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Mach's Principle and the Locally Measured Gravitational Constant in General Relativity*

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It has been conjectured that a "Mach's principle" might lead to a dependence of the local Newtonian gravitational constant, K, on universe structure, $K^{-1} \sim M/R$. Einstein and others have suggested that general relativity predicts such a result. A closer analysis, however, including the carrying out of the geodesic equations to second order, seems to indicate that this is not true and that the apparent "Mach's principle" terms involving total universe structure are really only coordinate effects. Further, the measure of gravitating mass obtained in a local, proper Newtonian gravitational experiment is compared in a coordinate-free way to an experimentally measurable inertial mass and found to be related to it in a way independent of the rest of the universe. A generalization of these results is given. It is based on the fact that in general relativity the only way the universe can influence experiments done in an electrically shielded laboratory is through the metric and that this can be "transformed away" to any degree of accuracy for a sufficiently small laboratory. Consequences of this are summarized in Dicke's "strong principle of equivalence." It is noted, however, that there are other statements which might be called "Mach's principles" which are satisfied in general relativity.

I. INTRODUCTION

THE principal idea which guided Einstein in formulating the general theory of relativity was the local equivalence of gravitational and inertial effects, that is, the equivalence of a uniform gravitational force field and a constant acceleration of the reference frame. Another idea relating gravity and inertia is Mach's principle. This is less precisely formulated but suggests that the inertial properties of a body are determined by the distribution of matter in the universe. Since the gravitational field interacts with *all* matter, one could hope to see the Mach principle relationship between inertial and distant matter described in terms of the gravitational field. To state this in a way independent of units, consider the ratio of the inertial mass of a body to its active gravitational mass.[1]

In particular, let us see that this ratio might be in a static universe consisting only of a mass shell of radius R and inertial mass M together with a relatively small body of inertial mass m at its center. If we probe the gravitational field of m with a small test particle, we might expect from the Eötvös experiment that the acceleration of the test particle is independent of its mass. It certainly depends, however, on m and r and conceivably on M and R. The fact that the Newtonian theory of gravity is valid to a high degree of accuracy suggests that for $m \ll M$, $r \ll R$, the acceleration is

$$a = -[m/r^2 F(M,R)], \tag{1.1}$$

where F is a function of dimensions mass over length (velocity of light $c=1$). Dimensional analysis then suggests

$$F = AM/R, \tag{1.2}$$

where A is a constant dimensionless number. For a more general type of universe with masses m_a at distances r_a from some point x, this might be extended to

$$F(x) = A \sum_a m_a/r_a. \tag{1.3}$$

Until recently, experimental determinations of F from (1.1) were possible only on the earth. The value found is not inconsistent with (1.3), a positive value of A in the neighborhood of 10^0 or 10^1, and present astronomical knowledge of m_a and r_a. It is clear that in a uniform universe, $m_a \sim r_a^2$, so that the dominant contribution to the sum on the right side of (1.3) comes from distant matter and the resulting $F(x)$ is fairly constant in space and time. This also is consistent with present observations.

A comparison of (1.1) with the standard classical Newtonian theory of gravity shows that F^{-1} plays the role of Newton's "universal gravitational constant." However, if (1.3) is true, this number is not a universal constant but depends on the distribution of mass in the universe about the point where it is measured. To investigate possible resulting changes in value of this number, it is convenient to introduce a standard value and refer variations to it. Specifically, let $K_0/8\pi$ be defined as the presently observed terrestrial value of $F(x)^{-1}$. $K_0/8\pi$ is thus a constant number of dimensions length over mass. Then rewrite (1.1) as

$$a = -\frac{K_0}{8\pi r^2}\left(\frac{8\pi m}{K_0 F}\right). \tag{1.4}$$

This equation is identical with Newton's if the quantity in parentheses, $m_g \equiv 8\pi m/K_0 F$, is taken to be the active gravitational mass[1] associated with m. Notice that by definition of K_0, this gives $m_g = m$ at the present time on earth. However, if (1.3) is true, a Cavendish-type experiment interpreted in the context of a Newtonian theory with fixed gravitational constant $K_0/8\pi$ would give a measurement of active gravitational mass m_g

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[1] H. Bondi, Revs. Modern Phys. 29, 423 (1957).

yielding a ratio

$$\frac{m}{m_g}=A\sum_a \frac{K_0 m_a}{8\pi r_a}, \qquad (1.5)$$

which would not necessarily always be unity.

Einstein[2] claims to find such a result in general relativity. In order to study this problem, consider the creation of relatively small masses m_a', at distances r_a'. from the present standard laboratory in which, prior to the creation of m_a', $m/m_g=1$ by definition of K_0. With m_a' present, however, (1.5) then yields

$$\frac{m}{m_g}=1+A\sum_{\text{"new matter"}} \frac{K_0 m_a'}{8\pi r_a'}. \qquad (1.6)$$

If it is assumed that each $K_0 m_a'/8\pi r_a'$ is small compared to unity, the weak-field equations might be used to check (1.6). Einstein does this and arrives at

$$\left(1+\sum_{\text{"new matter"}} \frac{K_0 m_a}{8\pi r_a}\right)a\cong -\frac{K_0 m}{8\pi r^2}. \qquad (1.7)$$

Thus

$$\frac{m}{m_g}=1+\sum_{\text{"new matter"}} \frac{K_0 m_a}{8\pi r_a}, \qquad (1.8)$$

which is identical with (1.6) if $A=1$. Einstein argued from this that since some matter contributes to the ratio, m/m_g, all the universe probably does (Sec. II). There has been some discussion[3] of what the numerical coefficient A of the sum in the right side of (1.8) should be, and indeed the first approximation procedure seems inadequate to resolve this. Consequently, the equations of motion through second order will be applied to this problem in Sec. II.

This result (1.8), or its corrected form (2.11), is clearly coordinate dependent, however. Hence the relationship between its numerical description of the path of a particle and the actually observed path is not defined without further analysis. The usual interpretation of general relativity is based on the identification of the *invariant* theoretical measure of an interval, proper time, with time experimentally measured in some fundamental way, e.g., on an atomic clock. An invariant measure of distance and thus acceleration can be obtained from this by setting the velocity of light equal to one. When this is done, the invariant description of the path of a test particle relative to a central mass is found to be approximately Newtonian with coefficients independent of the rest of the universe. (See Sec. III.)

However, the number m appearing in the left side of (1.5) has not yet been related to an experimentally measured inertial mass. To remedy this, a description of a process for invariantly studying the acceleration of charged bodies in a known electric field is given. The resultant ratio of "force" to acceleration is defined as the inertial mass. For a simple theory of matter m_{inert} is found to be just the m appearing in (1.5) (Sec. IV). This procedure assumes given standards of charge and time interval.

The independence of the relationship between the two numbers, m_g and m_{inert}, from the rest of the universe is more generally true than the above special case might indicate. In fact, assume that the space in the neighborhood of an electrically shielded laboratory is sufficiently flat that in a certain coordinate system the differences between the metric components and those of the Minkowskian, together with the first two derivatives of these differences, are negligible over the laboratory. Then, according to general relativity, if small masses, charged or uncharged, are introduced into the laboratory, the description of their motions and interactions in this coordinate system is independent of the rest of the universe. This is due to the fact that once the laboratory is shielded, the only way the rest of the universe could influence it, according to general relativity, is through the metric. If this is sensibly flat within, its influence can be transformed away by a coordinate transformation, thus eliminating any effects from the rest of the universe. This is Dicke's "strong principle of equivalence.[4]" (See Sec. V.)

There are, however, other statements which might be considered Mach's principles. These are based on the fact that in general relativity gravitational and inertial forces have the same formal origin. (See Sec. V.)

II. EINSTEIN'S RESULTS

Gravity and general relativity being largely concerned with the interaction between masses as masses, Einstein was naturally interested in whether or not Mach's principle as discussed in Sec. I above was satisfied in general relativity. Specifically, is the attraction and resultant relative motion of two gravitating bodies influenced by the rest of the universe?

Einstein investigated this in the weak-field approximation.[2] The metric he found to represent the gravitational field due to a distribution of small masses corresponding to a "density" σ and having small velocities, dx^i/ds, can be written as

$$g_{00}=1-\frac{K}{4\pi}\int\frac{\sigma dV}{r},$$

$$g_{0i}=\frac{K}{2\pi}\int\frac{\sigma(dx^i/ds)}{r}dV, \qquad (2.1)$$

$$g_{ij}=-\delta_{ij}\left(1+\frac{K}{4\pi}\int\frac{\sigma dV}{r}\right),$$

[2] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1955), 5th ed., pp. 99–108.

[3] W. Davidson, Monthly Notices Roy. Astron. Soc. **117**, 212 (1957).

[4] R. H. Dicke, Science **129**, 621 (1959). See also Revs. Modern Phys. **29**, 355 (1957); J. Wash. Acad. Sci. **48**, 213 (1958); Am. J. Phys. **28**, 344 (1960).

on replacing Einstein's imaginary time x^4 by the real $x^0=-ix^4$. Here K is just the constant introduced in the Einstein field equations and thus not yet related to K_0 or other observed numbers. Equation (2.1) is correct only to first order in $K\int \sigma dV/r$, and dx^i/ds. The geodesic equation for a test particle in this field becomes

$$\frac{d}{dx^0}[(1+\bar{\sigma})\mathbf{v}]=\nabla\bar{\sigma}+\frac{\partial \mathbf{A}}{\partial x^0}+(\nabla\times\mathbf{A})\times\mathbf{v}, \tag{2.2}$$

where

$$\mathbf{v}\equiv d\mathbf{x}/ds,\quad \bar{\sigma}\equiv\frac{K}{8\pi}\int\frac{\sigma dV}{r},\quad \mathbf{A}\equiv\frac{K}{2\pi}\int\frac{\sigma\mathbf{v}}{r}dV. \tag{2.3}$$

For simplicity, consider the application of these results to the case of the motion of a test particle near a small mass m at rest at the origin, all inside a static, spherical shell of mass M_s and radius R_s,[5] (2.2) now becomes

$$\frac{d}{dx^0}\left[\left(1+\frac{KM_s}{8\pi R_s}+\frac{Km}{8\pi r}\right)v^i\right]=\frac{Km}{8\pi}\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right). \tag{2.4}$$

Thus, $(1+KM_s/8\pi R_s+Km/8\pi r)$ times the coordinate acceleration of the test particle is just the Newtonian term, to this approximation. Einstein interpreted this by saying that the "inert mass is proportional to $1+\bar{\sigma}$,"[2] or in (2.4) to $1+(K/8\pi)(M_s/R_s+m/r)$. However, an equivalent statement, more convenient for this discussion and in keeping with that of Sec. I, can be made. Specifically, dividing (2.4) by $[1+(K/8\pi)\times(M_s/R_s+m/r)]$ gives, for v_i instantaneously zero,

$$\frac{d}{dx^0}v^i=\frac{Km}{8\pi[1+(K/8\pi)(M_s/R_s+m/r)]}\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right). \tag{2.5}$$

This, in keeping with Einstein's interpretation above, would suggest that the locally measured Newtonian active gravitational mass of m is

$$m_g=m/[1+(K/8\pi)(M_s/R_s+m/r)], \tag{2.6}$$

or that the effective, locally measured Newtonian gravitational constant is

$$K_E=K/[1+(K/8\pi)(M_s/R_s+m/r)]. \tag{2.7}$$

If this is true, a comparison of (2.6) with (1.5) would show that a Mach's principle in the sense of Sec. I would be satisfied in general relativity, since the number K_E in (2.7) measuring the attraction of m for test particles would depend on the mass distribution M_s/R_s in the rest of the universe. To clarify the relation of (2.6) and (2.7) to the discussion in Sec. I, it is necessary to consider M_s and m as small additions to a background universe [i.e., as the m_a' were in the discussion preceding (1.6) above]. For the background universe assume that K has been chosen equal to K_0. Thus, (2.6) will coincide with (1.6) if $A=1$ in the latter.

One objection[6] that might be raised against the above procedure is based on the fact that (2.4) and thus (2.5) are true only to first order in KM_s/R_s and Km/r. Hence the "Mach's principle" terms in (2.5) are of higher order than can be consistently retained.

In other words, the difference between (2.5) and

$$\frac{d}{dx^0}v^i=\frac{Km}{8\pi}\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right) \tag{2.8}$$

is too small to be retained in view of the approximations made in deriving (2.5). Consequently, to the accuracy assumed, (2.6) should be written

$$m_g=m, \tag{2.9}$$

and (2.7)

$$K_E=K. \tag{2.10}$$

This objection, however, can be overcome by studying the equations of motion to higher order.[7] The result, for the same type of universe, is

$$\frac{d}{dx^0}v^i=\frac{Km}{8\pi(1+5KM_s/8\pi R_s)}\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right). \tag{2.11}$$

In Eq. (2.11), terms of order $(Km/r)^2$ have been neglected as well as terms of order r/R_s. These are not relevant to this discussion and for a situation of physical interest would be small compared to the terms kept. Terms of order $(KM_s/R_s)^2$ and $(KM_s/R_s)(Km/r)$ have not been neglected, however, so that equations analogous to (2.6) and (2.7) might be written

$$m_g=\frac{m}{1+(5KM_s/8\pi R_s)}, \tag{2.12}$$

and

$$K_E=\frac{K}{1+(5KM_s/8\pi R_s)}. \tag{2.13}$$

There is, however, another objection that might be raised against these results. This is discussed in the following section.

[5] This example, while admittedly rather specialized, is sufficient to illustrate the ideas under consideration. It should also be noted that here $KM_s/R_s\ll 1$ so that this does not correspond to the total "universe mass shell" discussed in Sec. I and for which $KM/R\sim 1$.

[6] See also in this connection reference 3. Davidson criticized Einstein's retention of the $\bar{\sigma}v$ term because of the assumed smallness of v. He "corrects" this by retaining all velocity terms in the geodesic equation. His result is still questionable, however, on the basis of the discussion following in the text.

[7] A. Papapetrou, Proc. Phys. Soc. (London) **A64**, 57 (1951); V. Fock, J. Phys. (U.S.S.R.) **1**, 81 (1939); L. Infeld, Revs. Modern Phys. **29**, 398 (1957).

III. LOCAL, PROPER GRAVITATIONAL CONSTANT

Since (2.11) contains coordinate acceleration and distance, there is a question associated with the interpretation of it in Sec. II. This question concerns the meaning of coordinates and the metric tensor. The usual interpretation of general relativity rests on the identification of

$$d\tau=(-g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}};\quad (\text{if } d\tau^2\geqslant 0), \tag{3.1}$$

as the differential of "proper time," or time read on some basic, e.g., atomic, clock associated with the coordinate interval dx^μ. Defining the velocity of light to be 1, and assuming a light ray to be a null geodesic, provides the basis for a method of obtaining a "proper" measurement of a "distance" between particles. Specifically, the proper distance between two time-like paths will be taken as one-half the proper time of flight (measured along one path) of a light ray from one path to the other and back again.[8] This provides a coordinate-free, if impractical, method for obtaining a measurable, numerical description of the relative motion of two bodies.

An application of this method to (2.11) yields

$$\frac{d^2}{dx_p{}^{02}}x_p{}^i=\frac{Km}{8\pi}\frac{\partial}{\partial x_p{}^i}\left(\frac{1}{r_p}\right), \tag{3.2}$$

where $x_p{}^i$ is proper distance as measured from the test particle to m and $x_p{}^0$ is proper time along the test particle. In obtaining (3.2) higher order terms in Km/r and r/R_s were neglected but the $(KM_s/R_s)(Km/r)$ term was kept and cancelled out.

Thus, a coordinate-free description of the motion shows that it is independent of the mass distribution in the rest of the universe, at least to the order of approximation for which (3.2) is valid. Hence, this example does not seem to indicate the validity of a physically detectable Mach's principle in general relativity in the sense of Secs. I and II.

The rest of this section will be devoted to a sketch of the calculations leading from (2.11) to (3.2). First of all, it should be noted that m must be replaced by a non-singular source as used in the Papapetrou-Fock method before a proper distance between its center and that of the test particle can be defined. However, for the purpose of the discussion above, terms of order $(Km/r)^2$ are neglected and, since both sides of (2.11) are already of first order in Km/r, this means that contributions to the metric from m can be neglected. Hence, Infeld's[7] renormalized delta function, which disregards self interaction could equally well be used.

Secondly, since (2.11) is accurate only through terms $(KM_s/R_s)^2$, $(KM_s/R_s)(Km/r)$ and both sides of (2.11) are already of order Km/r, only terms linear in KM_s/R_s in the metric need be kept in converting the distances and times in (2.11) to proper units. The fact that only the "first order" terms need be kept is important because the metric obtained by Infeld coincides to this order with that of Papapetrou-Fock. Further, the coordinate description of the motion of n bodies through second order, of which (2.11) is a special case, is the same in either method. Thus, the really observable prediction, a relation of proper relative accelerations to proper distances and velocities, is identical in both cases.

Finally, for the example at hand, all particles are instantaneously at rest and terms r/R_s are to be neglected. This essentially means that between the test particles and m, changes in the background metric, i.e., neglecting m, can be ignored. Thus

$$\begin{aligned} x_p{}^0&\cong[-\bar{g}_{00}(0)]^{\frac{1}{2}}x^0,\\ x_p{}^i&\cong[\bar{g}_{ii}(0)]^{\frac{1}{2}}x^i, \end{aligned} \tag{3.3}$$

$\bar{g}_{\alpha\beta}\equiv$background metric, i.e., with $m=0$, where $x_p{}^0$ and $x_p{}^i$ are proper time and distance as defined above. For the example at hand,

$$\begin{aligned} \bar{g}_{00}(0)&\cong-1+KM_s/4\pi R_s,\\ \bar{g}_{ij}(0)&\cong\delta_{ij}(1+KM_s/4\pi R_s). \end{aligned} \tag{3.4}$$

Thus (2.11) becomes

$$\left(1-\frac{KM_s}{4\pi R_s}\right)\frac{d^2}{dx_p{}^{02}}\frac{x_p{}^i}{(1+KM_s/8\pi R_s)} =\frac{Km(1+KM_s/4\pi R_s)}{8\pi(1+5KM_s/4\pi R_s)}\frac{\partial}{\partial x_p{}^i}\left(\frac{1}{r_p}\right), \tag{3.5}$$

which immediately reduces to (3.2).

The question of what the m appearing in (3.2) means and how it is to be measured will now be taken up.

IV. RELATIONSHIP OF GRAVITATIONAL TO INERTIAL MASS

The discussion in Sec. III did not say anything about the physical meaning of the number m appearing in the right side of (3.2). Mathematically this number m came from the stress tensor of matter in the Einstein equations and in fact, if $w^\beta=$ four-velocity of the particle, it is

$$m=-\int_{x^0=\text{const}} w^0T_\alpha{}^\alpha(-g)^{\frac{1}{2}}d^3x. \tag{4.1}$$

However, this is still not enough and merely replaces one unknown by another, m by $T_\alpha{}^\beta$, leaving the physical meaning and method of measurement of the latter undefined.

This section will consider one of two coordinate-free methods for obtaining physically measurable numbers associated with the tensor $T_\alpha{}^\beta$, namely, the measurement of inertial mass. The other, active gravitational mass, was considered in Secs. II and III. Thus $T_\alpha{}^\beta$ is considered as a mathematical intermediary between two observed numbers.

[8] E. Wigner, Revs. Modern Phys. **29**, 255 (1957).

In order to measure inertial mass, "standard" electromagnetic theory will be assumed with its conserved current density. This gives a constant total charge and it is assumed that quantities of this charge are physically available in arbitrarily small amounts. A standard Coulomb law experiment, in proper units, will then provide a method for measuring inertial mass, the units of which will thus be determined solely in terms of a unit of time and charge.

The two numbers, inertial and active gravitational mass, will be proved approximately equal for the case of a "fluid," irrespective of what the mass distribution in the rest of the universe is, provided, of course, that it does not encroach on the laboratory and that the latter and the masses and charges in it are sufficiently small.

As far as this paper is concerned, the main significance of this result is not so much the equality of active gravitational and inertial mass but the fact that the "function" expressing one in terms of the other is independent of the rest of the universe. A generalization of this result will be sketched in Sec. V.

Standard electromagnetic theory in general relativity is based on the following equations:

$$\begin{aligned} F^{\alpha\beta}{}_{;\beta}&=\sigma w^\beta; \quad w^\beta w_\beta=-1, \\ F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}&+F_{\gamma\alpha,\beta}=0, \end{aligned} \tag{4.2}$$

where σ is a scalar, $F_{\alpha\beta}$ an antisymmetric tensor, and w^β the four-velocity of the charge. As is well known, if on each surface $x^0=\text{constant}$ σ is zero outside a bounded region, then

$$q\equiv\int_{x^0=t}(-g)^{\frac{1}{2}}\sigma w^0 d^3x \tag{4.3}$$

is independent of t. This constant number q is defined as the total charge of the distribution represented by σw^α. It is assumed that a unit of charge as defined by (4.3) is physically available. The operational definition of inertial mass as described later in this section is fundamentally based on this association of the theoretical number (4.3) with a given, physical "charged" particle.

The equations of motion follow from conservation of the total stress-energy tensor given by

$$T_{\text{total}}{}^{\alpha\beta}={}_mT^{\alpha\beta}+\tfrac{1}{4}g^{\alpha\beta}F_{\lambda\mu}F^{\lambda\mu}-F^\alpha{}_\lambda F^{\beta\lambda}. \tag{4.4}$$

Thus, $(T_{\text{total}}{}^{\alpha\beta})_{;\beta}=0$ becomes

$${}_mT^{\alpha\beta}{}_{;\beta}=\sigma w^\rho F^\alpha{}_\rho. \tag{4.5}$$

Papapetrou's derivation of the equations of motion in general relativity is based on the conservation equations, which in the absence of charge become

$${}_m\mathcal{T}_\alpha{}^\beta{}_{,\beta}-\tfrac{1}{2}\,{}_m\mathcal{T}^{\mu\nu}g_{\mu\nu,\alpha}=0, \tag{4.6}$$

where

$${}_m\mathcal{T}_\alpha{}^\beta\equiv(-g)^{\frac{1}{2}}\,{}_m\mathcal{T}_\alpha{}^\beta. \tag{4.7}$$

The standard choice of ${}_mT_\alpha{}^\beta$ for a fluid is

$$\begin{aligned} {}_mT^{\alpha\beta}&=(\rho+p)w^\alpha w^\beta+pg^{\alpha\beta}, \\ w^\alpha&=dx^\alpha/d\tau; \quad d\tau\equiv(-g_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}}, \end{aligned} \tag{4.8}$$

with ρ and p scalars and p much smaller than ρ. Papapetrou defines his choice of ${}_m\mathcal{T}^{\alpha\beta}$ only through the approximations necessary to derive the equation of motion through second order in terms Km/r. He assumes that for a situation represented by n "particles," at points $x_a{}^i$, ${}_m\mathcal{T}^{\alpha\beta}$ is the sum of n terms,

$${}_m\mathcal{T}^{\alpha\beta}=\sum_{a=1}^{n}{}_a\mathcal{T}^{\alpha\beta}, \tag{4.9}$$

with each ${}_a\mathcal{T}^{\alpha\beta}$ vanishing outside a small region around $x_a{}^i$ with the radius of this region much smaller than the separation $|x_a{}^i-x_b{}^i|$ between any pair of particles. Changing to a metric of signature $(-,+++)$ his choice for ${}_a\mathcal{T}_\alpha{}^\beta$ to the necessary order becomes

$$\begin{aligned} {}_a\mathcal{T}_0{}^0&=-\bar\rho_a(1+\tfrac{1}{2}v_a{}^2-U_a+\tfrac{1}{2}u_a), \\ {}_a\mathcal{T}_0{}^k&=-\bar\rho_a\dot x_a{}^k(1+\tfrac{1}{2}v_a{}^2-U_a+\tfrac{1}{2}u_a)-p_a\dot x_a{}^k, \\ {}_a\mathcal{T}_i{}^k&=-\bar\rho_a\dot x_a{}^i\dot x_a{}^k+\delta_i{}^kp_a, \\ {}_a\mathcal{T}_i{}^0&=-{}_a\mathcal{T}_0{}^i+4\textstyle\sum_b\bar\rho_a u_b(\dot x_a{}^i-\dot x_b{}^i), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} U_a\equiv U(x_a{}^i)&\equiv\textstyle\sum_b u_b(x_a{}^i), \\ \nabla^2u_b(x^i)&=-\tfrac{1}{2}K\bar\rho_b, \\ v_a{}^2&\equiv\delta_{ik}\dot x_a{}^i\dot x_a{}^k. \end{aligned} \tag{4.11}$$

He assumes that the functions $\bar\rho_a(x)$ and $p_a(x)$ are spherically symmetric about $x_a{}^i$. This is not an unreasonable assumption since "distorting" forces on a from gravitating mass M at a distance R would be of the order $(KM/R^2)(l_a/R)$ where l_a is the distance across the ath mass. From the assumption above, $l_a\ll R$, so that this "distorting" force can be neglected compared to the gravitational force KM/R^2. Finally, he also assumes that the velocities v_a are small compared to one, $v_a{}^2$ being of the same order as U_a.

The mass m which appeared as the active gravitational mass in (2.11) is defined for the ath particle by

$$m_a\equiv\int_{R_a}\bar\rho_a d^3x, \tag{4.12}$$

where R_a is a region containing all points at which $\bar\rho_a\neq0$ but none where $\rho_b\neq0$ for $b\neq a$. He obtains

$$d\bar\rho_a/dt=0=dm_a/dt, \tag{4.13}$$

and

$$p_{a,i}=\bar\rho_a u_{a,i} \tag{4.14}$$

from the lowest order equations of motion. Using (4.14) and the spherical symmetry of p_a, $\bar\rho_a$, a little manipulation shows that

$$\int_{R_a}p_a d^3x=\tfrac{1}{6}\int_{R_a}\bar\rho_a u_a d^3x. \tag{4.15}$$

The lowest order terms in the metric tensor can then be written

$$\begin{aligned} g_{00}&=-(1-2U),\\ g_{0i}&=-4\sum_a u_a\dot{x}_a{}^i,\\ g_{ik}&=\delta_{ik}(1+2U),\\ (-g)^{\frac{1}{2}}&=1+2U. \end{aligned} \tag{4.16}$$

The comparison between Papapetrou's choice (4.10) and the standard one in Eq. (4.8) now follows immediately. Inserting (4.16) into (4.8) and carrying out the operations to an order comparable to (4.10) gives

$$\begin{aligned} {}_a\mathcal{T}_0{}^0&=-\rho_a(1+2U_a+v_a{}^2),\\ {}_a\mathcal{T}_0{}^k&=-\rho_a\dot{x}_a{}^k(1+2U_a+v_a{}^2)-p_a\dot{x}_a{}^k,\\ {}_a\mathcal{T}_i{}^k&=\rho_a\dot{x}_a{}^i\dot{x}_a{}^k+p_a\delta_i{}^k,\\ {}_a\mathcal{T}_k{}^0&=-{}_a\mathcal{T}_0{}^k+4\sum_b u_b\rho_a(\dot{x}_a{}^k-\dot{x}_b{}^k). \end{aligned} \tag{4.17}$$

Hence, the pressure terms in (4.17) and (4.10) can be identified while the densities are related by

$$\bar{\rho}_a=\rho_a(1+3U_a-\tfrac{1}{2}u_a+\tfrac{1}{2}v_a{}^2), \tag{4.18}$$

or, from (4.16) and (4.8)

$$\bar{\rho}_a=\rho_a w_a{}^0(-g)^{\frac{1}{2}}(1-u_a/2). \tag{4.19}$$

Thus Papapetrou's active gravitational mass is just

$$m_a\equiv\int_{R_a}\bar{\rho}_a d^3x=\int_{R_a}\rho_a w_a{}^0(-g)^{\frac{1}{2}}d^3x -\tfrac{1}{2}\int_{R_a}\rho_a w_a{}^0(-g)^{\frac{1}{2}}u_a d^3x. \tag{4.20}$$

The similarity of the first term on the right side of (4.20) to total charge is significant. In fact, if $p_a=0$, $\rho_a w_a{}^\alpha$ is conserved so that the argument used to define a constant charge in Sec. IV could also be used here. From (4.15) the second term on the right can be replaced to this order of approximation by $-3\int(-g)^{\frac{1}{2}}\times p_a w_a{}^0 d^3x$. Hence

$$m_a=\int_{R_a}(\rho_a-3p_a)w_a{}^0(-g)^{\frac{1}{2}}d^3x, \tag{4.21}$$

or

$$m_a=-\int_{R_a}{}_a\mathcal{T}_\alpha{}^\alpha w_a{}^0 d^3x. \tag{4.22}$$

The arrangement for measuring an inertial mass associated with m_a is as follows. First of all, add a spherically symmetric charge distribution to m_a giving it a total charge e_a. Add a charge e_b to another mass m_b much nearer to m_a than the other bodies in the universe. Further, these particles are assumed to be in an electrically shielded laboratory. Then measure that part of the proper relative acceleration of m_a to m_b due to the presence of e_a. Determine this acceleration as a function of proper distance between m_a and m_b when they are instantaneously at rest.

The limit ($A_p{}^i\equiv i$th component of relative proper acceleration)

$$\bar{m}_a\equiv\lim_{\text{"small"}\ |x_a{}^i-x_b{}^i|}\ \lim_{e_b\to 0}\ \lim_{e_a\to 0}\frac{e_b(x_a{}^i-x_b{}^i)_p/4\pi(r_{ab})_p{}^3}{\partial A_p{}^i/\partial e_a} \tag{4.23}$$

will then be called the inertia mass of m_a, provided m_b is much greater than m_a.[9] In (4.23), "small" under "lim" means that while each $|x_a{}^i-x_b{}^i|$ is assumed so small as to eliminate background metric curvature effects, Km_a and Km_b are still negligible in comparison to each $|x_a{}^i-x_b{}^i|$. This will be discussed in the following paragraph.

This choice for $\bar{m}_a$ is motivated by the fact that in flat space it reduces to the ratio of the electrostatic force per unit charge between two charges to the resulting relative acceleration. Here, however, proper distances, times, and accelerations must be used to correspond to the results of real measurements. The partial derivative with respect to e_a is used in the denominator since only that part of the acceleration of m_a due to the presence of e_a is desired. The limit $e_a\to 0$ is required to eliminate any higher order contributions that might arise from finite e_a. Such contributions might come through the metric, for example. Finally, $x_a{}^i$ must approach $x_b{}^i$ so that the space over which the interaction occurs can be considered to have a metric which is nearly constant. It is clear that an electrostatic interaction over a large distance for which curvature effects due to the background metric are not negligible cannot be expected to behave according to Coulomb. Of course, it is to be understood that these limits are to be taken in a practical physical sense, i.e., the numbers involved must be made only so small that decreasing them further would not observably change the value of the ratio being measured. Further, it is assumed that this limit is reached by a value of $|x_a{}^i-x_b{}^i|$ which is still much larger than Km_a and Km_b.

To carry out this program in Papapetrou's formalism, write (4.5) as

$${}_a\mathcal{T}_\alpha{}^\beta{}_{,\beta}-\tfrac{1}{2}\,{}_a\mathcal{T}^{\mu\nu}g_{\mu\nu,\alpha}=\sigma_a w_a{}^\mu F_{\alpha\mu}(-g)^{\frac{1}{2}}. \tag{4.24}$$

Equation (4.24) with $\alpha=i$ will be integrated over R_a under the assumption that the rest of the universe is instantaneously at rest. Further, only first-order terms in m_a will be kept. The determination of the electromagnetic field on the right of (4.24) is based on (4.2). If all particles are instantaneously at rest, $g_{0i}=0$, to the necessary order and that part of the field due to b

[9] This is to eliminate the necessity of converting from reduced mass and is only for computational convenience in the example at hand. A more accurate definition taking this reduced-mass effect into consideration could easily be made, but its complexity would unnecessarily confuse the point of this example, namely, the independence from the rest of the universe of the relationship between active gravitational mass and a reasonably defined inertial mass.

can be written

$$(-g)^{\frac{1}{2}}F^{i0}=e_b(x^i-x_b{}^i)/4\pi r_b{}^3,\qquad F^{ik}=0. \tag{4.25}$$

The shielded laboratory walls have eliminated any radiation contributions to (4.25). In integrating over (4.24) only that part of the electromagnetic field due to b, as given in (4.25), need be kept since the self terms, due to a, would integrate to zero as a result of the spherical symmetry valid in this approximation. Since only first-order results in m_a are desired, the contribution of m_a to the metric will be neglected. Further, it will be assumed that m_b, while much larger than m_a, is still so small that its contribution to the metric can be neglected in comparison to that of the rest of the universe. Finally, using (4.16) for the metric, integration of (4.24) over R_a gives

$$(m_a+\pi_a)(1+3U_a)\frac{d^2x_a{}^i}{dx^{02}}+m_aG_1{}^i+\pi_aG_2{}^i=\frac{e_ae_b(x_a{}^i-x_b{}^i)}{4\pi r_{ab}{}^3}, \tag{4.26}$$

where $\pi_a\equiv\int_{R_a}p_ad^3x$ and $G_1{}^i$ and $G_2{}^i$ are functions arising from the second term in the left side of (4.24) and thus are proportional to derivatives of the metric tensor. It is easy to verify that the neglect of the electromagnetic contribution to the metric tensor used in obtaining (4.26) is justified because of the limit $e_a\rightarrow 0$, $e_b\rightarrow 0$ in the definition.[10] The terms in π_a may not cancel now as they did in Papapetrou's work where $e_a=e_b=0$ and

$$\int_{R_a}(p_a-\tfrac{1}{6}\rho_au_a)d^3x=0. \tag{4.27}$$

However, here the left side of (4.27) while not necessarily vanishing will be at least of second order in charge, so that the difference between the π_a terms in (4.26) and those in this case are of order $A_p{}^i$ times charge squared, and does not contribute to the terms in the definition.[10]

The conversion of (4.26) to proper units proceeds precisely as in the transition from (2.11) to (3.5). The final result is, to the necessary approximation,

$$m_a\frac{\partial A_p{}^i}{\partial e_a}=\frac{e_b}{4\pi}\frac{(x_a{}^i-x_b{}^i)_p}{(r_{ab})_p{}^3}. \tag{4.28}$$

hence, (4.23) yields

$$\bar{m}_a=m_a. \tag{4.29}$$

This then is the required result for the case of a particle representable by a fluid-type tensor.

While (4.29) has been derived only through order one in m_a, terms involving products of m_a and m_b with masses in the "rest of the universe" have *not* been neglected. It may be true that nonzero terms of order $m_a{}^2$ will appear on the right side of (4.29), but these are not really relevant here. The whole purpose of this section and the calculation leading to (4.29) is to present an example of a relationship between active gravitational mass and a reasonably defined inertial mass that is independent of the rest of the universe. This was done using a unit of charge as a standard so that in these units, using time and charge, the locally measured Newtonian gravitational constant in general relativity is independent of the rest of the universe. This is consistent with Dicke's strong principle of equivalence.[4]

By now the reader is undoubtedly aware that all of the calculations leading to the coordinate-free results (3.2) or (4.29) could most conveniently have been done directly in a coordinate system in which the background metric has already been "transformed away." A study of this approach to the problem is contained in the next section.

V. SUMMARY AND GENERALIZATION

This section will be mainly concerned with investigating some of the consequences of the fact that in general relativity the entire gravitational interaction between masses is carried by the metric tensor which can be "transformed away" to any desired degree of accuracy over a sufficiently small neighborhood of any point. This fact leads naturally to the following definition relating a standard physical laboratory to a mathematical "coordinate patch." A locally almost Minkowskian coordinate system is one in which test particles of any velocity experience no observable acceleration when there is no matter or radiation present in the laboratory. The description of experiments done in a standard physical laboratory is assumed to correspond to the mathematical description given by such a coordinate system.

Using this definition, Dicke's[4] strong principle of equivalence can be defined as the assertion that as far as inertial and gravitational effects are concerned, the numerical content of experiments described in a locally almost Minkowskian coordinate system is independent of any characteristics of the mass distribution in the rest of the universe. It is important to realize that this is a definite extension of such results of the Eötvös experiment as generalized in the weak principle, i.e., the assertion that the acceleration of a test particle instantaneously at rest relative to a small gravitating body is independent of the mass of the test particle in the limit as this mass goes to zero. In other words, the Eötvös experiment suggests that the acceleration effects of an external gravitating body on a sufficiently

[10] This argument can be sketched as follows. Changing to proper acceleration and keeping second-order terms in charge would put (4.26) in the form $(m_a+\pi_a+e_a{}^2f_1+e_ae_bf_2+e_b{}^2f_3)g_1A_p{}^i=e_ae_bg_2{}^i+K(m_a+\pi_a)h^i$, where g_1 and $g_2{}^i$ are independent of e_a, e_b, and h^i is the contribution of charge to the derivative of the metric tensor. It is easily seen that $e_a\rightarrow 0$ and $e_b\rightarrow 0$ allows the neglect of f_1, f_2, and f_3 in (4.26). The $K(m_a+\pi_a)h^i$ term can be shown to be of the order $e_ae_bg_2{}^i$ times Km_a/r_{ab} and is thus negligible compared to $e_ae_bg_2{}^i$ in view of the assumption that r_{ab} while small is still much greater than Km_a.

small laboratory can be at least approximately eliminated by allowing the laboratory to fall "freely" since it seems to imply that all parts of the laboratory would fall with very little, if any, relative acceleration. However, it contains nothing to suggest that the only effect of the gravitating body on the laboratory is accelerative, which is the basis for the strong principle.

A sketch of an argument generalizing the results of Sec. IV and suggesting the validity of a strong principle in general relativity follows.

Consider a region having space-time dimensions, in arbitrary but fixed units, bounded by a number ϵ. This region is to represent the space contained in a laboratory in which standard experiments are to be performed. Let the matter tensor in the laboratory be represented by λT_L (here and in the following, to avoid unnecessary clutter, tensor indices will be suppressed when no confusion will arise), where λ is a positive number. Further, let the matter tensor for the rest of the universe be T_U and assume that $T_U=0$ within the laboratory, while $T_L=0$ outside it. The total matter tensor is thus $T_U+\lambda T_L$ everywhere. The purpose of the following discussion is then to show that under certain conditions the influence of the "rest of the universe" on real, proper experiments done in such a laboratory can be made arbitrarily small by making ϵ sufficiently small. The crux of the argument is the fact that the observable outcome of such experiments cannot depend on the purely mathematical choice of coordinate systems in which the calculations are performed.

To this end, let ρ (again suppressing indices) stand for all the matter variables other than the metric, ρ_L referring to matter in the laboratory, and ρ_U to all other matter. Thus, λT_L is a function of ρ_L and T_U is a function of ρ_U. Assume the variables satisfy "equations of motion"

$$f(\rho,g,g')=0, \tag{5.1}$$

where g' stands for all first derivatives of g. Further, let the metric, g, be written as the sum of two parts ${}^0g+\gamma(\lambda)$, with ${}^0g$ independent of λ and where $\lim\gamma(\lambda)=0$ as $\lambda\rightarrow 0$.

Let ${}^0\rho$ represent the functional form of ρ when $\lambda=0$. Hence, when there is no matter within the laboratory, $\lambda=0$, and ${}^0\rho$ and ${}^0g$ satisfy

$$f({}^0\rho,{}^0g,{}^0g')=0, \tag{5.2}$$

$$S({}^0g)-T_U({}^0\rho)=0 \quad (S^{\alpha\beta}\equiv R^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}R). \tag{5.3}$$

If ρ_{null} represents the form of ρ corresponding to the vacuum and η is the Minkowski metric, then it will be assumed that

$$f(\rho_{\text{null}},\eta,0)=0, \tag{5.4}$$

$$T_U(\rho_{\text{null}})=0. \tag{5.5}$$

The two most important assumptions will now be made. Within the space of the laboratory it is assumed that (1) ${}^0\rho=\rho_{\text{null}}$, and (2) the differences between ${}^0g+\gamma(\lambda)$ and η together with the first two derivatives of ${}^0g+\gamma(\lambda)$ go continuously to zero with ϵ and λ. The first assumption is simply that when $\lambda=0$ there are really no matter or fields within the laboratory. In other words, it ensures that when the tensor for matter in the laboratory, λT_L, is zero, the matter variables, ρ_L, actually correspond to the vacuum. This assumption is probably unnecessary for the ordinary descriptions of matter. The second assumption may seem strong in its requirement on the second derivatives of the metric. However, it will be used in the argument following Eq. (5.13).

Similarly, let ${}^0\rho_L$ and $\eta+{}^0\gamma$ be the matter variables and metric describing the situation inside the laboratory in the absence of any matter outside, i.e., when $\rho_U=\rho_{\text{null}}$. Thus, by definition,

$$f({}^0\rho_L,\eta+{}^0\gamma,{}^0\gamma')=0, \tag{5.6}$$

$$S(\eta+{}^0\gamma)-\lambda T_L({}^0\rho_L)=0. \tag{5.7}$$

Finally, the full field equations can be written

$$f(\rho,{}^0g+\gamma,{}^0g'+\gamma')=0, \tag{5.8}$$

$$S({}^0g+\gamma)-\lambda T_L(\rho_L)-T_U(\rho_U)=0. \tag{5.9}$$

In particular, within the laboratory,

$$f(\rho_L,{}^0g+\gamma,{}^0g'+\gamma')=0, \tag{5.10}$$

$$S({}^0g+\gamma)-\lambda T_L(\rho_L)=0. \tag{5.11}$$

However, by assumption, within the laboratory ${}^0g$ differs from η, and its first two derivatives from zero, only by numbers which go to zero as $\epsilon\rightarrow 0$. Thus (5.10) and (5.11) can be rewritten as

$$f(\rho_L,\eta+\gamma,\gamma')=H, \tag{5.12}$$

$$S(\eta+\gamma)-\lambda T_L(\rho_L)=E, \tag{5.13}$$

where $H\rightarrow 0$ as $\epsilon\rightarrow 0$ and $E\rightarrow 0$ as $\epsilon\rightarrow 0$. Notice that since S depends on the second derivatives of ${}^0g+\gamma$, it is sufficient that these vanish as $\epsilon\rightarrow 0$ for $E\rightarrow 0$ as $\epsilon\rightarrow 0$. Actually, this condition may not also be necessary, but this point is irrelevant to the main argument.

The final result is thus that the variables, ρ_L and $\eta+\gamma$, satisfy, within the laboratory, Eqs. (5.12) and (5.13) which differ from those, (5.6) and (5.7), satisfied by the corresponding variables in the absence of matter in the rest of the universe only by functions H and E which can be made arbitrarily small by making ϵ sufficiently small.

Thus, it seems reasonable to expect that for each λ, the solutions with matter in the rest of the universe, ${}^0g+\gamma$ and ρ_L, and those with matter only in the laboratory, $\eta+{}^0\gamma$ and ${}^0\rho_L$, can be brought arbitrarily close together by making the laboratory sufficiently small. Further, the outcome of proper, local experiments done in such a laboratory can depend only on the behavior of the metric and matter variables within it. Thus, the results of such experiments can be made as nearly independent of the matter in the rest of the universe as desired by making ϵ sufficiently small.

Of course, the definition of quantities to be measured[11] and local laws to be tested within the laboratory may require $\lambda \rightarrow 0$. It might then be thought that for λ small enough the effects of the matter in the rest of the universe would become comparable to those of matter within the laboratory, vitiating the above argument. To prevent this, a lower limit for λ is demanded. This limit could be determined by the lower bound of available experimental accuracy for the measurements requiring $\lambda \rightarrow 0$. That is, values of λ below this limit would not produce observable differences in measurements. For this fixed λ, ϵ can then be determined as above.

There are, however, other statements which might possibly be called "Mach's principles" which are valid in general relativity. For example, inertial and gravitational forces have a common formal origin in general relativity. Specifically, for a test particle of mass m and velocity w^β,

$$F^\mu \equiv -m\Gamma_{\alpha\beta}{}^\mu w^\alpha w^\beta \qquad (5.14)$$

might be identified with the gravitational force acting on m. On the other hand, this quantity transforms just as an inertial force should, i.e., in going to a relatively accelerated system, the acceleration enters F^μ linearly. For example, in a coordinate system rotating relatively to a Lorentz system in a flat space, F^μ as defined in (5.14) contains the centrifugal and Coriolis forces experienced by particles in this rotating system. Thus, F^μ might also be identified with "inertial force" acting on m. Inertial coordinate systems would then be those in which F^μ vanishes or equivalently, those in which "free" uncharged test particles are unaccelerated. This coincides with the definition of locally almost Minkowskian coordinate systems above. Another way of saying this is that the locally almost Minkowskian or inertial coordinate systems are those in which the total gravitational force vanishes.

If suitable boundary conditions could then be exhibited for a general type of universe, the Einstein equation would predict the over-all state of motion of inerital frames relative to the total mass distribution in the universe. This statement alone has been mentioned as a "Mach's principle."[12] However, once it is required that fundamental, standard experiments be done within such frames, the rest of the universe cannot, in general relativity, influence their results.

Another paper[13] will discuss modifications of general relativity violating the strong principle of equivalence by the introduction of a variable gravitational "constant" determined through field equations by the mass distribution in the universe.

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[11] For example, inertial mass. See Eq. (4.23) and the discussion following it.

[12] For a general discussion see F. A. E. Pirani, Helv. Phys. Acta, Suppl. IV, 198 (1956). Actually, Pirani's "Mach's principle" is stronger than that mentioned above. His requires that inertial systems be nonrotating relative to some average mass density in the universe.

[13] C. H. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).