10$. Each curve has been arbitrarily normalized.

(4.6) in the following way:

$$\mathrm{Im}G_1(t)=g_1J_1(t)/|D(t)|^2, \tag{4.7}$$

$$\mathrm{Im}G_2(t)=g_2J_2(t)/|D(t)|^2+g_1J_3(t)/|D(t)|^2. \tag{4.8}$$

We note that

$$J_3(t)=[M/2(\tfrac{1}{4}t-M^2)][J_2(t)-J_1(t)].$$

We also note that if we should choose $d_1=\text{const}$, cf. Eqs. (3.3) and (3.6), then we would obtain $J_3=0$. This conclusion is of course expected, from general invariance considerations, if the annihilation amplitude M does not depend on the crossed invariants.

The functions J_1, J_2, and J_3 will be discussed in the next section. Further, the function $D(t)$ is given by the integral

$$D(t)=1-\frac{t}{\pi}\int_9^\infty \frac{dt'\sigma(t')}{t'(t'-t-i\epsilon)}J_0(t'), \tag{4.9}$$

where

$$J_0(t)=\int ds_{23}\kappa_1{}^3Q^3(t/s_{23})^{\frac{1}{2}}\bar{F}(s_{23},t). \tag{4.10}$$

These two functions will also be presented in the next section.

V. NUMERICAL RESULTS AND CONCLUSIONS

As stated in the introduction, we have carried out the form factor calculations for two values of the pion-pion scattering resonance, $t_r=10$ and 20. The precise functions assumed for the resonant phase shift integrals $|e^{\Delta}|^2$ are shown in Fig. 6. Before explaining the numerical results in detail, let us examine the form factors in the case of a delta function resonance. If we set

$$|e^{\Delta}|^2=a\delta(s-t_r), \tag{5.1}$$

then we find, with the normalization $G_l(0)=1$,

$$G_1(t)=G_2(t)=\frac{3(t_r-1)}{3(t_r-1)-t}. \tag{5.2}$$

The mean square radius is

$$\langle r^2\rangle=2/(t_r-1). \tag{5.3}$$

For $t_r=10$, or 20, the root mean square radius is 0.67f, or 0.48f, respectively. It will be possible to increase these radii by giving the resonance a finite width and by appealing to a 3π-3π interaction. The finite width also allows the singularities in s_{15} to shift the effective resonance position to lower energies.

Let us now turn to the details of a more realistic calculation. It is instructive to compare the form factors which follow from different approximations for the two relevant amplitudes. The simplest approximation which will be discussed is the statistical model and corresponds to setting $D=1$, $d_l=1$, so that $H_1=1$ and $H_2=0$. Then, except for over-all constant factors, we find

$$\mathrm{Im}G_1=\mathrm{Im}G_2=J_0(t).$$

The function $J_0(t)$ is shown in Fig. 7 for $t_r=10$. The radius turns out to be 0.61f for $t_r=10$ and 0.4f for $t_r=20$. Either case is an improvement over perturbation theory.[11]

Let us now set $d_l(s)=g_l(s-\bar{M}^2)^{-1}$, as in Eq. (3.6), but keep $D(t)=1$. We then have to deal with the three integrals J_1, J_2, and J_3. These functions are plotted in Fig. 7. Since they are so similar in shape, we will focus our attention on J_1 only, and we predict

$$G_1(t)=G_2(t). \tag{5.4}$$

If one wants to make an explicit subtraction, to compensate for the crudeness of our calculation, this statement is changed to

$$\begin{aligned}G_1(t)&=(1-a)+aG(t),\\ G_2(t)&=(1-b)+bG(t).\end{aligned} \tag{5.5}$$

The parameters a and b may be interpreted as measuring the effect of higher mass and hence uncalculable states on the form factors. The first equation, (5.4), is in disagreement with the results of references (1) and (2). However, the values of $G_2(t)$ given by these respective

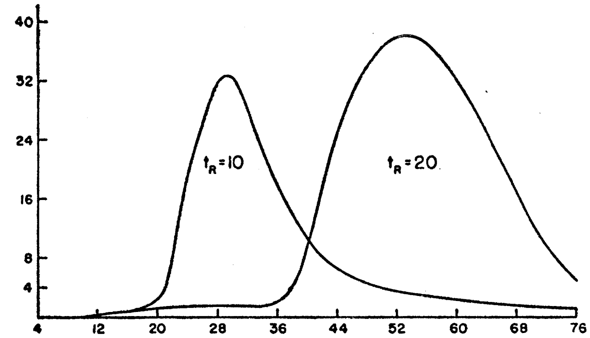

FIG. 8. The functions J_1 for $t_r=10$ and 20.

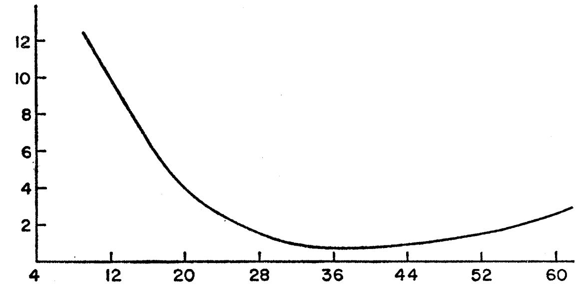

FIG. 9. Three-pion rescattering function, $|D(t)|^{-2}$.

authors are in striking disagreement, and our choice $G_2(t)=G_1(t)$ can be considered as a compromise. Obviously further work is needed.

The functions $J_1(t)$ are plotted in Fig. 8 for $t_r=10$ and 20. The radii turn out to be 0.66f for $t_r=10$ and 0.48f for $t_r=20$.

Finally, let us consider the effect of a $3\pi\rightarrow 3\pi$ interaction. For this, we use Eq. (2.5) for $D(t)$ and Eq. (2.11) for $\sigma(t)$. The integral to be evaluated here is, except for $\sigma(t)$, an integral similar to that giving the form factors. A numerical evaluation, however, is tedious, since the integral in $D(t)$ is to be evaluated where the integrand is singular. We therefore approximated $J_0(t)$ by a resonance-type formula and carried out the integration analytically. A plot of $|D(t)|^{-2}$ for $s_0=8$, $\Gamma=-\frac{1}{4}$, and $t_r=10$ is shown in Fig. 9. The final absorptive part, $\mathrm{Im}G_1(t)$, including the effect of this type of $|D(t)|^{-2}$, yields a root mean square radius of 0.75f for $t_r=10$ and of 0.55f for $t_r=20$.

The plot of Fig. 9 suggests that $D(t)$, or its analytic continuation through the physical cut $D^{II}(t)$, has a zero for some $t=t_0<9$, and this point deserves a brief comment. If $D(t_0)=0$, then there is a three-pion bound state, and its effect has to be included as a delta function in $\mathrm{Im}G_1(t)$. However, if $D^{II}(t_0)=0$, then there is a virtual bound state[17] and its effect need not be included explicitly. It can be quite effective in increasing the radius, however.

Let us conclude with a discussion of the form factors for $t_r=20$. The radius cannot be increased much by adjusting the parameter s_0. Increasing $|\Gamma|$ would correspond to a stronger $3\pi\rightarrow 3\pi$ interaction, but it is not at all clear whether this would affect the radius greatly. Since there is some experimental preference[7] for a resonance above 20, we would like to point out two modifications which would lead to a reasonable value of the radius in this case. The most drastic involves a three-pion bound state.[9] Its effects need not be discussed further. The second modification involves using a rescattering function $D^{-1}(t)$ which shows a strong resonance in the vicinity of the π-π resonance position. Such a resonance, if it exists, should be seen experimentally.[28]

To summarize, we have seen that if the π-π resonance were in the vicinity of 10, the scalar radius could be fitted without any spectacular $3\pi\rightarrow 3\pi$ scattering effects. On the other hand, since the resonance is higher than 20, it is perhaps impossible to fit the radius unless a three-pion resonance or a bound state is present.

At this point, let us mention only two of the many shortcomings of this work: First, no attempt was made to explain at the same time the smallness of the scalar magnetic moment and the largeness of the radius. Second, no attempt was made to calculate the function $D(t)$ in terms of an assumed π-π scattering without introducing new parameters. Our results, however, specifically point to the importance of such a calculation. These two difficult points certainly deserve further study.

ACKNOWLEDGMENT

One of us (J.T.) wishes to thank Professor J. R. Oppenheimer for his hospitality at the Institute for Advanced Study.

[28] A search for three-pion correlations in the process $p\bar{p}\rightarrow 5\pi$ has been carried out by B. Maglić, L. Alvarez, A. Rosenfeld, and M. Stevenson, Phys. Rev. Letters **7**, 178 (1961). Their results indicate a narrow 3π resonance at $t=32$. We thank the authors for informing us of their work before publication.