Nonlinear Interaction of an Electromagnetic Wave with a Plasma Layer in the Presence of a Static Magnetic Field. II. Higher Harmonics and a Nonlinear Propagation Theory

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Following the theory developed in Part I, a discussion is given of the properties of the third- and fourth-harmonic waves generated within a plasma layer. The conversion loss for the hth harmonic is presented. Conditions for the validity of the small-signal analysis are derived. Also, a technique for using the harmonics as a means of measuring the plasma parameters is presented. A method is then given for reiterating the general solution for the field at the hth harmonic. This reiteration procedure extends the range of validity of the small-signal analysis and also allows the determination of a correction to the field at the fundamental frequency. This correction takes into account the effects of the nonlinear terms, in the Boltzmann transport equation, on propagation phenomena. A discussion is given of the corrected field as a function of the plasma parameters. Under certain conditions the correction can be as large as 50%.

1. INTRODUCTION

IN Part I (hereafter referred to as I) of this paper[1] the theory of electromagnetic wave propagation through an anisotropic ionized layer, including the effects of the nonlinear terms in the Boltzmann transport equation, was presented. The method of solution of the nonlinear equations involved an expansion of the dependent variables in the transport equation in a Fourier series in time. The differential equations describing wave propagation were then solved for the field at each frequency in the series, including the effects of the reflections from the boundaries of all the waves within the plasma layer. A solution in closed form was obtained, under small-signal conditions, for the field at the hth harmonic in the Fourier series. An iteration technique was employed in order to solve the equations and an approximation made such that the electric field at the hth harmonic only depended upon the electric fields at frequencies less than $h\omega$, where ω was the frequency of the incident wave. A discussion of the properties of the wave at the second-harmonic frequency was given.

In this paper a brief discussion is given of the properties of the waves at the third- and fourth-harmonic frequencies under the same set of assumptions as in I. The harmonics are discussed as a function of the magnitude of the external dc magnetic field, the steady-state electron density, the electron-neutral particle collision frequency, the field strength of the incident wave, and the thickness of the plasma layer. From these results the conversion loss per harmonic is deduced for the hth harmonic.

Then a method for reiterating the general solution for the field at the hth harmonic is presented. This reiteration procedure accounts for the fact that the field at a particular harmonic depends not only upon the fields at the lower harmonics but also on the fields at the higher harmonics. For example, the nonlinear terms involving products of the fields at 2ω and 4ω not only produce a wave at 6ω, but also a wave at 2ω. In I, the wave at 2ω was derived solely from products of waves at ω and the other contributions were neglected on the basis of the small-signal analysis. The effects of these additional waves are now taken into account by a reiteration procedure. A general equation for computing the corrected value of the electric field at the hth harmonic is given.

Of particular interest are the effects of the nonlinear terms on wave propagation at the fundamental frequency. These effects are contained within the general equation for the electric field, after the reiteration technique has been employed. In this paper the effects of the nonlinearities on propagation at frequency ω are discussed within the approximation that the electric field at the second-harmonic frequency is the only field that modifies appreciably the wave at frequency ω.

2. HIGHER HARMONICS

The theoretical model to be discussed is as follows. A wave at frequency ω is incident on the plasma layer from the left. The plasma layer is assumed to be of thickness d in the direction of propagation of the incident signal and infinite in all other directions. A uniform dc magnetic field is impressed upon the plasma layer in a direction normal to the direction of propagation. The plasma is assumed to be electrically neutral and of uniform electron density in the absence of electromagnetic forces. The motion of the ions, as well as any thermal forces, is neglected. Plane-wave solutions for the fields are examined under the assumption of a constant electron-neutral particle collision frequency. Due to the nonlinear nature of the Boltzmann transport equation a series of waves at harmonic frequencies of the fundamental wave is assumed to exist within the plasma layer. Refer to I for a detailed discussion of the assumptions and the method for solving the nonlinear equations.

In I, an equation [Eq. (I.10)] was given for the solution for the electric field of the hth harmonic as a function of the plasma parameters. Through the use of

[1] R. F. Whitmer and E. B. Barrett, Phys. Rev. **121**, 661 (1960).

Eq. (I.10), the power density in the hth harmonic can be found to be

$$P_h=\left(\frac{2\mu_0 e^2}{m^2 c\omega^2}\right)^{h-1} P_0{}^h Q_h\left(\frac{\omega_p}{\omega},\frac{\omega_c}{\omega},\frac{\nu}{\omega},\frac{\omega d}{c},h\right), \qquad (1)$$

where μ_0 is the permeability and c is the velocity of light in free space, e is the charge and m the mass of an electron, ω_p is the plasma frequency, ω_c is the electron cyclotron frequency, ν is the electron-neutral particle collision frequency, d is the thickness of the plasma layer, and ω is the frequency and P_0 the power density of the wave incident on the plasma layer. Q_h is a dimensionless function of the normalized plasma parameters and the harmonic number under consideration. The discussions in I were centered around the dimensionless function Q which, in the present notation, now becomes Q_2.

It can be seen that the power density in the hth harmonic is proportional to the incident power density to the hth power and inversely proportional to the $(h-1)$ power of ω^2. Q_h is the parameter which varies as a function of the electron density, the electron-neutral particle collision frequency, the magnitude of the external magnetic field, and the thickness of the plasma layer, as well as the harmonic under discussion. For a given value of the incident power and at a fixed frequency, the efficiency with which power is converted from the fundamental frequency to frequency $h\omega$ depends on Q_h. A brief discussion of Q_3 and Q_4 will be presented and from this deductions on Q_h can be made. All these discussions will be given for the typical case $\omega d/c=18.63$, since $\omega d/c$ is not a particularly important parameter in determining the harmonics.

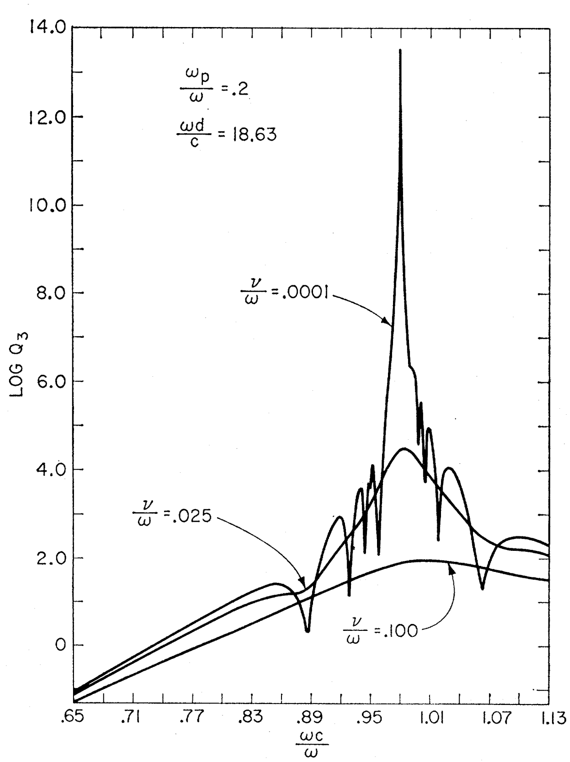

FIG. 1. Third-harmonic power vs ω_c/ω for $\omega_p/\omega=0.2$.

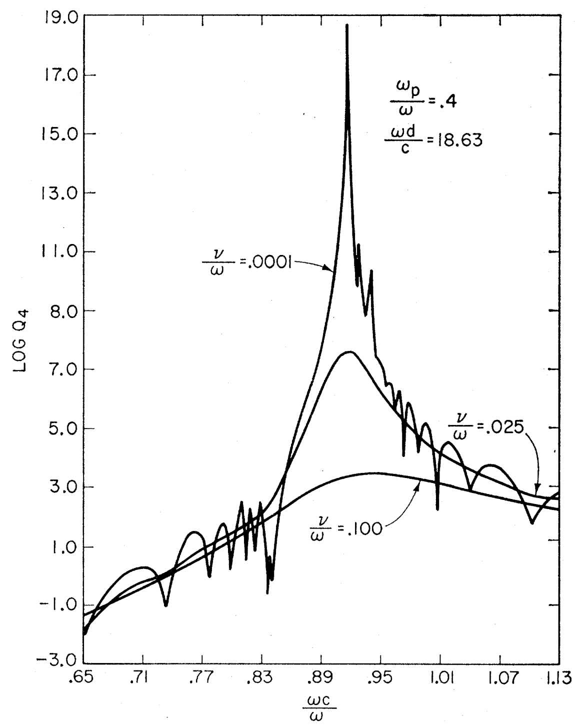

FIG. 2. Fourth-harmonic power vs ω_c/ω for $\omega_p/\omega=0.4$.

A plot of Q_3 vs ω_c/ω for $\omega_p/\omega=0.2$ is given in Fig. 1 for several values of ν/ω. A similar plot of Q_4 for $\omega_p/\omega=0.4$ is given in Fig. 2. In general appearance Q_2, Q_3, and Q_4 are very similar. There is one major peak in each of the Q's and the position of this peak shifts to lower values of ω_c/ω as ω_p/ω is increased. In each case the magnitude of the peak value decreases as ν/ω is increased. The smaller resonances about the main peak are due to internal reflections within the plasma layer and these disappear as ν/ω is increased. The values of ω_c/ω for which Q_2, Q_3, and Q_4 are a maximum are identical and, therefore, by induction from Q_2,

$$(\omega_c/\omega)Q_{h\ \max}=[1-(\omega_p/\omega)^2]^{\frac{1}{2}}+(\nu/\omega)^2. \qquad (2)$$

Equation (2) gives the correct value of ω_c/ω for maximum harmonic output to an accuracy of 5%, providing $\omega_p/\omega<0.8$, $\nu/\omega<0.2$, and $\omega d/c>2\pi$. This is well within the range of interest, since the harmonic output decreases rapidly outside of this range. By examining the width of the resonance lines at the half-power points, it is found that

$$(\Delta\omega_c/\omega)_{\frac{1}{2}}=(\nu/\omega)[1+e^{-(h-2)}]. \qquad (3)$$

This states that the linewidth of the second harmonic is $2\nu/\omega$, which is also approximately the width of the absorption line of the wave at the fundamental frequency. As the harmonic number increases the linewidth decreases, approaching the value ν/ω for $h \geq 4$.

Equations (2) and (3) indicate that the harmonics could be used to measure the electron density and collision frequency in a plasma. By measuring the linewidth, ν can be obtained; and by measuring ω_c/ω for peak harmonic power, the electron density can be obtained. At first thought, this scheme appears somewhat academic since the same measurements can be made on the fundamental. However, the absorption of the fundamental becomes very large near resonance and the peak of the absorption line is difficult to measure accurately. However, the second harmonic is a maximum in the region of interest and the measurement of the linewidth can be made accurately.

In the case of $\omega_c/\omega=0$, Q_2 and Q_4 are zero, while Q_3 is finite. In the absence of a dc magnetic field the waves at frequency 2ω and 4ω are purely longitudinal, or electrostatic, and therefore do not radiate. However, the interaction of a longitudinal wave at 2ω with the transverse wave at frequency ω results in an electromagnetic wave at frequency 3ω through the term n_2v_1. Therefore, $Q_3 \neq 0$ for $\omega_c/\omega=0$, although the magnitude of Q_3 for this case is much less than the peak magnitude in the presence of a dc magnetic field. This generation of a third harmonic was predicted by Margenau[2] and discussed recently by Rosen.[3] However, they did not obtain quantitative information on the harmonic as a function of the plasma parameters.

A plot of Q_3 as a function of ω_p/ω is given in Fig. 3 for the case of $\omega_c/\omega=0$. It can be seen that Q_3 has a maximum for $\omega_p/\omega=2.0$. This can be understood by the following argument.[4] In the case $\omega_c/\omega=0$ the

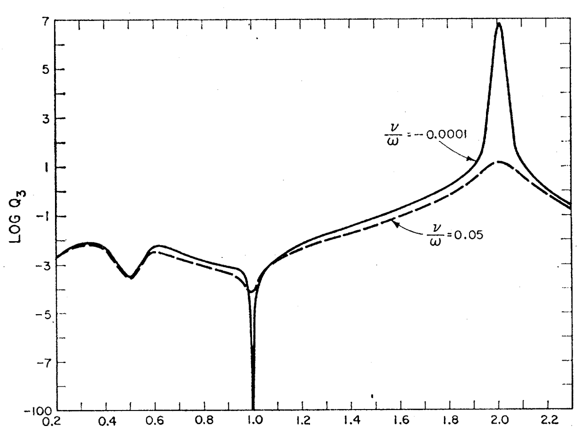

FIG. 3. Third-harmonic power vs ω_p/ω for $\omega_c/\omega=0$.

[2] H. Margenau and L. Hartman, Phys. Rev. **73**, 309 (1948).
[3] P. Rosen, Phys. Fluids **4**, 341 (1961).
[4] J. E. Hopson, (private communication).

Boltzmann transport equation becomes (assuming $\nu=0$)

$$\partial \mathbf{v}_t/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}_t=(e/m)(\mathbf{E}_t)+(e/m)(\mathbf{v}\times\mathbf{B})_t, \quad (4)$$

$$\partial v_x/\partial t+(\mathbf{v}\cdot\nabla)v_x=(e/m)(E_x+\hat{x}\cdot\mathbf{v}_t\times\mathbf{B}_t), \quad (5)$$

where the subscript t indicates the transverse component and x the longitudinal component of the wave. For a transverse wave $B_x \equiv 0$ and Eq. (4) can be solved exactly, even though it appears to be nonlinear, since

$$(\mathbf{v}\cdot\nabla)\mathbf{v}_t=(e/m)(\mathbf{v}\times\mathbf{B})_t. \quad (6)$$

For a longitudinal wave, $B\equiv 0$ and Maxwell's equations give

$$env_x+\partial D_x/\partial t=0. \quad (7)$$

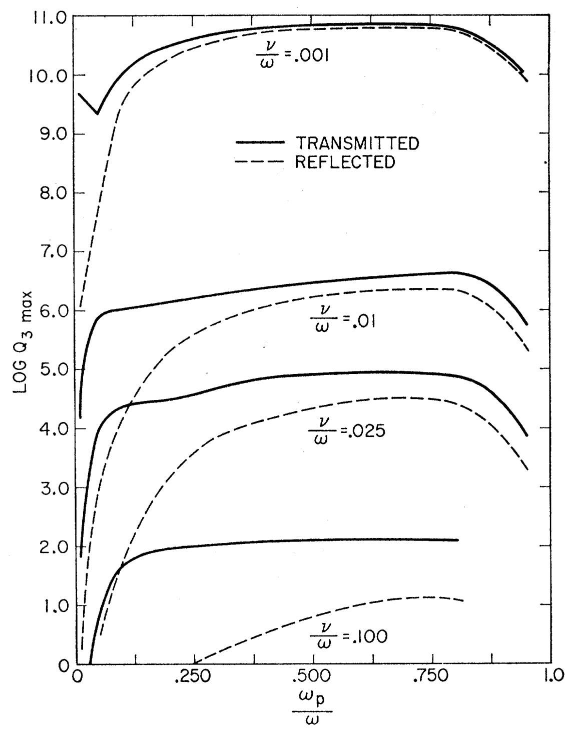

FIG. 4. Third-harmonic power at resonance vs ω_p/ω (dashed curves indicate reflected power).

After substituting Eq. (7) into Eq. (5) and linearizing with respect to E_x, Eq. (5) becomes

$$\partial^2 E_x/\partial t+\omega_p^2 E_x=-\omega_p^2(\mathbf{v}_t\times\mathbf{B}_t). \quad (8)$$

It can be seen that the transverse wave drives the longitudinal wave. Now let us examine a wave at frequency 2ω. In this case, Eq. (8) yields

$$E_{2x}=\omega_p^2(\mathbf{v}_{1t}\times\mathbf{B}_{1t})_x/(4\omega^2-\omega_p^2), \quad (9)$$

which indicates that E_{2x} is infinite (in the absence of collisions) when $\omega_p/\omega=2.0$. Recall that it is the transverse component of velocity, at frequency ω, multiplied by the electron density variation, at frequency 2ω, which produces the third harmonic. Equation (9) then indicates why the power in the third harmonic is a maximum at $\omega_p/\omega=2.0$ (when $\omega_c/\omega=0$) since the electron density is proportional to E_{2x}. The minimum at $\omega_p/\omega=1.0$ can also be explained since $v_{1t}=0$ at this point (for $\nu=0$). No explanation has been found for the dip when $\omega_p/\omega=0.5$, although it appears to be due to a boundary effect.

Returning to the case of a finite external magnetic field, $Q_{3\,\max}$ and $Q_{4\,\max}$ are plotted vs ω_p/ω in Figs. 4 and 5, respectively. The solid lines indicate the Q for the transmitted waves and are similar to the curves for $Q_{2\,\max}$. Through an examination of these curves it can be shown that

$$\log_{10}Q_{h\,\max}\approx(h-1.2)[-1.1-2.5\log_{10}(\nu/\omega)]. \qquad (10)$$

These results are valid to within 10% provided $0.1<\omega_p/\omega<0.8$, $\omega d/c>2\pi$, and $\nu/\omega>0.001$. For $\nu/\omega<0.001$, internal reflections within the plasma layer can modify $Q_{h\,\max}$.

The dashed curves in Figs. 4 and 5 are plots of $Q_{3\,\max}$ and $Q_{4\,\max}$ for the reflected power. It can be seen that the power emitted at the harmonic frequencies in the direction opposite to the direction of the incident wave is almost equal to the harmonic power emitted in the direction of the transmitted wave. This occurs because the moving electrons, which produce the harmonics, radiate in both the positive and the negative x directions. The detailed curves for the reflected harmonic powers are very similar to those for the transmitted powers plotted in Figs. 1 and 2 and will not be presented here (the only difference is a slight shift toward lower values of ω_c/ω for the peak in the reflected power for $\nu/\omega>0.01$).

The question of the convergence of the series used to solve the nonlinear equations still remains. The total power in all harmonics, and in the fundamental, must equal the incident power. That is,

$$P_0=TP_0+LP_0+RP_0+\sum_{h=2}^{\infty}(P_{hT}+P_{hR}+P_{hL}), \qquad (11)$$

where T is the transmission, R the reflection, and L the loss coefficients for the wave at frequency ω. From Figs. 4 and 5 it can be seen that P_{hR} is, at most, equal to P_{hT}. Also, P_{hL} must be small since the medium is practically transparent to waves at the higher frequencies. In the linear theory $T+R+L=1$; in the present theory, because of the small signal assumption, $T+R+L=1-\epsilon$, where ϵ is a small quantity. Therefore, by combining Eqs. (11), (10), and (1), one obtains

$$2q\sum_{h=2}^{\infty}(10\Omega P_0)^{h-1}<\epsilon, \qquad (12)$$

where

$$q=10^{0.2[1.1+2.5\log(\nu/\omega)]}, \qquad (13)$$

and

$$\Omega=2\mu_0e^2/m^2c\omega^2. \qquad (14)$$

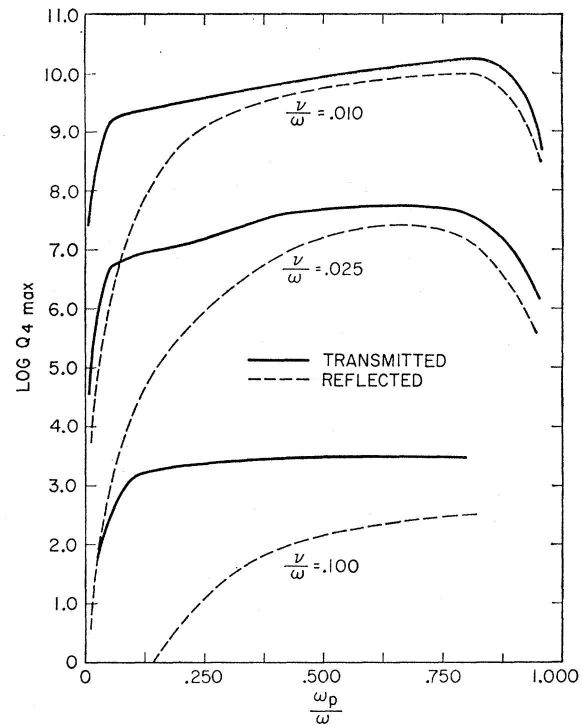

FIG. 5. Fourth-harmonic power at resonance vs ω_p/ω (dashed curves indicate reflected power).

The series on the left-hand side of Eq. (12) must converge for the theory to be valid; therefore, applying d'Alembert's test for convergence, one has

$$10\Omega P_0<1. \qquad (15)$$

This places a limit on P_0. Since Eq. (15) must hold, the series may be summed, and Eq. (12) becomes

$$P_0<\epsilon/10\Omega(2q+\epsilon). \qquad (16)$$

In the region of interest $0.1<q<1.0$ and if ϵ is assumed to be less than 0.01, then Eq. (16) becomes, approximately,

$$P_0<10^{-12}\omega^2\epsilon. \qquad (17)$$

Equation (17) is only a necessary condition on P_0 such that the Fourier series expansion is valid. The experimental results (to be discussed in a subsequent paper) indicate that $\epsilon<10^{-7}$ in the experimental arrangement.

3. NONLINEAR PROPAGATION THEORY

In I the theory of harmonic generation was derived on the assumption that the field at frequency $h\omega$ only depended upon fields at frequencies less than $h\omega$. Using this approximation to determine the field $\mathbf{E}_h$ at the hth harmonic, a reiteration technique will now be employed to obtain a corrected value for $\mathbf{E}_h$. This corrected field will account for the dependence of $\mathbf{E}_h$ on fields at frequencies greater than $h\omega$.

From Eqs. (I.6) and (I.8) the equation to be solved becomes, including terms at all frequencies,

$$[\nu-ih\omega-\boldsymbol{\omega}_c\times][\nabla\times\nabla\times\mathbf{E}_h'+(ih\omega/c)^2\mathbf{E}_h']-(ih\omega\omega_p^2/2)\mathbf{E}_h'\cong\frac{ih\omega\omega_p^2}{2c^2}\Bigg\{(1-\delta_{h1})\sum_{s=1}^{h-1}\Bigg[\mathbf{v}_s\times\mathbf{B}_{h-s}-\frac{m}{e}(\mathbf{v}_s\cdot\nabla)\mathbf{v}_{h-s}+\frac{m}{en_0}(\nu-ih\omega-\boldsymbol{\omega}_c\times)n_s\mathbf{v}_{h-s}\Bigg]+\sum_{s=1}^{\infty}[\mathbf{C}_{s+h,s}+\mathbf{C}_{s,s+h}{}^*]\Bigg\}, \quad (18)$$

where $\mathbf{E}_h'$ is the corrected field,

$$\mathbf{C}_{s+h,s}=\mathbf{v}_{s+h}\times\mathbf{B}_s{}^*-(m/e)(\mathbf{v}_{s+h}\cdot\nabla)\mathbf{v}_s{}^*+(m/en_0)(\nu-ih\omega-\boldsymbol{\omega}_c\times)n_{s+h}\mathbf{v}_s{}^*, \quad (19)$$

and $\mathbf{C}_{s,s+h}{}^*$ is obtained by reversing the subscripts and conjugate signs in (19). In I the C's were neglected since they involve fields at frequencies greater than $h\omega$ and it was assumed that these fields were negligible in comparison with the fields at frequencies less than $h\omega$. $\mathbf{E}_h'$ can be obtained by solving (18) after the terms on the right-hand side have been determined by means of the technique in I. Writing

$$\mathbf{E}_h'=\mathbf{E}_h+\mathbf{E}_h{}^c, \quad (20)$$

where $\mathbf{E}_h{}^c$ is the correction to be added to $\mathbf{E}_h$, then $\mathbf{E}_h{}^c$ is given by, following I,

$$\mathbf{E}_h{}^c=\sum_{a,b}\sum_{m,n}\sum_{\pm}\mathbf{E}_{mn}{}^{\pm}\times\prod_{ij}\exp[\pm i\mathbf{k}_{(q_i)_m}\cdot\mathbf{r}\pm i\mathbf{k}_{(p_j)_n}\cdot\mathbf{r}]; \quad (21)$$

where a, b are all pairs of numbers such that $a-b=h$, for $a,b>0$; m and n are the numbers of particular partitions of a and b, respectively, (arbitrarily ordered); and

$$\textstyle\sum_i (q_i)_m=a, \quad \sum_j (p_j)_n=b, \quad (22)$$

where q_i and p_j are numbers within the partitions m and n, respectively. The product is taken over all i, j and the sums over all $\pm$ combinations, then over all m, n, and then for all a, b. Although Eq. (21) appears to be somewhat complicated, it does provide a formula by means of which the solution for a particular case can be written down in closed form. Using (21) the boundary value problem for the plasma layer then can be solved exactly, as was done in I, and this discussion will not be repeated here.

As an example of the results one obtains for $\mathbf{E}_h{}^c$ from such a procedure, a reiteration has been performed for the case $h=1$. This is the case of most interest because it describes the effects of the nonlinear terms on propagation phenomena at the fundamental frequency. $\mathbf{E}_1$ is identical with the field obtained from the usual linearized equations[5] which describe propagation at frequency ω. $\mathbf{E}_1{}^c$ is then the correction to the linear propagation theory which accounts for the fact that harmonics are generated within the plasma layer. These waves, at the harmonic frequencies, interact and modify the wave at the fundamental frequency giving rise to $\mathbf{E}_1{}^c$.

Referring to Eq. (18), this equation can be rewritten as, for $h=1$,

$$L(\mathbf{E}_1')=(1/\omega)[G_1(\mathbf{E}_2\mathbf{E}_1{}^*)+G_2(\mathbf{E}_3\mathbf{E}_2{}^*)+G_3(\mathbf{E}_4\mathbf{E}_3{}^*)+\cdots], \quad (23)$$

where L is a differential operator and the G's are known functions of the uncorrected fields. Because of the small signal assumption used in I, G_1 is the dominant term on the right-hand side of Eq. (23). Therefore, in order to simplify the calculation, the reiteration to be discussed here will only include the effects of the second harmonic on propagation at frequency ω. Since $E_h\propto E_0{}^h/\omega^{h-1}$, where E_0 is the incident field strength, then from (23)

$$P_1'=P_0\sum_{n=0}^{\infty}A_n(P_0/\omega^2)^n \quad (24)$$

where P_1' is the corrected power density in the wave at frequency ω and A_n is a function of the plasma parameters.

Assuming

$$G_2=G_3=\cdots=G_n=0, \quad (25)$$

Eq. (24) then reduces to

$$P_1'=AP_0+BP_0{}^2/\omega^2+CP_0{}^3/\omega^4, \quad (26)$$

where A, B, and C are functions of the normalized plasma parameters. In fact, A is the usual transmission coefficient for a plasma layer derived from the linearized propagation equations.

Plots of B and C for the transmitted power at frequency ω are given in Figs. 6 and 7. It can be seen that B may be either positive or negative, whereas C is always positive. Both B and C have a minimum, whose width depends upon ν/ω, where Q_2 is a maximum and both B and C have a maximum immediately to the right (increasing ω_c/ω) of this minimum. Depending upon ν/ω there is a value of ω_c/ω below which B is always negative. For ω_c/ω approximately twice the value for which Q_2 is a maximum, B oscillates about zero and then remains negative for increasing values of ω_c/ω.

Equation (26) seems to indicate that, for fixed values of ω_p/ω, ν/ω, ω_c/ω, and $\omega d/c$, the correction due to the nonlinear effects increases as ω decreases, for a fixed value of P_0. This, however, is not correct because P_0 cannot remain fixed over a wide range of ω because of

[5] R. F. Whitmer, Microwave J. 2, 47 (1959).

the requirement for the convergence of the iteration technique given by Eq. (17). Equation (17) gives the maximum allowable value of P_0, and substituting this into (26) yields

$$P_1/AP_0<1+10^{-12}\omega^2\epsilon B/A+10^{-24}\omega^2\epsilon^2C/A. \quad (27)$$

Equation (27) indicates that the upper bound for the corrected power, within the assumptions of the analysis, is independent of ω over any large variation in ω. Assuming $\epsilon\approx10^{-2}$, then B/A must be of the order of 10^{14} and C/A of the order of 10^{28} for the correction terms to be of much importance. $(B/A)_{\max}$ and $(C/A)_{\max}$ are plotted vs ω_p/ω in Fig. 8. This figure indicates that the correction terms are important in the region of $\omega_c/\omega=1.0$ when $\omega_p/\omega<0.2$ and $\nu/\omega<0.005$. In this case the correction terms can modify the transmitted power by as much as 50%. Similar corrections exist for the reflected power. These results indicate that the effect of the nonlinear terms may be important when using propagation phenomena as a probe to measure the

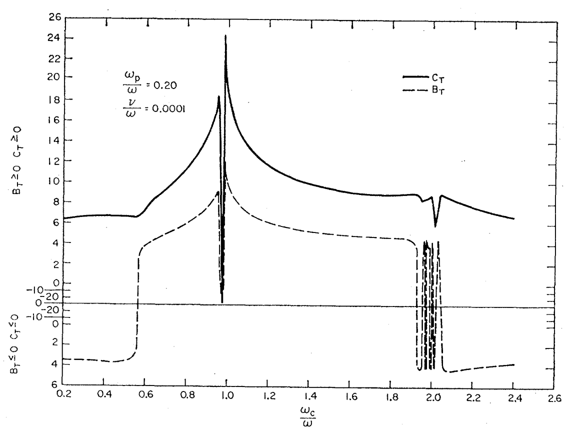

FIG. 6. Correction to transmitted power at frequency ω vs ω_c/ω for $\omega_p/\omega=0.2$.

plasma parameters and also in understanding propagation through the ionosphere.

4. SUMMARY

The Boltzmann transport equation, coupled with Maxwell's equations, has been solved, under a small-signal plane-wave assumption, for a plasma layer including the effects of the nonlinear terms in the equations. Employing a Fourier series expansion in time for all the dependent variables, a solution to the equations has been obtained, in closed form, for the wave at the hth harmonic of the Fourier series, including the effects of the reflections of each wave within the plasma layer. The second harmonic was discussed in detail in I. In this paper the third and fourth harmonics were discussed. The theory predicts a major peak in the power at any harmonic for a value of the dc magnetic field such that

$$\omega_c/\omega=[1-(\omega_p/\omega)^2]^{\frac{1}{2}}+(\nu/\omega)^2,$$

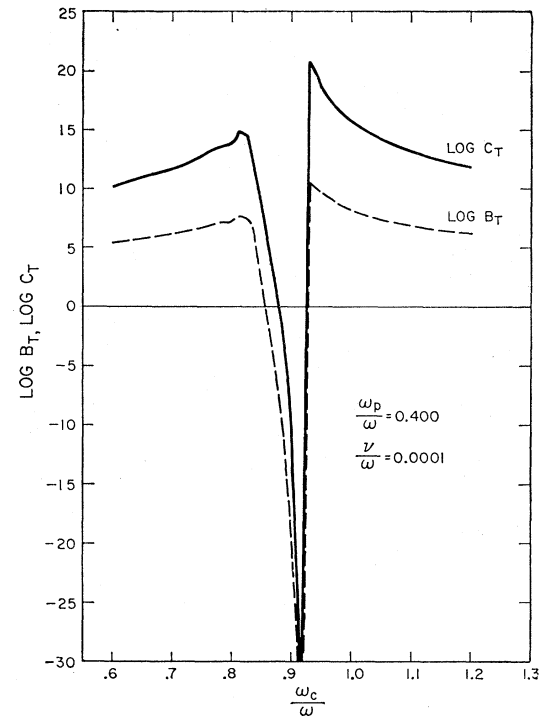

FIG. 7. Correction to transmitted power at frequency ω vs ω_c/ω for $\omega_p/\omega=0.4$.

and the width of the resonance line is proportional to ν/ω and h approaching ν/ω for $h\geq4$. Minor resonances occur on either side of the major peak because of the

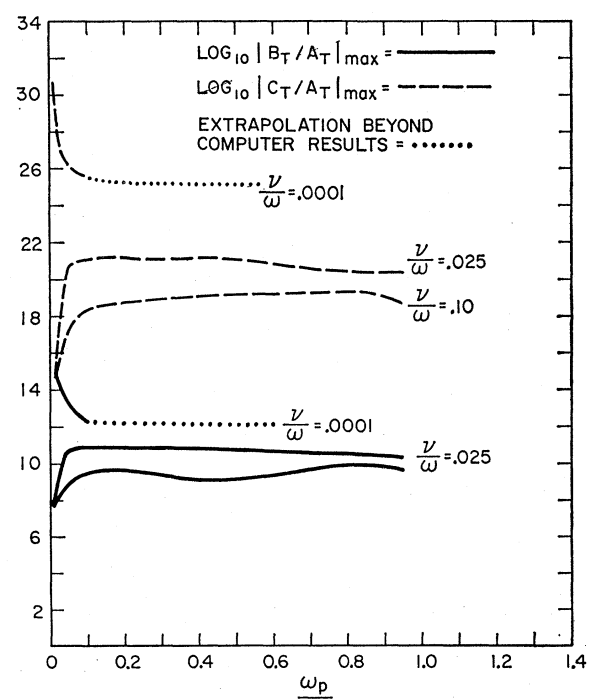

FIG. 8. Relative magnitude of the peak value of the correction terms vs ω_p/ω.

existence of standing waves within the plasma layer. The power density at the hth harmonic varies as the input power density to the hth power for fixed values of the normalized plasma parameters. The peak values of the third- and fourth-harmonic powers, $Q_{3\,\max}$ and $Q_{4\,\max}$, have been discussed, indicating that they are independent of ω_p/ω in the range $0.1<\omega_p/\omega<0.8$. Outside this range both decrease rapidly. The peak values also vary inversely with ν/ω. Similar statements hold for the harmonic powers reflected from the plasma layer.

The question of the convergence of the series used to solve the equations has been examined. A condition on P_0, the incident power density at frequency ω, is derived such that the small signal analysis is valid.

An analysis of the effects of the nonlinear terms on propagation at the incident frequency ω is then discussed. This is accomplished through a reiteration procedure, including only the effects of the second harmonic, which yields a correction to the equation for the power transmitted through the layer calculated from the linearized equations. The results indicate that the correction can be as much as 50% for $\omega_p/\omega<0.2$ and $\nu/\omega<0.005$. This may be of importance when using an electromagnetic wave to measure the properties of a plasma and also when considering ionospheric propagation phenomena.

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Thermodynamic Behavior of Liquid Helium-Three in Its Possible Superfluid Phase. I*

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The thermodynamic behavior of liquid He^3 in its possible superfluid phase is investigated by extending the methods of Brueckner *et al.* They suggest that such a correlated phase can exist at very low temperatures due to the fact that there exist attractive D-state interactions near the Fermi surface. The free energy and the energy gap of the system for D-state interactions corresponding to different pure azimuthal modes are calculated at different temperatures. It is found that $l=2$, $m=2$ and $l=2$, $m=1$ modes correspond to the lowest free energy of the system near the critical temperature. In the intermediate range of temperatures the free-energy curves for the two modes, when the computations are made numerically, come out to be very nearly the same. But actually it can be shown by an analytical method that they are identical. The $l=2$, $m=0$ mode yields a higher free energy for all temperatures less than the critical temperature. The mixing of modes is investigated near the critical temperature. Any linear combination of all the modes $l=2$, $m=0$, 1, -1, 2, and -2 does not seem to lead to a lower free energy than that of the $l=2$, $m=\pm2$, and $m=\pm1$ modes. The correlation lengths at different temperatures are also analyzed. The specific heat and entropy curves for the $l=2$, $m=2$ mode are given.

I. INTRODUCTION

RECENTLY, it has been suggested by Brueckner, Anderson, Morel, and Soda[1,2] and Emery and Sessler[3] that liquid He^3 may have a superfluid phase at very low temperatures. They extended the method of Bardeen *et al.*[4] to a system in which the interactions are represented by non-spherically-symmetric potentials and found that a fermion system such as He^3 can become superfluid due to the attractive interaction in the $l=2$ state very close to the Fermi surface.

In the above-mentioned papers, the total energy and the energy gap of the system for the ground state have been calculated. The transition temperature T_c as well as the discontinuity of the specific heat at T_c have been

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[1] K. A. Brueckner, T. Soda, P. W. Anderson and P. Morel, Phys. Rev. **118**, 1442 (1960).

[2] P. W. Anderson and P. Morel, Phys. Rev. Letters **5**, 136, 282 (1960).

[3] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 43 (1960).

[4] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957). Hereafter we refer to this as BCS.