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Chirality Conservation and Soft Pion Production*

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A formally γ_5-invariant system consisting of a Dirac field and a massless pseudoscalar field allows chirality conservation in the sense that its expectation value is a constant of motion. This leads to the consequence that in any reaction a change in the fermion chirality ($\sim$helicity$\times$velocity) is compensated for by the emission of a massless boson at zero energy, which can be expressed by a simple formula relating the radiative amplitude to the elastic amplitude. Assuming the pion-nucleon system to be γ_5-invariant when the pion mass can be neglected, the formula is applied to the processes $N+\pi \rightarrow N+\pi$ and $N+2\pi$. A reasonable agreement with experiment is obtained in a case dominated by the 3–3 resonance.

I. INTRODUCTION

THE observed equality of the vector coupling constant G_V in nuclear and muon β decays has led to the conserved current theory of Feynman and Gell-Mann,[1] according to which the vector part of the nuclear β-decay interaction is proportional to the total isotopic spin current. The nonrenormalization of G_V due to strong interactions is then guaranteed by the isospin conservation in strong interactions.

One expects further a close proportionality of the nucleon electromagnetic and β-decay matrix elements, including the Pauli magnetic term, and the pion β decay $\pi^{\pm} \rightarrow \pi^0$ would also proceed at the "universal" rate. Although these predictions are yet to be confirmed experimentally, there seems to be enough theoretical motivation for such speculations. For we would then be able to regard the weak interactions, like the electromagnetic interaction, as an agent which reveals the basic symmetries that might exist beneath the confusing effects of strong interactions.

As the nucleon β decay interaction also contains an axial vector part with a comparable strength ($G_A \sim 1.2 G_V$), several authors[2] have naturally tried to extend the principle by postulating axial vector current conservation or invariance under the so-called γ_5 transformation. In this case, however, it has not been found possible to guarantee the nonrenormalization of G_A by the axial vector conservation, and besides the conservation law seems to be only approximate under strong interactions. Nevertheless, it has led to one interesting result, namely the Goldberger-Treiman[3] relation between the Gamow-Teller constant G_A, the pion-nucleon constant, and the pion decay constant, which is known to be satisfied.

Recently one of us[4] has proposed a composite model of elementary particles based on an analogy with superconductivity. This model is built on the essential assumption that the Lagrangian describing the nucleons is invariant under the γ_5 transformation but that the physical vacuum state need not be so. As an interesting consequence, we observe that there exist nucleon-antinucleon bound states which behave as massless pseudoscalar mesons. The Goldberger-Treiman relation follows immediately from this if we identify them with the pions.

In the present paper we shall study another consequence of the axial vector conservation which can be used as an experimental test of the assumption. The main point is that for any reaction involving nucleons and mesons, the axial vector current conservation implies a close relation between the elastic amplitude and the "radiative" amplitude where an extra massless pion is emitted at zero energy.

Since the real pion has a finite mass, such a relation cannot actually be satisfied, but one may expect it to be approximately true at sufficiently high energies where the pion mass is negligible.

We shall first consider a simple model due to Nishijima and show how the above-mentioned relation is expected for a process involving the scattering of a nucleon by an external γ_5-invariant potential (Sec. II). We shall verify the relation explicitly in the lowest order perturbation (Sec. III), and then suggest a possible way of proving it in general (Sec. IV). Finally we shall discuss some experiments, in particular those involving pion-nucleon scattering, which would test the above-mentioned relation and hence the assumption of the (approximate) γ_5 invariance of strong interactions.

II. THE NISHIJIMA MODEL

In the actual β decay problem, the relevant symmetry to be associated with the axial vector part is the invariance under the $\gamma_5\times$isospin gauge transformation which acts on the nucleon (proton-neutron) field ψ as

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] S. Bludman, Nuovo cimento **9**, 433 (1958); F. Gürsey, Nuovo cimento **16**, 230 (1960); Ann. Phys. **12**, 91 (1961); M. Gell-Mann and M. Lévy, Nuovo cimento **16**, 705 (1960); J. Bernstein, M. Gell-Mann and L. Michel, Nuovo cimento **16**, 560 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo cimento **17**, 757 (1960); Chou Kang-Chao, J. Exptl. and Theor. Phys. (U.S.S.R.) **39**, 703 (1960) [Soviet Phys. JETP **12**, 492 (1961)]; Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[3] M. L. Goldberger and S. M. Treiman, Phys. Rev. **111**, 354 (1958).
[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

follows:

$$\psi(x) \to \exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}\gamma_5)\psi(x), \tag{2.1}$$

where $\boldsymbol{\alpha}$ is an arbitrary vector in isospace.

In this and following sections we shall study a simplified model proposed by Nishijima[5] because there would be no loss of the essential features.

The system consists of a single nucleon field ψ and a massless neutral pseudoscalar field ϕ ("pion") coupled through

$$L_{\text{int}} = -m\bar{\psi}\exp[(ig/m)\gamma_5\phi]\psi, \tag{2.2}$$

which also incorporates the (bare) mass term for the nucleon. By expanding the exponential, we recognize that the first two terms are the mass and the ordinary meson-nucleon coupling terms. (It is also clear that with the transformation $\psi \to \exp[-(ig/2m)\gamma_5\phi]\psi$, the theory is equivalent to the simple derivative coupling model. We prefer the Nishijima representation because it brings in the nucleon γ_5 transformation explicitly.) In addition, we shall introduce an external vector (or axial vector) potential $V_\mu(A_\mu)$,

$$L_{\text{ext}} = i\bar{\psi}\gamma_\mu\psi V_\mu \quad (\text{or } i\bar{\psi}\gamma_\mu\gamma_5\psi A_\mu). \tag{2.2$'$}$$

The entire Lagrangian is invariant under the γ_5 transformation,

$$\psi \to e^{i\alpha\gamma_5}\psi, \tag{2.3a}$$

and

$$\phi \to \phi - (2m/g)\alpha, \tag{2.3b}$$

for constant α. Of course it is essential for the invariance that the pion is massless. One easily verifies the axial vector current conservation which follows from the aforementioned invariance:

$$\begin{aligned} &\partial_\mu j_\mu(x) = 0, \\ &j_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi - (2m/g)\partial_\mu\phi. \end{aligned} \tag{2.4}$$

An important feature of the theory is that the existence of the "pion" field is necessary to preserve the γ_5 invariance in the presence of finite nucleon mass. In this respect it is similar to the model proposed by Nambu and Jona-Lasinio.[4] There the pion field and finite bare nucleon mass are not assumed, but, in the end, we find that, if the nucleon has a finite observed mass, it must also be accompanied by a massless pion field which is to be interpreted as nucleon-antinucleon bound states, or a collective excitation of such pairs. In this case, the conserved current j_μ is simply

$$j_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi, \tag{2.5}$$

which, however, implicitly contains the pion contribution.

Let us now turn to the meaning of the conservation law associated with Eq. (2.4). We call this conserved quantity the chirality χ, defined by

$$\chi = -i\int j_4 d^3x = \int \bar{\psi}\gamma_4\gamma_5\psi d^3x + (2im/g)\int \partial_4\phi d^3x = \chi_N + \chi_\pi. \tag{2.6}$$

Equation (2.4) shows that χ is a constant of motion, i.e., a conserved quantity. However, we can easily see that a one-nucleon state or a one-meson state is not an eigenstate of χ. As was analyzed in the model of Nambu and Jona-Lasinio, this seemingly paradoxical situation is related to the fact that the γ_5 transformation generated by $\exp[i\chi\alpha]$ is not a proper operation in the Hilbert space of real particles, but it carries a Hilbert space into another which is orthogonal to it.

Since χ is thus not diagonal, we shall, in the following, work with the expectation value $\langle\chi\rangle$, which should be conserved in any reaction, i.e.,

$$\langle\alpha^{\text{in}}|\chi|\alpha^{\text{in}}\rangle = \langle\alpha^{\text{out}}|\chi|\alpha^{\text{out}}\rangle.$$

For a one-nucleon state of momentum p, in particular, we have

$$\langle\chi\rangle = Z\bar{u}_p\gamma_4\gamma_5 u_p = -Zhv_p, \tag{2.7}$$

where u_p is the Dirac spinor normalized to $\bar{u}u = m/E$, h the helicity $\langle\boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|\rangle$, v_p the velocity $p/(p^2+m^2)^{\frac{1}{2}}$, and Z is a renormalization constant. For convenience, we can replace χ by $Z^{-1}\chi$ and call it chirality. Then Z will drop out in Eq. (2.7). For a one-pion state, we have, naturally, $\langle\chi\rangle = 0$.

Let us next consider the scattering of a nucleon by the external potentials. For a static potential, the velocity will not change, so that one might conclude that the helicity remains unchanged: $h_i = h_f$. This, however, is clearly not the case as one can easily check by perturbation calculation. The contradiction is resolved by noting that any scattering can always be accompanied by emission of pions which are massless. The final asymptotic state is then a linear combination.

$$|f\rangle = C_0|N\rangle + C_1|N+\pi\rangle + C_2|N+2\pi\rangle + \cdots, \tag{2.8}$$

and the expectation value $\langle\chi\rangle_f$ must be taken with respect to this complete amplitude. From Eq. (2.6) we further recognize that $\langle\chi\rangle_f$ will have contributions both diagonal and nondiagonal in the decomposition (2.8), the latter being between states $|N+n\pi\rangle$ and $|N+(n\pm1)\pi\rangle$ differing by one zero-energy meson (spurion). We are led to the conclusion that taking the expectation value is essential in interpreting the conservation of chirality. In this sense, we can call it a weak conservation in contrast to the usual "strong" conservation such as charge conservation.

It is clear, furthermore, that one can distinguish various degrees of "weakness" depending on the number of final state degrees of freedom over which the expectation value $\langle\chi\rangle_f$ must be taken in order to satisfy

[5] K. Nishijima, Nuovo cimento **11**, 698 (1959).

$\langle\chi\rangle_f = \langle\chi\rangle_i$. The weakest case would involve a summation over all spins, scattering angles, and many-pion production processes. A less weak case might involve only a spin summation and up to one-spurion emission processes. In the following section we shall verify the chirality conservation by perturbation calculation and show that the stronger type of weak conservation (which we might call detailed weak conservation) can sometimes be valid.

III. PERTURBATION CALCULATION

We shall consider the scattering of a nucleon by an external potential in the Born approximation and check the chirality conservation in the lowest order in the meson coupling constant g. For simplicity, the potential will be assumed to be of the vector type

$$L_V = i\bar{\psi}\gamma_\mu\psi V_\mu,$$

where V_μ is static and has an inversion symmetry: $V(\mathbf{x}) = V(-\mathbf{x})$. (Of course one can work with an axial vector potential equally well.) We first note that the initial nucleon state can be specified by means of the covariant projection operator

$$\begin{aligned} P(p,n) &= \Lambda(p)S(n) = S(n)\Lambda(p), \\ \Lambda(p) &= (m - i\gamma\cdot p)/2E_p, \quad [\Lambda^2 = (m/E_p)\Lambda], \\ S(n) &= (1 + i\gamma\cdot n\gamma_5)/2. \end{aligned} \tag{3.1}$$

Here n is the covariant polarization vector which reduces to $(\mathbf{n}',0)$ in the nucleon rest system. In the laboratory frame one has $n = (\mathbf{n}, n_0)$ where

$$\mathbf{n} = \mathbf{n}' + \mathbf{p}(\mathbf{p}\cdot\mathbf{n}')/m(E+m), \quad n_0 = \mathbf{p}\cdot\mathbf{n}'/m. \tag{3.2}$$

Naturally

$$n\cdot p = 0, \quad n^2 = 1. \tag{3.3}$$

The initial chirality of the nucleon is then given by

$$\begin{aligned} \chi_i &= \bar{u}_i\gamma_4\gamma_5 u_i = \mathrm{Tr}[\gamma_4\gamma_5 P(p,n)] \\ &= n_0 m/E_p = \mathbf{n}'\cdot\mathbf{v}. \end{aligned} \tag{3.4}$$

It is clear from Eq. (2.6) that in our present approximation (which is the zeroth order in g) only the lowest order elastic and one-pion inelastic processes need be considered. The corresponding diagrams are shown in Fig. 1. These amplitudes are given by

$$\begin{aligned} M_{\mathrm{el}}(p',p) &= i\gamma\cdot V(q), \quad q = p' - p \\ M_{\mathrm{rad}}(p',p;k) &= i\gamma\cdot V(q+k)S(p-k)ig\gamma_5 \\ &\quad + ig\gamma_5 S(p'-k)i\gamma\cdot V(q-k). \end{aligned} \tag{3.5}$$

The chirality of the scattered wave at a given angle may be evaluated with the aid of the formula

$$\begin{aligned} \langle O\rangle_f &= [\bar{u}_i\bar{M}\Lambda(p')O\Lambda(p')Mu_i]/[\bar{u}_i\bar{M}\Lambda(p')Mu_i] \\ &= \mathrm{Tr}[\bar{M}\Lambda(p')OMP(p,n)]/\mathrm{Tr}[\bar{M}\Lambda(p')MP(p,n)], \end{aligned}$$

where O is a Dirac matrix and $\bar{M} = \gamma_4 M^\dagger\gamma_4$. Adapting

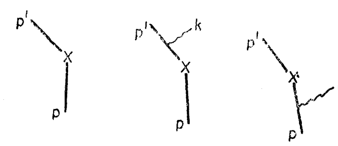

FIG. 1. Lowest order diagrams to be considered for chirality conservation in potential scattering. The point × is where the potential acts.

this to the present case, we get[6]

$$\begin{aligned} \langle\chi\rangle_f = \{&\mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')\chi_N'\Lambda(p')M_{\mathrm{el}}P(p,n)] \\ &+ \mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')\chi_\pi'\Lambda(p')M_{\mathrm{rad}}P(p,n)] \\ &+ \mathrm{Tr}[\bar{M}_{\mathrm{rad}}\Lambda(p')\chi_\pi'^\dagger\Lambda(p')M_{\mathrm{el}}P(p,n)]\} \\ &\times\{\mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')M_{\mathrm{el}}P(p,n)]\}^{-1}, \end{aligned}$$

where

$$\begin{aligned} \chi_N' &= \gamma_4\gamma_5, \\ \chi_\pi' &= (2m/g)\gamma_4\left\langle 0\left|\int \dot{\phi}d^3x\,\phi(y)\right|0\right\rangle = -(im/g)\gamma_4. \end{aligned} \tag{3.6}$$

M_{rad} does not contribute to the denominator because of its vanishing weight. The elastic contribution can be easily evaluated. We find

$$\begin{aligned} &\mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')\chi_N'\Lambda(p')M_{\mathrm{el}}P(p,n)] \\ &\quad = (-im/2E^3)[V\cdot V(m^2+p\cdot p') - 2p\cdot Vp'\cdot V]n_4 \\ &\qquad + (im/E^3)[V\cdot n(m^2+p\cdot p') - p\cdot Vp'\cdot n]V_4, \\ &\mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')M_{\mathrm{el}}P(p,n)] \\ &\quad = (1/2E^2)[V\cdot V(m^2+p\cdot p') - 2p\cdot Vp'\cdot V], \quad V_\mu \equiv V_\mu(q), \end{aligned}$$

so that

$$\begin{aligned} \langle\chi_N\rangle_f = (m/E)n_0& \\ &- (2m/E)V_0\frac{V\cdot n(m^2+p\cdot p') - p\cdot Vp'\cdot n}{V\cdot V(m^2+p\cdot p') - 2p\cdot Vp'\cdot V}. \end{aligned} \tag{3.7}$$

Here we have made use of the previous assumptions about V, which means that $E_f = E_i = E$, and the Fourier transform $V_\mu(\mathbf{q})$ is a real vector

$$\begin{aligned} V_\mu(\mathbf{q}) &= V_\mu(-\mathbf{q}) = \eta_\mu V_\mu^*(\mathbf{q}), \\ \eta_\mu &= 1,\ \mu = 1, 2, 3; \quad \eta_\mu = -1,\ \mu = 4. \end{aligned}$$

For a general V_μ, the result is more complicated.

As for the inelastic contribution, we first note that although the matrix element of $\chi_\pi \propto \int\partial_4\phi d^3x$ vanishes like $k_0/(2k_0)^{\frac{1}{2}}$ as $k_0 \to 0$, the inelastic amplitude contains a factor $1/(2k_0)^{\frac{1}{2}}$ which will cancel the former. Thus there is no trouble in this respect. A more serious difficulty is that the results of the calculation depend on the way the limit $k=0$ is defined. In fact we get different answers depending on whether (a) we approach the limit $k=0$ staying on the zero-mass shell ($k_0 = |\mathbf{k}|$), or (b) first we allow a finite mass μ_0, and go to the limit $\mathbf{k}\to 0$, $k_0 = \mu_0 \to 0$ successively.

We shall show below that the second procedure gives the correct result. Note also that this is the more

[6] Note that χ_π acts as unity operator for the nucleon, which means $O = \gamma_4$. Hence the form of χ_π' below.

appropriate one for application to the real pion problem where μ_0 is indeed finite.

Accordingly we calculate the χ_π part thus:

$$\mathrm{Tr}[\bar{M}_{\mathrm{el}}\Lambda(p')\chi_\pi'\Lambda(p')M_{\mathrm{rad}}P(p,n)]$$

$$= \lim_{\mathbf{k}\to 0,\mu_0\to 0} m\left\{\int \mathrm{Tr}\left[\gamma\cdot V\frac{m-i\gamma\cdot p'}{2E}\gamma_5\frac{m-i\gamma\cdot(p'+k)}{(p'+k)^2+m^2}\gamma\right.\right.$$

$$\left.\cdot V\frac{m-i\gamma\cdot p}{2E}\frac{1+i\gamma\cdot n\gamma_5}{2}\right]k_0\delta(k^2+\mu_0^2)\theta(k_0)dk_0$$

$$+\int \mathrm{Tr}\left[\gamma\cdot V\frac{m-i\gamma\cdot p'}{2E}\gamma\cdot V\frac{m-i\gamma\cdot(p-k)}{(p-k)^2+m^2}\gamma_5\frac{m-i\gamma\cdot p}{2E}\right.$$

$$\left.\left.\times\frac{1+i\gamma\cdot n\gamma_5}{2}\right]k_0\delta(k^2+\mu_0^2)\theta(k_0)dk_0\right\},$$

which becomes

$$-(m/2E^3)[V\cdot n(m^2+p\cdot p')-p\cdot Vp'\cdot n]V_0.$$

The other interference term in Eq. (3.6) gives an equal contribution so that

$$\langle\chi_\pi\rangle_f=\frac{2m}{E}V_0\frac{V\cdot n(m^2+p\cdot p')-p\cdot Vp'\cdot n}{V\cdot V(m^2+p\cdot p')-2p\cdot Vp'\cdot V}. \quad (3.8)$$

Combining Eqs. (3.7) and (3.8) we get

$$\langle\chi\rangle_f=\langle\chi_N\rangle_f+\langle\chi_\pi\rangle_f=(m/E)n_0=\langle\chi\rangle_i. \quad (3.9)$$

It is interesting to see how the two contributions to $\langle\chi\rangle_f$ add up in the nonrelativistic approximation. The chirality of the nucleon is in this case

$$\langle\chi_N\rangle=-\langle\boldsymbol{\sigma}\rangle_i\cdot\mathbf{p}/m. \quad (3.10)$$

The scattering takes place through V_4 which does not flip the spin but changes the direction of motion so that, after the scattering, we have

$$\langle\chi_N\rangle_f=-\langle\boldsymbol{\sigma}\rangle_i\cdot\mathbf{p}'/m. \quad (3.11)$$

The difference is

$$\Delta\langle\chi_N\rangle=-\langle\boldsymbol{\sigma}\rangle_i\cdot\mathbf{q}/m. \quad (3.12)$$

On the other hand, the above-mentioned limiting procedure for $\langle\chi_\pi\rangle$ corresponds to the S-wave pion production process which goes through the negative energy states, and this is easily seen to cancel $\Delta\langle\chi_N\rangle$.

IV. GENERAL PROOF

The previous particular example shows that the weak conservation of chirality holds in detail, i.e., if summed only over the final nucleon spins, and over elastic and one-pion bremsstrahlung processes at a fixed scattering angle. This is a stronger result than the general statement $\langle\chi\rangle_i=\langle\chi\rangle_f$, and is due to the Born approximation and the special assumption made about V_μ. (In fact, for a completely general static potential V_μ, we do not obtain conservation unless we sum over all the scattered and unscattered waves.)

We expect that the chirality conservation in our sense will hold true to all orders in the pion coupling g, where many-pion processes will also come in. But the calculation is made difficult because of the self-energy effects. In the following, we will therefore try to derive a general relation which follows from the γ_5 invariance and certain other assumptions and is expressed in terms of directly observable (renormalized) quantities.

We assume that there exists a conserved quantity called chirality $\chi=-i\int\chi_4 d^3x$ $(\partial_\mu\chi_\mu=0)$. This means that

$$\chi^{\mathrm{in}}=\chi^{\mathrm{out}}=S^{-1}\chi^{\mathrm{in}}S,$$

where S is the S matrix. We rewrite it as

$$S\chi^{\mathrm{in}}-\chi^{\mathrm{in}}S=0. \quad (4.1)$$

We will further assume, in accordance with the previous discussion, that χ consists of the nucleon part χ_N and the pion part $\chi_\pi=\lambda\int\dot{\phi}d^3x$. Asymptotically, χ_N will be an operator that does not change the number of pions, whereas χ_π, being linear in the pion field, will lead to the absorption or emission of a zero-energy pion. Accordingly Eq. (4.1) becomes

$$S\chi_N^{\mathrm{in}}-\chi_N^{\mathrm{in}}S=-S\chi_\pi^{\mathrm{in}}+\chi_\pi^{\mathrm{in}}S.$$

Let us apply this to the case of the potential scattering. Taking the matrix element between elastic states, we get

$$\begin{aligned} i\langle p'|M_{\mathrm{el}}\chi_N^{\mathrm{in}}-\chi_N^{\mathrm{in}}M_{\mathrm{el}}|p\rangle &=-\langle p'|S\chi_\pi^{\mathrm{in}}-\chi_\pi^{\mathrm{in}}S|p\rangle \\ &=\textstyle\sum_k[-\langle p'|S|pk\rangle\langle k|\chi_\pi|0\rangle \\ &\quad+\langle 0|\chi_\pi|k\rangle\langle p'k|S|p\rangle], \end{aligned} \quad (4.2)$$

because χ_π^{in} results only in the creation or absorption of a pion. Furthermore,

$$\begin{aligned}\langle p'|S|p,0\rangle=\langle p',0|S|p\rangle&=[i/(2k_0)^{\frac{1}{2}}]\langle p'|k^2\phi(k)|p\rangle|_{k=0}\\ &\equiv[i/(2k_0)^{\frac{1}{2}}]M_{\mathrm{rad}}(p',p;k)|_{k=0},\end{aligned}$$

and

$$\langle k|\chi_\pi|0\rangle=-\langle 0|\chi_\pi|k\rangle=\lambda i(k_0/2)^{\frac{1}{2}}\delta(\mathbf{k}). \quad (4.3)$$

Equation (4.2) thus may be written

$$\chi_N M_{\mathrm{el}}(p',p)-M_{\mathrm{el}}(p',p)\chi_N=i\lambda M_{\mathrm{rad}}(p',p;0). \quad (4.4)$$

If χ is so normalized that $\chi_N=\gamma_4\gamma_5$ for a free nucleon, then the continuity equation

$$\partial_\mu\chi_{N\mu}-\lambda\Box\phi=0,$$

which follows from χ conservation, means that $1/\lambda$ is more or less the conventional pion coupling constant

$$1/\lambda=f=g/2m. \quad (4.5)$$

It is not proven, however, that this agrees with the coupling constant defined in the dispersion theory. For the time being, we assume it to be the case.

Equation (4.4) represents a general relation between M_{el} and the one-pion emission amplitude M_{rad}. The origin of such a simple relation is easily traced back to our starting assumption about the asymptotic behavior of χ. From the above derivation, it is also clear that a similar relation will exist for any reaction amplitude M and the accompanying inelastic amplitude resulting in the emission of an extra zero-energy meson.

We shall next bring the relation (4.4) into a more explicit form. In the covariant form, M is usually defined without the projection operator Λ, so that the left-hand side of Eq. (4.4) should be written

$$\gamma_4\gamma_5\Lambda(p')M_{\rm el}(p',p)-M_{\rm el}(p',p)\Lambda(p)\gamma_4\gamma_5.$$

Observing that

$$\gamma_4\gamma_5\Lambda(p)-\Lambda(p)\gamma_4\gamma_5=-\gamma_5,$$

Eq. (4.4) becomes

$$iM_{\rm rad}(p',p;0)=(g/2m)[(\gamma_4\gamma_5 m/E-\gamma_5)M_{\rm el}(p',p)+M_{\rm el}(p',p)(\gamma_5\gamma_4 m/E-\gamma_5)]. \quad (4.6)$$

If we use the positive energy two-component spinors, the above equation simplifies to

$$iM_{\rm rad}(p',p;0)=-\frac{g}{2m}\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E}M_{\rm el}(p',p)-M_{\rm el}(p',p)\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}\right]. \quad (4.7)$$

In this form the meaning of the relation is easy to understand. Suppose the initial nucleon is in an eigenstate of helicity: $(\sigma\cdot p/E)u_p=\pm vu_p$. The helicity change in the elastically scattered state p' is

$$\left(\bar{u}_pM_{\rm el}^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E}M_{\rm el}u_p\right)\Big/(\bar{u}_pM_{\rm el}^\dagger M_{\rm el}u_p)-\left(\bar{u}_p\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}u_p\right)$$
$$=\left(\bar{u}_pM_{\rm el}^\dagger\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E}M_{\rm el}-M_{\rm el}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}\right]u_p\right)\Big/(\bar{u}_pM_{\rm el}^\dagger M_{\rm el}u_p)$$
$$=-i\lambda(\bar{u}_pM_{\rm el}^\dagger M_{\rm rad}u_p)/(\bar{u}_pM_{\rm el}^\dagger M_{\rm el}u_p).$$

The last expression just corresponds to the interference term $\langle\chi_\pi\rangle$ in the final state with a particular momentum p'. This shows that the weak chirality conservation holds *in detail*, i.e., after summing only over the final nucleon spins, and over elastic and associated inelastic amplitudes *provided that* the initial nucleon is in a helicity eigenstate. Otherwise a more general weak conservation will prevail in general.

V. $\gamma_5\times$ ISOSPIN INVARIANCE AND THE EXPERIMENTAL TEST

The results obtained in previous sections can be generalized to the $\gamma_5\times$ isospin gauge transformation [Eq. (2.1)]. Models that possess this invariance have been considered by Gürsey, Gell-Mann, *et al.*[2] and Nambu and Jona-Lasinio.[4] In this case there are three conserved quantities χ^i, $i=1, 2, 3$, corresponding to the three components of the isospin. The simplest example having such conservation is the conventional meson theory with derivative coupling. The results of Sec. IV can be directly taken over if we replace χ_N by $\chi_N{}^i=\chi_N\tau^i$, and χ_π by $\chi_\pi{}^i=\lambda\int\dot{\phi}^i d^3x$.

In some of the models, the expression for χ is actually more complicated. In general it contains nonlinear terms in the pion field, and terms involving a neutral scalar meson field can also occur. We may assume, however, that these additional terms do not contribute to the asymptotic values $\langle\chi^{\rm in}\rangle$ and $\langle\chi^{\rm out}\rangle$ for the following reason. First, the contribution from non-linear terms $\sim\phi(x)^n$, $n>2$, which involves creation and annihilation of n mesons in the neighborhood of point x, will tend to zero when, before or after the scattering, the particles are well separated from each other, just as the interaction energy is supposed to vanish in this asymptotic region. As for the neutral scalar field, a meson of this kind (σ meson) may exist in nature, but it would be, in any case, quite massive and unstable. We can therefore exclude σ from the fields that contribute to $\chi^{\rm in}$ and $\chi^{\rm out}$.

With this observation, Eqs. (4.6) and (4.7) are replaced in the present case by

$$iM_{\rm rad}{}^i=(g/2m)[(\gamma_4\gamma_5 m/E-\gamma_5)\tau^iM_{\rm el}+M_{\rm el}\tau^i(\gamma_5\gamma_4 m/E-\gamma_5)], \quad (5.1)$$

$$iM_{\rm rad}{}^i=-(g/2m)[(\boldsymbol{\sigma}\cdot\mathbf{p}'/E)\tau^iM_{\rm el}-M_{\rm el}\tau^i(\boldsymbol{\sigma}\cdot\mathbf{p}/E)]. \quad (5.2)$$

It is convenient to quantize the spins of the initial and final nucleons along their own direction of motion. Equation (5.2) then reduces simply to

$$iM_{\rm rad}{}^{h'h,i}=-(gv_p/2m)[h'\tau^iM_{\rm el}{}^{h'h}-M_{\rm el}{}^{h'h}\tau^ih], \quad (5.3)$$

where h and h' are the initial and final helicities, and $M^{h'h}$ are the corresponding amplitudes introduced by Jacob and Wick.[7]

We shall apply the above relation to the pion nucleon scattering, which seems to be the simplest case of physical interest. Naturally $M_{\rm el}$ has to be identified with the elastic scattering amplitude $M(p'q',pq)$ for the process $N_p+\pi_q\to N_{p'}+\pi_{q'}$, whereas $M_{\rm rad}{}^i(p'q',pq;k)$ corresponds to the production process $N_p+\pi_q\to N_{p'}+\pi_{q'}+\pi_k{}^i$. The fundamental assumption here is that the pion-nucleon system is γ_5 invariant to the extent that the pion mass can be neglected.

To exhibit the isotopic dependence of $M_{\rm el}$, we write

$$iM_{{\rm el}\beta\alpha}=M^{(+)}\delta_{\beta\alpha}+M^{(-)}\tfrac{1}{2}[\tau_\beta,\tau_\alpha], \quad (5.4)$$

so that $M_{\rm rad}{}^{i=3}$ for the process $\pi^\pm+p\to\pi^\pm+p+\pi^0$ takes the form

$$iM_{\rm rad}{}^{h'h}(\pi^0)=-(gv_p/2m)(h'-h)M^{h'h}(\pi^\pm p\to\pi^\pm p)$$
$$=-(gv_p/2m)(h'-h)\times[M^{(+)h'h}\mp M^{(-)h'h}]. \quad (5.5)$$

[7] M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

The relation becomes somewhat more complicated when $\pi^{\pm}$ are produced. For $\pi^- + p \rightarrow \pi^- + n + \pi^+$, we have, for example

$$iM_{\rm rad}{}^{h'h}(\pi^+) = -\frac{\sqrt{2}gv_p}{2m}[h'M^{h'h}(\pi^- p \rightarrow \pi^- p) - hM^{h'h}(\pi^- n \rightarrow \pi^- n)]$$
$$= -\frac{\sqrt{2}gv_p}{2m}[h'(M^{(+)h'h} + M^{(-)h'h}) - h(M^{(+)h'h} - M^{(-)h'h})]. \quad (5.6)$$

Let us now compare the elastic and inelastic cross sections. The simplest way to apply our formula will be to take the case where the meson k is produced with small energy ($|\mathbf{k}| \approx 0$). We easily find

$$\left(\frac{d^2\sigma_{\rm rad}}{d\omega_k d\Omega_{q'}}\right) \bigg/ \left(\frac{d\sigma_{\rm el}}{d\Omega_{q'}}\right) = \frac{|\mathbf{k}|}{(2\pi)^2} |\bar{u}_{p'} M_{\rm rad} u_p|^2 / |\bar{u}_{p'} M_{\rm el} u_p|^2. \quad (5.7)$$

For the process $\pi^{\pm} + p \rightarrow \pi^{\pm} + p + \pi^0$ [Eq. (5.5)] this reduces to

$$\left(\frac{d^2\sigma_{\rm rad}}{d\omega_k d\Omega_{q'}}\right) \bigg/ \left(\frac{d\sigma_{\rm el}}{d\Omega_{q'}}\right) = \frac{1}{\pi}\frac{g^2}{4\pi}\frac{|\mathbf{k}|}{4m^2} v_p{}^2 (h'-h)^2. \quad (5.8)$$

Averaging over the nucleon spin, we get

$$\langle (h'-h)^2 \rangle_{\rm av} = 2[1 - A'(\Omega_{q'})], \quad (5.9)$$

where $A' \equiv \langle h'h \rangle_{\rm av}$ is one of the polarization parameters introduced by Wolfenstein.[8] Equations (5.7) and (5.8) are the relations that should hold at each scattering angle. If we integrate Eq. (5.8) over the angles, we get

$$\left(\frac{d^2\sigma_{\rm rad}}{d\omega_k}\right) \bigg/ \sigma_{\rm el} = \frac{2}{\pi}\frac{g^2}{4\pi}\frac{|\mathbf{k}|}{4m^2} v_p{}^2 (1 - \bar{A}'), \quad (5.10)$$

where $\bar{A}'$ means an angular average.

In the actual case of finite meson mass, the low-energy limit $\mathbf{k} = 0$ does not have a Lorentz invariant meaning, so that the analysis will depend on the choice of the coordinate system. For example, a meson at rest in the c.m. system will have a momentum $|\mathbf{k}_{\rm L}| = v_{\rm L}\mu/(1-v_{\rm L}{}^2)^{\frac{1}{2}}$ in the laboratory system. In order to keep such an ambiguity reasonably small, e.g., $|\mathbf{k}_{\rm L}| \lesssim \mu$, we must demand that $1/(1-v_{\rm L}{}^2)^{\frac{1}{2}}$ is not too large. In other words, the energy at which we carry out the experiment should not be large compared to the nucleon rest energy.

An alternative way to test our relation is to consider the energy distribution of the inelastically scattered meson near its maximum energy since the produced pion would come out with low energy. In this case, the ratio of cross sections is calculated to be

$$\left(\frac{d^2\sigma_{\rm rad}}{d\omega_{q'} d\Omega_{q'}}\right) \bigg/ \left(\frac{d\sigma_{\rm el}}{d\Omega_{q'}}\right) = \frac{1}{(2\pi)^2}\left(\frac{E}{m}\right)^{\frac{3}{2}} [2\mu(\omega_{\rm m} - \omega_{q'})]^{\frac{1}{2}} \times |\bar{u}_{p'} M_{\rm rad} u_p|^2 / |\bar{u}_{p'} M_{\rm el} u_p|^2, \quad (5.11)$$

where $\omega_{\rm m}$ is the maximum meson energy at the particular angle, and E is the total c.m. energy of the system. Again for the process $\pi^{\pm} + p \rightarrow \pi^{\pm} + p + \pi^0$, this yields

$$\left(\frac{d^2\sigma_{\rm rad}}{d\omega_{q'} d\Omega_{q'}}\right) \bigg/ \left(\frac{d\sigma_{\rm el}}{d\Omega_{q'}}\right) = \frac{1}{\pi}\frac{g^2}{4\pi}\left(\frac{E}{m}\right)^{\frac{3}{2}} \frac{[2\mu(\omega_{\rm m} - \omega_{q'})]^{\frac{1}{2}}}{4m^2} v_p{}^2 (h'-h)^2. \quad (5.12)$$

Equations (5.8) and (5.11) or the specific forms (5.8) and (5.12) give a relation between elastic and inelastic cross sections in terms of directly measurable quantities alone. When the polarization of the nucleon is not measured, they still can give an inequality since

$$(h'-h)^2 \leq 4.$$

Unfortunately, these formulas are supposed to apply only at the extreme ends of the meson energy spectrum, for which experimental data are scarce and difficult to obtain.

On the other hand, if the elastic matrix elements are precisely known, we can directly calculate the inelastic amplitudes from Eq. (5.3) [or Eqs. (5.5) and (5.6)]. When, for example, the elastic scattering is dominated

TABLE I. Angular dependence and magnitude of the radiative cross section $\pi + N \rightarrow \pi + N + \pi'$, where the last pion is produced nearly at rest, for the case of $T = \frac{3}{2}$, $J = \frac{3}{2}{}^{\pm}$ elastic channel, $Z = \cos\theta_\pi$.

Process	$\frac{d^2\sigma_{\rm rad}}{d\Omega_{q'} d\omega_k}$	$\left(\frac{d\sigma_{\rm rad}}{d\omega_k}\right) \bigg/ \left(\frac{g^2}{4\pi^2}\frac{v_p{}^2}{4m^2} k\sigma_{\rm el}\right)$
$\pi^{\pm} + p \rightarrow \pi^{\pm} + p + \pi^0$	$(1+3Z)^2(1-Z)$	2
$\pi^+ + p \rightarrow \pi^+ + n + \pi^+$	$1 + 3Z + 3Z^2 - (27/5)Z^3$	20/9
$\pi^- + p \rightarrow \pi^- + n + \pi^+$		20

[8] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954).

TABLE II. Angular dependence and magnitude of the radiative cross section for the case of $T=\frac{1}{2}$, $J=\frac{3}{2}^{\pm}$ and $\frac{5}{2}^{\pm}$ channels.

Process	$\dfrac{d^2\sigma_{\text{rad}}}{d\Omega_{q'}d\omega_k}$	$\left(\dfrac{d\sigma_{\text{rad}}}{d\omega_k}\right)\Big/\left(\dfrac{g^2}{4\pi^2}\dfrac{v_p^2}{4m^2}k\sigma_{\text{el}}\right)$	
$\pi^-+p\rightarrow\pi^-+p+\pi^0$	$(1+3Z)^2(1-Z)$	2	$J=\frac{3}{2}$
	$(1-2Z-5Z^2)^2(1-Z)$		$J=\frac{5}{2}$
$\pi^-+p\rightarrow\pi^-+n+\pi^+$	$1+3Z^2$	2	$J=\frac{3}{2}$
	$1-2Z^2+5Z^4$		$J=\frac{5}{2}$

by a resonance, the associated production amplitude is easily obtained.

In Table I we give the results for the $\frac{3}{2}-\frac{3}{2}$ resonance. This is not an ideal case because the resonance energy is only twice the pion mass, which makes the neglecting of the pion mass somewhat dubious. But perhaps one could argue that we should compare the elastic scattering at energy T with the inelastic scattering at incident energy $T+\mu$. For the 3–3 resonance, the latter comes out to be ≈ 300 Mev.

In Fig. 2 we show the measured angular distribution of the inelastically scattered π^- from the reaction[9] $\pi^-+p\rightarrow\pi^-+n+\pi^+$ at 290 Mev laboratory energy which is to be compared with the theoretical curve. The agreement is quite reasonable. We note here that, although the experimental points are integral distributions over pion energy, the $\pi^- - n$ resonance in the final system would tend to produce low energy π^+ as is wanted in the comparison with theory.[9a]

As for the magnitude of the cross section, we can make only a crude comparison. We estimate the theoretical production cross section $\sigma(\pi^-p\rightarrow\pi^-n\pi^+)$ by extrapolating Eq. (5.7) over the entire energy range. This gives the ratio (with $g^2/4\pi\approx 15$)

$$r=\frac{\sigma(\pi^-p\rightarrow\pi^-n\pi^+)}{\sigma(\pi^-p\rightarrow\pi^-p)}\approx 1/100$$

at 290 Mev. Since the theoretical peak elastic cross section due to the 3–3 resonance is 20 mb, the above ratio means

$$\sigma(\pi^-p\rightarrow\pi^-n\pi^+)\approx 0.2 \text{ mb}.$$

The corresponding experimental value is 0.61 ± 0.13 mb at[9] 290 Mev and 0.71 ± 0.10 mb at 317 Mev.[10] These are quite compatible with the prediction considering the crudeness of the estimation. (We also note from Table I that the ratio r for the other production processes would be only 1/10 of the present case. In such an event there may be more contaminations from other partial waves.)

Similar considerations can be made for the higher resonances. We list in Table II the corresponding predictions for resonances with $T=\frac{1}{2}$, $J=\frac{3}{2}^{\pm}$ and $\frac{5}{2}^{\pm}$. Finally it is interesting to compare the preceding results with those of the statistical model. We may identify the Fermi interaction volume Ω with the ratio[11]

$$R=\frac{1}{2\mu}\frac{|M_{\text{rad}}|^2}{|M_{\text{el}}|^2}\approx\frac{g^2}{4\mu m^2}v_p^2\approx\frac{v_p^2}{\mu^3},$$

as far as the soft meson emission is concerned. The corresponding interaction radius R is then

$$R\approx 0.6 v_p^{\frac{2}{3}}\mu^{-1},$$

which is energy dependent, and approaches $0.6\mu^{-1}$ at high energies.

VI. FURTHER REMARKS

The main theme of this paper is to show that, under the assumption of γ_5 invariance in strong interactions, the change of nucleon helicity in any reaction will result in the bremsstrahlung of soft pions. This relation is characterized specifically by Eq. (5.1) or (5.2), which expresses a kind of low energy limit theorem. A comparison with the pion-nucleon scattering data around the 3–3 resonance shows an agreement with the formula.

Other interesting applications or tests of our relations may be found in nucleon-nucleon and nucleon-antinucleon scattering and general high energy multiple production processes. Our formulas would give us a way to analyze these events by observing low-energy

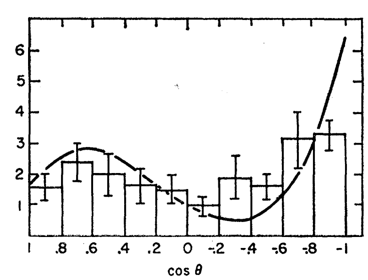

FIG. 2. Angular distribution of π^- from the reaction $\pi^-+p\rightarrow n+\pi^++\pi^-$ at 290 Mev.[9] The curve is calculated from the last line in Table I.

[9] Ya. A. Batusov, S. A. Bunyatov, V. M. Sidorov, and V. A. Yarba, *Proceedings of the 1960 International Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), p. 77; Ya. A. Batusov *et al.*, Doklady Nauk U.S.S.R., **133**, 52 (1960) [Soviet Phys.-Doklady **5**, 731 (1961)].

[9a] *Note added in proof.* However, the agreement may be fortuitous since there are various angular momentum channels that are neglected in this simplified approach. Our formula shows a characteristic dip in the forward direction. But the corresponding behavior of the experimental data may or may not be real.

[10] W. A. Perkins, III, J. C. Caris, R. W. Kenney, and V. Perez-Mendez, Phys. Rev. **118**, 1364 (1960).

[11] E. Fermi, Progr. Theoret. Phys. (Kyoto) **5**, 570 (1950).

mesons emitted. For example, in the case of nucleon-antinucleon annihilation at rest, the initial chirality is zero, and, since the final state contains no nucleons, we would have $\langle X_\pi \rangle_f = 0$, i.e., the amplitude for the emission of very low energy pions would be unusually small.

The notion of γ_5 invariance or chirality conservation can be extended to composite systems and strange particles.[4] There is also a possibility that the K meson plays a role similar to the pion in the conservation of strangeness-changing chirality current. It is likely, however, that even if such a symmetry existed in essence, the large mass of the K meson would tend to make it more approximate in nature than for the case involving pions, except perhaps at sufficiently high energies.

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Perturbation Theory of Pion-Pion Interaction. I. Renormalization

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The problem of pion-pion scattering is studied on the basis of the model of a four-particle direct interaction without derivative coupling. Renormalization is carried out for this model with a detailed analysis of overlap insertions. To every finite order in the renormalized coupling constant, it is shown that the unitarity relation holds and that the Feynman integral representation is still valid, and hence renormalization has no effect on analytic properties.

1. INTRODUCTION

SINCE the pion is a pseudoscalar boson, the simplest coupling among pions is a local ϕ^4 coupling. Furthermore, this leads to a dimensionless coupling constant. If this coupling is taken to be correct, then the problem of pion-pion interaction is the simplest among all problems involving strongly-interacting particles. It is the purpose here to study the pion-pion interaction under this coupling using perturbation theory.

In order that the perturbation theory be meaningful, it is necessary to have a consistent procedure to remove the infinities due to integrations over large momenta and to interpret this removal as mass renormalization and coupling-constant renormalization.[1] In the much more familiar case of electrodynamics, the procedure of Ward[2] seems simpler than that of Salam[3]; hence, in the present case, differentiation with respect to external momenta is to be used for the purpose of treating overlap divergences, which are of main concern here. However, the problem of which path to use in carrying out the differentiation is quite complicated in the present case. In quantum electrodynamics, the treatment of the photon self-energy has been carried out by Mills and Yang,[4] and their treatment is the starting point for the present consideration. Thus, this case of electrodynamics is considered first in Sec. 4 after a preliminary study of the case of the ϕ^4 coupling. Renormalization is completed in Sec. 7, and some properties of this procedure are discussed in Secs. 8–11. In particular, the validity of the Feynman integral representation implies that renormalization does not change the domains of analyticity to every order of the coupling constant.

This paper is concerned mainly with the formal question of renormalization within the framework of perturbation theory. Thus, on the basis of the particular Lagrangian under consideration, all the equations here are exact in the sense of being true to every finite order of the coupling constant. In a later paper, the problem is considered concerning the derivation of a closed system of equations for the approximate description of the pion-pion system at low energies.

2. STATEMENT OF THE PROBLEM

Let ϕ_+, ϕ_0, and ϕ_-, respectively, be the field operators for the creation of the pions π^+, π^0, and π^-. Let $\phi_3=\phi_0$, and the triplet of operators (ϕ_1,ϕ_2,ϕ_3) transform as a vector in the space of isotopic spin; then with the usual phase conventions

$$\phi_\pm = \mp(\phi_1 \pm i\phi_2)/\sqrt{2}. \qquad (1)$$

Since π^0 is its own antiparticle, ϕ_i are Hermitian. Throughout this paper, the Lagrangian density is

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[1] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[2] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951).
[3] A. Salam, Phys. Rev. **82**, 217 (1951).
[4] R. L. Mills and C. N. Yang (private communication from Professor Yang).