Particle Theory Approach to the Two-Pion and Three-Pion Systems*

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The relation between the two-pion and three-pion resonances is discussed in terms of a model in which the motion of the pions is described by a partially relativistic Schrödinger-type wave equation, and the interaction between them is represented by a static potential. An attractive square-well potential that is almost strong enough to bind the di-pion gives a satisfactory account of the observed P-wave pion-pion scattering. If this potential is assumed to be additive between all pairs in the three-pion $I=0$, $J=1^-$ state, it produces far too much binding to agree with observation. As an alternative to additivity, the interaction may be assumed to saturate, so that the total potential never exceeds that between any pair of pions. It is found that the model provides a qualitatively consistent explanation both of pion-pion scattering and the three-pion system if the pair interactions saturate or nearly saturate.

I. INTRODUCTION

THE possible existence of a neutral vector meson was suggested several years ago by Nambu[1] as an explanation for the apparent absence of neutron charge structure at that time.[2] More recently, Chew[3] pointed out that a bound or resonant three-pion state with $I=0$, $J=1^-$ would serve the same purpose. Furthermore, he suggested that since such a system can be formed with each pair of pions in a relative $I=1$, $J=1^-$ state, for which a resonant interaction is expected,[4] the three-pion and two-pion resonances should have comparable total energies. Subsequent[5] experimental work on the nucleon electromagnetic form factors,[6] and on their theoretical interpretation,[7] has confirmed this view. Quite recently, more direct experimental evidence has been found for the two-pion[8,9] and three-pion[10] resonances. The first of these almost certainly has $I=1$ and is consistent with $J=1$, in agreement with the prediction of Frazer and Fulco.[4] The three-pion resonance almost certainly has $I=0$ and probably has $J=1^-$. The two-pion resonance has a total energy of about 750 Mev and a full width at half maximum of about[9] 150 Mev; the corresponding parameters for the three-pion resonance are 790 Mev and 30 Mev, respectively.[10]

The only quantitative theoretical attempt at correlating the two-pion and three-pion resonances made thus far is based on dispersion theory.[11] As an alternative approach, we assume in the present paper that the two-pion resonance arises from a potential, and ask what effect this potential has on the three-pion system. The use of potentials in field theory must be regarded as an approximation at relativistic energies,[12] although it does possess a considerable intuitive appeal.[13] The potential model used here differs from field theory in three respects: (1) the number of particles is fixed at two or three pions, as the case may be, with no nucleon pairs; (2) a partially relativistic Schrödinger-type wave equation is used to describe the motion of these particles; (3) the interaction between particles is represented by a static potential. The first assumption appears in somewhat weaker form in all calculations based on dispersion theory, where only the lowest mass states are included; no further attempt will be made to justify it here. In the development of point (2), the wave function is assumed to depend on the average time of the several particles, not on the time differences. This means that the center of mass of the system moves relativistically, but retardation effects are neglected in the internal motion. Since the use of a potential also implies that retardation is neglected, this is the principal physical difference between the present work and a dispersion theory treatment deriving from field theory.[11] The only justification for this neglect, apart from the simplicity and intuitive appeal of the resulting calculation, is that retardation should be less important in a strongly resonant or bound system, in which the particles are close together most of the time, than in nonresonant scattering.

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Our procedure consists first in setting up a plausible wave equation (Sec. II). With the added assumption of an attractive short-range interaction, which for convenience is chosen to be of square-well form, the P-wave pion-pion scattering is calculated and the potential depth and range determined to agree with observation[9] (Sec. III). An upper limit on the energy of the three-pion ground state is then obtained by means of the variational method, assuming that it is bound (Sec. IV). Since the energy estimate obtained in this way turns out to be a very rapidly-varying function of the square-well parameters, it cannot be determined reliably. By the same token, the square-well parameters are determined quite precisely even if the three-pion energy is allowed to vary considerably around zero binding. The calculations are carried through with two forms of variational trial function and with two assumptions concerning the additivity of the potentials between different pairs.

II. WAVE EQUATION

A single free pion satisfies the Klein-Gordon or Schrödinger relativistic equation,

$$(-\nabla^2+m^2)\phi=-\partial^2\phi/\partial t^2, \tag{1}$$

where $\phi=\phi(\mathbf{r},t)$, m is the pion rest mass, and units are chosen such that $\hbar=c=1$. For two noninteracting pions, omitting isotopic spin dependence, an unsymmetrized wave function may be written

$$\psi(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\phi_1(\mathbf{r}_1,t_1)\phi_2(\mathbf{r}_2,t_2),$$

where each of the ϕ's satisfies an equation of the form (1). Then ψ evidently satisfies the wave equation

$$(-\nabla_1^2-\nabla_2^2+2m^2)\psi=(-\partial^2/\partial t_1^2-\partial^2/\partial t_2^2)\psi. \tag{2}$$

It is convenient to rewrite Eq. (2) in terms of the average time $t=\frac{1}{2}(t_1+t_2)$, the relative time $\tau=t_1-t_2$, the center-of-mass coordinate $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, and the relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$:

$$(-\tfrac{1}{2}\nabla_R^2-2\nabla_r^2+2m^2)\psi=(-\tfrac{1}{2}\partial^2/\partial t^2-2\partial^2/\partial\tau^2)\psi.$$

We now assume that the dependence of ψ on τ can be neglected. We also allow the pions to interact through a static potential $V_2(\mathbf{r})$, which is added to the rest-mass term (scalar potential) rather than to the total energy term $i\partial/\partial t$ (fourth component of a four-vector). retardation is neglected, and the wave equation becomes

$$[-\tfrac{1}{2}\nabla_R^2-2\nabla_r^2+V_2(\mathbf{r})+2m^2]\psi=-\tfrac{1}{2}\partial^2\psi/\partial t^2. \tag{3}$$

The dependence of ψ on $\mathbf{R}$ and t can be separated out by writing

$$\psi=\chi(\mathbf{r})\,\exp i(\mathbf{P}\cdot\mathbf{R}-Et),$$

where E and $\mathbf{P}$ are naturally interpreted as the total energy and momentum of the two-pion system. Equation (3) then becomes

$$[-2\nabla_r^2+V_2(\mathbf{r})+2m^2]\chi=\tfrac{1}{2}(E^2-\mathbf{P}^2)\chi.$$

This shows that the center of mass moves relativistically, like a particle with rest mass $M_2=(E^2-\mathbf{P})^{\frac{1}{2}}$; thus M_2 is the total internal energy of the system. The wave equation for the internal motion is conveniently written

$$[-2\nabla_r^2+V_2(\mathbf{r})]\chi=(\tfrac{1}{2}M_2^2-2m^2)\chi, \tag{4}$$

and is of Schrödinger type.

The foregoing procedure is readily extended to any number of pions. For the three-pion system, a convenient set of coordinates is $\mathbf{R}=\frac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)$, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, $\boldsymbol{\varrho}=\mathbf{r}_3-\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. We again assume that ψ depends only on the average time $t=\frac{1}{3}(t_1+t_2+t_3)$, separate out a factor $\exp i(\mathbf{P}\cdot\mathbf{R}-Et)$, and define the total internal energy $M_3=(E^2-\mathbf{P}^2)^{\frac{1}{2}}$. The wave equation for the internal motion, analogous to (4), is then

$$[-2\nabla_r^2-(\tfrac{3}{2})\nabla_\rho^2+V_3(\mathbf{r},\boldsymbol{\varrho})]\chi(\mathbf{r},\boldsymbol{\varrho})=(\tfrac{1}{3}M_3^2-3m^2)\chi(\mathbf{r},\boldsymbol{\varrho}). \tag{5}$$

We postpone a discussion of the relation between the two-pion potential $V_2(\mathbf{r})$ and the three-pion potential $V_3(\mathbf{r},\boldsymbol{\varrho})$ until Sec. IV.

The neglect of the dependence of ψ on the relative times can be looked at in the noninteracting forms of Eqs. (4) and (5). If we set $V_2=0$ in (4), the resulting plane-wave solution yields the correct energy M_2 of the internal motion. On the other hand, when V_3 is set equal to zero in (5), M_3 has the correct value in the nonrelativistic limit, but not for high relative energies unless all three momenta have equal magnitudes. Thus neglect of the relative time dependence causes a kinematical error when more than two noninteracting pions are present. However, this, as well as the neglect of retardation mentioned in Sec. I, should be less important in a strongly resonant or bound system of three pions, in which the momenta are expected to be comparable, than for free particles.

III. PION-PION SCATTERING

The isotopic spin dependence has been omitted from Eq. (4); it appears as a multiplicative factor in the wave function. For the two-pion state with $I=1$, J must be odd, and we deal with the experimentally interesting P state. This implies that $V_2(\mathbf{r})$ is spherically symmetric; it must also be attractive if there is to be a resonance. For convenience, we choose V_2 to be of square-well form, with depth V_0 and range a.

The scattering calculation is quite straightforward,[14] and the results can be expressed in terms of spherical Bessel functions, which are tabulated. There is no bound state so long as V_0a^2 is less than $2\pi^2=19.74$; in this case, the scattering phase shift increases from zero at zero kinetic energy ($M_2=2m$), goes through a maximum as the energy increases, and approaches zero at infinite energy. For values of V_0a^2 less than about 14, the maximum phase shift is less than $\frac{1}{2}\pi$, and the total-

[14] See for example L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), Sec. 19.

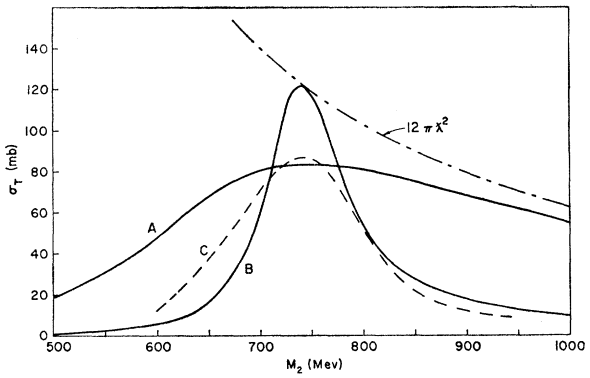

FIG. 1. Total P-wave pion-pion scattering cross section in millibarns plotted against total center-of-mass energy in Mev. Curves A and B are calculated on the basis of attractive square-well potentials of depth V_0 and range a: (A) $V_0a^2=12.50$, $V_0^{\frac{1}{2}}=6.5m$, $a=0.76$ fermi; (B) $V_0a^2=19.47$, $V_0^{\frac{1}{2}}=50m$, $a=0.12$ fermi. Curve C represents the experimental results of Erwin, March, Walker, and West.[9] The curve labeled $12\pi\lambda^2$ is an upper bound on the total cross section.

cross-section curve lies entirely below $12\pi\lambda^2$, where $1/\lambda^2=\frac{1}{4}M_2^2-m^2$; for larger values of V_0a^2, the maximum phase shift is between $\frac{1}{2}\pi$ and π, and the cross-section curve has a maximum at an energy slightly below the first point at which the phase shift is $\frac{1}{2}\pi$, which is the first point of tangency to $12\pi\lambda^2$.

The general shape of the scattering curve is determined by the value of V_0a^2, becoming sharper as this quantity approaches $2\pi^2$. The energy scale is determined by the value of V_0 or of a; for given V_0a^2, the energy of the cross-section peak increases as V_0 increases and a decreases. Figure 1 shows the total cross section in millibarns plotted against M_2 in Mev, for two sets of parameters:

curve A: $V_0a^2=12.50$, $V_0^{\frac{1}{2}}=6.5m$, $a=0.76$ fermi;
curve B: $V_0a^2=19.47$, $V_0^{\frac{1}{2}}=50m$, $a=0.12$ fermi.

For comparison, the experimental results[9] are plotted as curve C. The parameters of the two theoretical curves are chosen so that the positions of the maxima agree with observation. Curve A is clearly much too broad to fit the experiment. However, curve B, or something close to it, is sharp enough to be in agreement with curve C if the peak of the latter has been lowered slightly by instrumental broadening, as seems quite plausible.

We conclude, then, that an attractive square-well potential with V_0a^2 slightly less than the value needed for binding can provide at least a qualitative explanation of the experimental observations on the two-pion system.

IV. THREE-PION SYSTEM

It is easily seen that three pions, each of which has unit isotopic spin, can combine to form one state with $I=3$, two states with $I=2$, three states with $I=1$, and one state with $I=0$. Thus the $I=0$ state suggested by Chew[3] is unique; it may be represented as the triple scalar product of the three isospin vectors, and each pair of pions is in a relative $I=1$ state. Since it is antisymmetric in the interchange of any pair of pions, we require that the space part of the wave function be completely antisymmetric.

There are two independent vectors from which the space function χ, which is to be a solution of Eq. (5) with $J=1$, can be constructed. These may be taken to be $\mathbf{r}$ and $\boldsymbol{\varrho}$, which were defined in Sec. II; or they may be taken to be any two of the vectors $\mathbf{r}_{12}$, $\mathbf{r}_{23}$, $\mathbf{r}_{31}$ ($\mathbf{r}_{ij}\equiv\mathbf{r}_i-\mathbf{r}_j$), which satisfy the identity $\mathbf{r}_{12}+\mathbf{r}_{23}+\mathbf{r}_{31}\equiv 0$. If we avoid using spherical harmonics of higher order, there are only two completely antisymmetric states with $J=1$ that can be constructed:

$$\begin{gathered}\mathbf{A}f(r_{12},r_{23},r_{31}),\\ \mathbf{A}=\mathbf{r}\times\boldsymbol{\varrho}=-\mathbf{r}_{12}\times\mathbf{r}_{23}=-\mathbf{r}_{23}\times\mathbf{r}_{31}=-\mathbf{r}_{31}\times\mathbf{r}_{12},\end{gathered}\tag{6}$$

where f is symmetric in its arguments; and

$$\mathbf{r}_{12}g(r_{23},r_{31};r_{12})+\mathbf{r}_{23}g(r_{31},r_{12};r_{23})+\mathbf{r}_{31}g(r_{12},r_{23};r_{31}),\tag{7}$$

where g is symmetric in its first two arguments. The wave function (6) has odd parity, and each pair of pions is in a relative $J=1$ state; it is the one referred to by Chew,[3] and the one favored by experiment.[10] The wave function (7) has even parity; we shall not consider this state further.

Our objective is to determine whether or not the energy eigenvalue M_3 of Eq. (5), obtained with the help of the two-pion potential V_2 of Sec. III, is in agreement with observation. In order to accomplish this objective, we must relate V_3 to V_2, and then solve Eq. (5). For the latter step, we use the variational method. This assumes that the three-pion system is bound, and gives an upper limit for M_3. Experimentally, the state is not bound, since M_3 is approximately 790 Mev, and hence is greater than $3m$. However, we shall see that the variational energy is a very rapidly varying function of V_0 and a, so that a small change in these parameters suffices to change the variational estimate of M_3 by a large amount in the neighborhood of $M_3=3m$. For this reason, it does not seem worthwhile to improve the method of calculation of M_3. Instead, we shall set the variational estimate of M_3 equal to $3m$, and find the corresponding value of V_0a^2 (the separate values of V_0 and a do not enter in this case). Because of the strong dependence of the variational energy on the potential parameters, the value obtained for V_0a^2 is very insensitive to the value assumed for M_3. On the other hand, it does, of course, depend on the relation that is assumed to exist between V_2 and V_3.

The simplest assumption to make concerning V_3 is that it is the sum of the three pair interactions. Since each pair is in a relative $I=1$, $J=1$ state, only V_2 and not the interaction in other possible pair states enters. Our first assumption is, then, that

$$V_3=V_2(\mathbf{r}_{12})+V_2(\mathbf{r}_{23})+V_2(\mathbf{r}_{31}),\tag{8}$$

where V_2 is the attractive square-well potential of Sec. III. The variational calculation is carried through with an exponential trial function of the form (6), where

$$f=\exp-\alpha(r_{12}+r_{23}+r_{31}), \tag{9}$$

and with a Gaussian trial function

$$f=\exp-\beta(r_{12}^2+r_{23}^2+r_{31}^2). \tag{10}$$

Equations (6) and (9) may be used to calculate the expectation value W of the square bracket on the left side of Eq. (5), with the help of certain integrals that are discussed in the Appendix. The result is

$$\begin{aligned} &W=V_0[Kx^2+u(x)], \quad x=4\alpha a, \quad K=6/11V_0a^2, \\ &u(x)=-3+3e^{-x} \\ &\times\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{29x^5}{3960}+\frac{19x^6}{23\,760}+\frac{x^7}{23\,760}\right). \end{aligned} \tag{11}$$

The K term is the expectation value of the kinetic energy, and the u term is the expectation value of the potential energy. The corresponding expression for the Gaussian trial function, (6) and (10), is obtained in an elementary fashion:

$$\begin{aligned} &W=V_0[Ly^2+v(y)], \quad y=a(3\beta)^{\frac{1}{2}}, \quad L=10/V_0a^2, \\ &v(y)=-3\Phi(y)+4\pi^{-\frac{1}{2}}y(y^2+\tfrac{3}{2})\exp(-y^2), \\ &\Phi(y)=2\pi^{-\frac{1}{2}}\int_0^y \exp(-t^2)dt. \end{aligned} \tag{12}$$

The variational procedure consists in varying α and β while keeping V_0 and a fixed, so as to make W a minimum. The values of the dimensionless variables x and y that minimize W are called x_0 and y_0, and are functions of V_0a^2. Then V_0a^2 is chosen so that with the appropriate values of x_0 and y_0, W is made to vanish; this corresponds to zero binding energy, or $M_3=3m$. It is easy to see that the foregoing procedure in the exponential case is equivalent to solving the equations $W=0$, $dW/dx=0$ simultaneously for x_0 and V_0a^2; it can also be verified that $d^2W/dx^2>0$, so that W is a minimum. Both of the above equations can be handled quite simply; Eq. (11) leads to $x_0=6.052$, $V_0a^2=14.48$, and Eq. (12) leads to $y_0=1.613$, $V_0a^2=14.24$. Thus the two trial functions agree in giving a value for V_0a^2 that is much too small to explain the observed pion-pion scattering.

It is not difficult to see that a small increase in V_0a^2 would give a large increase in binding energy. One can show that for small variations Δ:

$$\Delta[W(x_0)/V_0]/\Delta(V_0a^2)=-(6/11)(x_0/V_0a^2)^2; \tag{13}$$

a similar expression holds for the Gaussian trial function, except that x_0 is replaced by y_0 and 6/11 is replaced by 10. The expression (13) is equal to -0.095 in the exponential case; thus if V_0a^2 were increased by unity, which is not nearly enough to account for pion-pion scattering, W would be decreased from zero to about $-0.1V_0$. Now W is an upper limit on $\frac{1}{3}M_3^2-3m^2$, and from the work of Sec. III we expect $V_0^{\frac{1}{2}}$ to be somewhat greater than $6.5m$; it follows that this small increase in V_0a^2 would be enough to make M_3^2 negative! The Gaussian case gives a similar result. Thus the variational estimate for M_3 is very sensitive to the value of V_0a^2, and there is no point in considering values of W other than $W=0$ when V_0a^2 is being estimated from the three-pion system.

It follows that within the framework of our potential model, Chew's conjecture[3] that the addition of pion pair interactions would give sufficient binding in the three-pion system grossly underestimates the effectiveness of the pair potential. An additive pair potential consistent with pion-pion scattering would give far too much binding in the three-pion $I=0$, $J=1^-$ state.

The opposite extreme to the additivity assumption expressed by Eq. (8) is the saturation assumption. According to this, $V_3=-V_0$ whenever any one, or any two, or all three of the interparticle distances r_{12}, r_{23}, r_{31} are less than a, and $V_3=0$ when all three of these distances are greater than a. This assumption has a certain intuitive appeal, since V_0 has been found to be so large that it is difficult to assign physical meaning to $2V_0$ or $3V_0$. The potential energy integral for the Gaussian trial function is impossibly complicated in the saturating case, although the exponential trial function is quite tractable (see the Appendix). The expression for W that is obtained with (6) and (9) is

$$\begin{aligned} &W=V_0[Mz^2+w(z)], \quad z=2\alpha a, \quad M=24/11V_0a^2, \\ &w(z)=-1+(4/33)[e^{-3z}(z^7+7z^6+23z^5+33z^4 \\ &\qquad -20z^3-132z^2-144z-48) \\ &\qquad +e^{-4z}(8z^6+52z^5+167z^4 \\ &\qquad\qquad +320z^3+366z^2+225z+225/4)]. \end{aligned}$$

The variational procedure described in connection with Eqs. (11) and (12) leads to $z_0=2.289$, $V_0a^2=22.70$. As before, W is a very rapidly-varying function of V_0a^2. This value of V_0a^2 is somewhat larger than that required to account for the observed pion-pion scattering, which is around 19.5, and would result in a bound di-pion. However, it must be remembered that the variational method overestimates the energy eigenvalue, and hence overestimates the value of V_0a^2 that is required to give the three-pion system zero binding energy.

We conclude that our potential model provides a qualitatively consistent explanation both of pion-pion scattering and the three-pion system if the pair interactions saturate or nearly saturate. The model also excludes the possibility that the pair interactions add without saturation.

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APPENDIX

The variational integrals involving the exponential trial function (6) and (9) are all of the form

$$\int\int\int r_{12}{}^{l}r_{23}{}^{m}r_{31}{}^{n}\exp[-(\alpha r_{23}+\beta r_{31}+\gamma r_{12})]d^3r_1d^3r_2d^3r_3,$$

where the over-all volume of integration Ω is very large, and the interparticle distances may or may not be restricted. Such integrals can be obtained by repeated differentiation with respect to α, β, and γ of the same integral with $l=m=n=-1$. When there is no restriction on the interparticle distances, the basic integral is[15]

$$\int\int\int (r_{12}r_{23}r_{31})^{-1}\times\exp[-(\alpha r_{23}+\beta r_{31}+\gamma r_{12})]d^3r_1d^3r_2d^3r_3=\frac{16\pi^2\Omega}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}. \quad \text{(A1)}$$

The evaluation of (A1) may be carried through by expressing each of the factors in the integrand in terms of its Fourier transform:

$$r^{-1}e^{-\alpha r}=(2\pi^2)^{-1}\int (k^2+\alpha^2)^{-1}e^{i\mathbf{k}\cdot\mathbf{r}}d^3k. \quad \text{(A2)}$$

The $\mathbf{r}$ integrations are performed by expressing the relative coordinates in terms of the orthogonal coordinates $\mathbf{R}$, $\mathbf{r}$, $\boldsymbol{\varrho}$ of Sec. II, and lead to δ functions in the $\mathbf{k}$ variables. Integration over two of the $\mathbf{k}$'s yields

$$32\pi\Omega\int_0^\infty [(k^2+\alpha^2)(k^2+\beta^2)(k^2+\gamma^2)]^{-1}k^2dk, \quad \text{(A3)}$$

which can be evaluated by residues to give the right side of Eq. (A1).

The same procedure works when one or more of the interparticle distances is restricted. The integral (A1) with only r_{23} restricted to be greater than a is evaluated by replacing the r_{12} and r_{31} parts of the integrand as in (A2), and replacing the r_{23} part by

$$(2\pi^2)^{-1}\int \phi_\alpha(k)(k^2+\alpha^2)^{-1}e^{i\mathbf{k}\cdot\mathbf{r}}d^3k, \quad \text{(A4)}$$

$$\phi_\alpha(k)=\tfrac{1}{2}e^{-\alpha a}[(1-i\alpha/k)e^{ika}+(1+i\alpha/k)e^{-ika}];$$

this is equal to $r^{-1}e^{-\alpha r}$ if $r>a$ and 0 if $r<a$. The integration over the space coordinates and two of the $\mathbf{k}$'s goes as before, and leads to (A3) except for a factor $\phi_\alpha(k)$ in the integrand. This integral can also be evaluated by residues, and yields

$$\left[\frac{16\pi^2\Omega}{(\beta^2-\gamma^2)}\right]\left[\frac{e^{-(\alpha+\gamma)a}}{(\alpha+\gamma)}-\frac{e^{-(\alpha+\beta)a}}{(\alpha+\beta)}\right].$$

Repeated derivatives of this integral are used in evaluating the three-pion potential energy in the nonsaturating case.

For the potential energy in the saturating case, we require the integral (A1) with all three interparticle distances restricted to be greater than a. As before, this leads to (A3) except for the factor $\phi_\alpha(k)\phi_\beta(k)\phi_\gamma(k)$ in the integrand; here, ϕ_β and ϕ_γ are given by (A4) with α replaced by β or γ. Evaluation of the integral yields

$$8\pi^2\Omega e^{-(\alpha+\beta+\gamma)a}\left[\frac{1}{\alpha\beta\gamma}-\frac{e^{-\alpha a}}{\alpha(\alpha+\beta)(\alpha+\gamma)}-\frac{e^{-\beta a}}{\beta(\beta+\alpha)(\beta+\gamma)}-\frac{e^{-\gamma a}}{\gamma(\gamma+\alpha)(\gamma+\beta)}\right].$$

[15] J. B. Fisk, L. I. Schiff, and W. Shockley, Phys. Rev. **50**, 1090 (1936); the value of the integral given in this paper should be divided by 4π.