for the original set, by examining the signs of the principal minors of **M**.

The simplification of the present section is the elimination of the requirement of orthonormality with respect to ρ of the trial functions Φ_{nt} and of the requirement that the Φ_{nt} diagonalize $(H-E)$, and the replacement by the simpler requirement on the signs of the minors of the determinant. The two requirements which were imposed in the problem of zero-energy scattering[2] were completely analogous but with the exception that ρ did not appear. It is however trivial, proceeding along lines almost identical to those above. to simplify the auxiliary conditions on the trial bound state functions.

The fact that the results are entirely independent of ρ suggests that ρ need never have been introduced, and it is indeed simple enough to derive bounds rather more directly than we have done by a derivation with ρ set equal to a constant. The details are given in a set of lectures[25] which also include the simplifications with regard to the zero energy case that were noted above.

The simplification at zero energy, which in turn was the starting point of the investigation of Sec. 4, is due to T. F. O'Malley (unpublished), to whom we would wish to express our thanks. The simplification at zero energy was obtained independently by Ohmura,[26] in a slightly different form.

[25] L. Spruch, in *Lectures in Theoretical Physics, Boulder, 1961* [Interscience Publishers, Inc., New York (to be published)], Vol. 4. The portion of these lectures devoted to variational minimum principles is basically a review of the present series of papers.

[26] T. Ohmura, Phys. Rev. **124**, 130 (1961).

PHYSICAL REVIEW VOLUME 125, NUMBER 4 FEBRUARY 15, 1962

Coherent Photoproduction of π^0 from Deuterium*

Fokion T. Hadjioannou†
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
(Received June 27, 1961)

The calculation of elastic photoproduction of π^0 mesons from deuterium, $\gamma+d\rightarrow\pi^0+d$, is carried out in the impulse approximation at photon energies around 500 Mev. The single nucleon photoproduction amplitudes are taken from dispersion formulas and are corrected for kinematic effects due to internal momentum of the nucleons in the deuteron. We include the D-state part of the deuteron wave function and use different models with Yukawa type or repulsive core wave functions. We give formulas connecting the cross section with the deuteron form factors. For small momentum transfers the formulas are of course model independent and reduce to the usual ones. The presence of a 7% D state in a repulsive-core model leads to a cross section which falls typically more slowly at high momentum transfers, e.g., at a momentum transfer 2.74 f^{-1}, the cross section is larger by about 40% than the cross section calculated in the absence of the D state. The experimental points favor this model.

I. INTRODUCTION

THIS work is a calculation of the elastic photoproduction of π^0 mesons from deuterium, $\gamma+d\rightarrow\pi^0+d$, at photon energies around 500 Mev.[1] Chew and Lewis[2] and Lax and Feshbach[3] first calculated the cross section for such a process in the impulse approximation. De Wire *et al.*[4] applied these results to analyze the $\gamma+d\rightarrow\pi^0+d$ cross section at energies 250–300 Mev, normalizing the theoretical formulas with the free-nucleon photoproduction cross sections. Multiple scattering corrections to the impulse approximation have been considered by Chappelear[5] but the analyses so far have taken the static limit of infinite nucleon mass.

* Supported in part by the U. S. Air Force through the Office of Scientific Research.

† Present address: Department of Physics, Stanford University, Stanford, California.

[1] Experiment of J. Friedman and H. Kendall (to be published).

[2] G. F. Chew and H. W. Lewis, Phys. Rev. **84**, 779 (1951).

[3] M. Lax and H. Feshbach, Phys. Rev. **88**, 509 (1952).

[4] J. W. De Wire, A. Silverman, and B. Wolfe, Phys. Rev. **92**, 520 (1953).

[5] John Chappelear, Phys. Rev. **99**, 254 (1955).

The present work can be described as follows: (1) As a refinement in the calculation we put much effort into maintaining relativistic covariance, at least on all kinematic quantities. For this we borrowed the free-nucleon dispersion formulas[6] and used them in the deuteron case. Of course the nucleons in the deuterium do not satisfy the free-particle energy-momentum relation $\mathbf{p}^2+m^2=E^2$; this is a familiar limitation to impulse approximation. (2) The transition matrix T will be a function of the two kinematic invariants $S=(k+P)^2$, the "invariant total energy" of the photon+deuteron, and $t=(k-q)^2$ the momentum transfer:

$$T=T(S,t). \tag{1.1}$$

k, P, and q are the photon, deuteron, and pion four-momenta, respectively.

Now we want to relate T to the photoproduction from single nucleons. Since the nucleons in the deuteron will be distributed over the internal momentum $\mathbf{p}$, we write

[6] G. F. Chew, F. E. Low, M. L. Goldberger, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

in the impulse approximation

$$T_d=\int d^3p\Psi_d(\mathbf{p})[T_1(s(\mathbf{p},S),t)+T_2(s(\mathbf{p},S),t)] \times\Psi_d(\mathbf{p}+\tfrac{1}{2}\mathbf{Q}), \quad (1.2)$$

where s refers to the one nucleon+photon "invariant energy," and T_1 and T_2 are the π^0 photoproduction amplitudes from a free proton and neutron, respectively. Only that part of T_1 and T_2 contributes which connects the spin triplet, isospin singlet initial deuteron ground state to the final deuteron ground state again. [E.g., the part of the T matrix proportional to $\mathcal{J}^0$ (see Chap. 3) will not contribute in the $\gamma+d\rightarrow\pi^0+d$ cross section. Also the peripheral $\gamma-3\pi$ interaction, an isotopic scalar, will not contribute either.] So it was more convenient to use the dispersion formulas instead of phenomological spin-flip and spin-nonflip amplitudes obtained from photoproduction of neutral pions from free nucleons. $\psi_d(\mathbf{p})$ is the ground-state deuteron wave function (all dependence besides $\mathbf{p}$ suppressed), and $\mathbf{Q}$ is the recoil momentum of the deuteron; since the deuteron is heavy we neglect its recoil kinetic energy and approximate $-t=\mathbf{Q}^2$. However the correct kinematics is preserved in the single-nucleon amplitudes.

If we integrate Eq. (1.2) over $\mathbf{p}$ we expect finally to get

$$\begin{aligned}T_d&=T_d(S',t)\\&=[\langle T_1\rangle_{\mathrm{av}}(s(t,S),t)+\langle T_2\rangle_{\mathrm{av}}(s(t,S),t)]\mathfrak{F}_d(Q), \quad (1.3)\end{aligned}$$

where $\langle T_1\rangle_{\mathrm{av}}$ and $\langle T_2\rangle_{\mathrm{av}}$ define averages over $\mathbf{p}$ for the one-nucleon transitions, and $\mathfrak{F}_d(Q)$ is the deuteron form factor, defined in detail in Sec. IV. In this experiment we are well above the meson nucleon resonance and the transition amplitudes T, in their energy dependence, are slowly varying functions of $\mathbf{p}$. Since the deuteron wave function is a much more rapidly varying function of $\mathbf{p}$, we can to a good approximation substitute for T in Eq. (2.3) its average at $\mathbf{p}=0$ and $\mathbf{p}+\frac{1}{2}\mathbf{Q}=0$ where the deuteron wave function has a sharp peak.

In principle, as a check on the impulse approximation applied here, one can keep the momentum transfer to the deuteron and hence the form factors constant and vary the photon energy to reproduce the energy dependence of the cross section predicted by the dispersion formulas.[6]

Our primary interest in this paper is to see what can be learned about the deuteron structure from these experiments and to compare with other sources of information, e.g., elastic scattering of electrons from deuterium. Therefore we shall try to calculate T explicitly.

The procedure we adopt here is to apply[6] the predicted momentum dependence of the single-nucleon dispersion formulas to the deuteron case, in the impulse approximation Fig. 1, with the nucleons in the deuteron distributed over the internal momentum $\mathbf{p}$ as indicated in Eq. (1.2). We then use formula (1.3) which separates the deuteron structure form factors from the rest of the T matrix to analyze the experimentally observed $|\mathbf{Q}|$ dependence in terms of the deuteron structure.

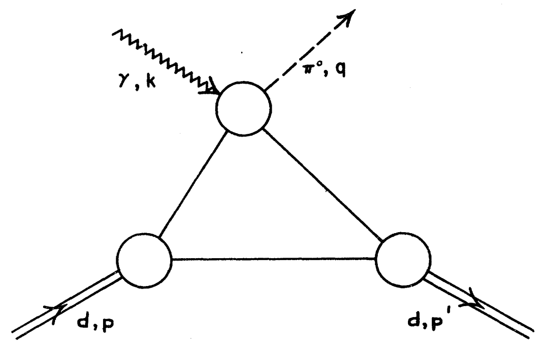

FIG. 1. The impulse approximation diagram for the elastic photoproduction of π^0 from deuterium, $\gamma+d\rightarrow\pi^0+d$.

We consider various percentages of D-state admixture in the ground-state wave function in different deuteron models, in order to test the sensitivity of the predicted cross sections to these parameters. We limit ourselves to nonrelativistic models obtained from static potentials which treat the nucleons with 2-component wave functions with Pauli spins. Since the meson field couples strongly with the nucleon spin, the presence of the spin-orbit tensor operator in the D state might show up D-state effects strongly. We find this to be the case so that this experiment provides a sensitive probe to the D state.

This calculation is of course limited by uncertainties of pion interaction amplitudes and the impulse approximation. For neutral pion photoproduction, however, the dispersion formula prediction agrees quite well with experiment in the energy region considered here, which is in our favor.

The results of the calculation are summarized as follows: (1) The agreement of the cross sections with the experimental points measured by Friedman and Kendall[1] is improved considerably by the care in treating the kinematics for $E_\gamma=500$ Mev. As mentioned earlier, in the coherent π^0 photoproduction, only that part of the T matrix contributes which connects the spin triplet and isotopic singlet deuteron states. Thus, in the energy region of this experiment, one calculates a cross section 15–20% smaller than that obtained by applying an impulse approximation with free proton-neutron cross sections.

The operation of averaging T over the internal momentum $\mathbf{p}$ has a small effect here since we are well above the resonance. The improvement over just taking $\mathbf{p}=0$ is not very important, the cross section being decreased about 20% for the highest momentum transfer. As a function of the momentum transfer this correction has the shape desired to fit the experimental points but is too small an effect to explain the observed slope without inclusion of D-state contributions. (2) When we include some D state in the deuteron wave function the form

TABLE I. Numerical values of the various quantities are calculated for the kinematics of Friedman and Kendall's experiment. The average is taken between $\mathbf{p}=0$ and $\mathbf{p}=-\mathbf{Q}/2$, where $\mathbf{p}$ is the relative momentum of the nucleons in the deuteron and $\mathbf{Q}$ the deuteron momentum transfer. (The phase shifts needed in evaluating the spin-flip and spin-nonflip amplitudes $|\mathbf{K}^{(+)}|$, $L^{(+)}$ are functions of the total energy of the pion+one nucleon in their own c.m. system. Therefore phase shifts are averaged for the two energies corresponding to $\mathbf{p}=0$ and $\mathbf{p}=-\mathbf{Q}/2$). $\cos\gamma\equiv(|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|/|\mathbf{K}^{(+)}|)_{\mathrm{av}}$. Because of inherent ambiguities in this calculation only the leading decimal figures have significance.[a]

Photon energy ω_k lab Mev	Photon energy ω_k c.m. Mev	Deuteron recoil Q (f^{-1})	$(\frac{2}{3}\lvert\mathbf{K}^{(+)}\rvert^2+\lvert L^{(+)}\rvert^2)_{\mathrm{av}}$ (10^{-30} cm^2)	$(\lvert\mathbf{K}^{(+)}\rvert^2)_{\mathrm{av}}$ (10^{-30} cm^2)	$2\cos^2\gamma-\frac{2}{3}$	Repulsive core model 7% D state: F_0^2	F_2^2	$d\sigma/d\Omega_q$ (10^{-30} cm^2)	Effective form factor $\dfrac{d\sigma/d\Omega_q}{(\frac{2}{3}\lvert\mathbf{K}^{(+)}\rvert^2+\lvert L^{(+)}\rvert^2)_{\mathrm{av}}}$	Apparent increase in form factor due to D state
456	374.8	1.748	22.7	12.2	0.613	0.092	0.005	2.09	0.101	1%
516	414	1.74	12.8	8.7	0.557	0.108	0.005	1.39	0.101	1%
542	431	1.74	6.7	4.3	0.549	0.108	0.005	0.79	0.101	1%
468	383	1.943	23.7	11.02	0.617	0.065	0.0055	1.61	0.068	5%
508	409	1.948	14.0	6.99	0.605	0.065	0.0055	0.965	0.068	5%
514	413	1.942	12.5	6.38	0.601	0.065	0.0055	0.85	0.068	5%
508	409	1.942	14.1	7.07	0.641	0.065	0.0055	0.97	0.068	5%
491	398	2.16	19.6	7.52	0.621	0.045	0.0059	0.97	0.05	10%
482	391	2.355	21.3	7.50	0.605	0.031	0.006	0.76	0.036	18%
473	385	2.555	23.6	6.63	0.725	0.020	0.0059	0.6	0.026	30%
470	383	2.74	23.6	5.50	0.381	0.014	0.0058	0.57	0.003	42%

[a] The phase shifts were taken from the Proceedings of the Ninth Annual Conference on High-Energy Physics, Kiev, 1959 (to be published). See report on pion-nucleon scattering, B. Pontecorvo, rapporteur, p. 30.

factors are modified. For Yukawa-type models the small percentage of 3–4% D state required to fit other low-energy parameters does not lead to any significant modifications and the situation can be equivalently described with a pure s-wave ground state appropriately normalized, as usual. A repulsive core, however, with a 7% D state brings the new terms forward. In addition to a high percentage of D state, the effect is here particularly enhanced, since for the repulsive core model the s-wave form factor[7] F_0 falls much more rapidly than F_2,[7] and the new terms which are proportional to $F_2{}^2$ acquire importance. Thus, for example, for momentum transfers of 2.74(f^{-1}), our highest recoil point, the new terms contribute 40% and increase with higher Q. (See Table I.) For small momentum transfers the new terms vanish and the different models give identical results.

The cross section predicted in the repulsive core model with a 7% D state seems to reproduce quite well the characteristic slope (Fig. 2) of the experimental form factors. It was this curving of the form factors, in fact, which stimulated the present calculation.

We do not make an effort to correct for multiple scattering. From Chappelear's work,[5] which takes into account on the energy shell contributions to multiple scattering only, it seems that the percentage corrections are angle independent, and thus Q independent, and will not affect the slope of the form factor. Rough extrapolation from Chappelear's work improves the experimental fit of the magnitude of the cross section with the repulsive core model.

From this work we conclude that the present experiment favors strongly a repulsive-core wave function with a correspondingly appreciable D-state admixture in a static deuteron model.

[7] J. A. McIntyre and S. Dhar, Phys. Rev. **106**, 1074 (1957).

II. CROSS SECTION

We use units $\hbar=c=\mu=1$ where μ is the pion mass. To follow the notation of reference 6 we give the S matrix for free-nucleon photoproduction of a π meson in the form

$$S_{fi}=\delta_{fi}-i(2\pi)^4\delta^4(k+p_i-q-p_f)\times\frac{m}{(4\epsilon_f\epsilon_i\omega_k\omega_q)^{\frac{1}{2}}}\bar{u}_f T u_i. \qquad (2.1)$$

Here p_i and p_f are the initial and final nucleon 4-momenta, ϵ_i, ϵ_f are the nucleon energies, m is the nucleon mass, k the photon, and q the meson 4-momenta, and ω_k, ω_q are their energies. $\bar{u}_f$ and u_i are nucleon spinors and $\bar{u}_f T u_i$ the transition matrix, which is a function of $s=(k+p_i)^2$, the invariant "total energy" squared, and $t=(p_f-p_i)^2$, the recoil momentum squared. With the above S matrix, we obtain in the c.m. (center-of-mass) system the cross section

$$d\sigma_{(\gamma+N\to\pi+N)}=\frac{1}{(4\pi)^2}\frac{m^2}{\epsilon_i\epsilon_f\omega_k}\frac{|T|^2}{(1+k/\epsilon_i)}\frac{q}{(1+\omega_g/\epsilon_f)}d\Omega_q$$
$$=\frac{1}{(4\pi)^2}\frac{m^2|T|^2}{w^2}\frac{q}{k}d\Omega_q, \qquad (2.2)$$

where w is the total energy in the c.m. system. Let us now come to photoproduction from deuterium. We chose the deuteron+photon c.m. system. To transform from the one-nucleon+photon c.m. system, in which the

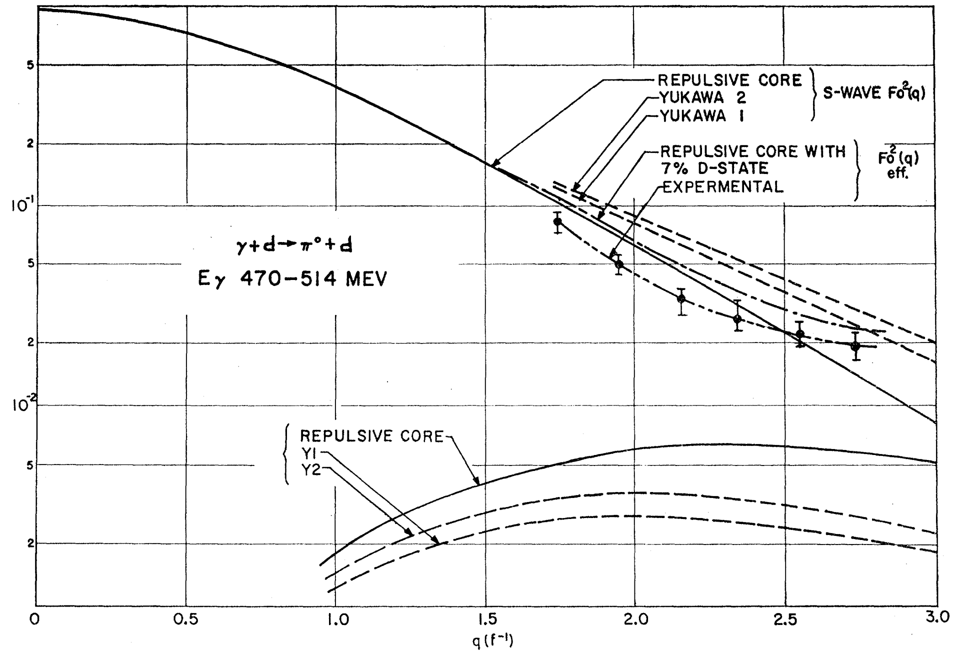

FIG. 2. The deuteron form factors together with the experimental data are plotted as functions of the deuteron momentum transfer Q. We plot the effective form factor $(F_0)_{\rm eff}(Q)$, only for the repulsive core model. For the Yukawa 1 and Yukawa 2 models, in this range of Q, $(F_0)_{\rm eff}(Q)$, practically coincides with the corresponding $F_0(Q)$.

dispersion formulas are written, to the deuteron+photon c.m. system, a Galilean transformation is sufficient since the velocities are small. In the deuteron+photon c.m. system corresponding to (2.2) we must have

$$d\sigma_{(\gamma+d\to\pi+d)}=\frac{1}{(4\pi)^2}\frac{m^2}{\langle\epsilon_i\epsilon_f\rangle_{\rm av}}\frac{1}{\omega_k}\frac{|T_d|^2}{(1+\beta_d)}\frac{q}{(1+\omega_\pi/E_f)}d\Omega_q$$

$$=\frac{1}{(4\pi)^2}\frac{m^2}{\langle\epsilon_i\epsilon_f\rangle_{\rm av}}\frac{E_iE_f}{W^2}|T_d|^2\frac{q}{\omega_k}d\Omega_q, \qquad (2.3)$$

where $\beta_d=k/E_i$ is the velocity of the initial deuteron, E_i, E_f are the initial and final deuteron energies, ω_q, q are the energy and momentum of the pion, W the total c.m. energy, and ω_k is the photon energy; all quantities refer to the deuteron+photon c.m. system. The normalization factor $m^2/\epsilon_i\epsilon_f$ is introduced to preserve formal covariance of $|T|$ in (2.1) since (2.2) and (2.3) will be connected through impulse approximation, i.e., $T_d=T_1+T_2$ is the sum of the photoproduction amplitudes from nucleons 1 and 2. ϵ_i is the single-nucleon initial energy as in (2.2), and ϵ_f is the corresponding final energy of the same one nucleon.

$\langle\epsilon_i\epsilon_f\rangle_{\rm av}$ with all other quantities is averaged over the internal momentum $\mathbf{p}$ at $\mathbf{p}=0$ and $\mathbf{p}=-\frac{1}{2}\mathbf{Q}$ as stated in the introduction. Because of slow variation it is adequate here to take $\langle\epsilon\rangle_{\rm av}=\epsilon(\mathbf{p}=0)$. We have not yet specified the transition matrix $|T_d|^2$, which implicitly contains form factors, isotopic spin sums, and kinematic quantities from the nucleon's internal motion. We proceed to calculate $|T_d|^2$ in the following section.

III. THE MATRIX ELEMENT

The single-nucleon pion photoproduction amplitude,[6] decomposed in angular momentum projections (constant momentum transfer t) in the c.m. system, is given in terms of $\mathfrak{F}$:

$$\langle f|\mathfrak{F}|i\rangle_{\rm c.m.}=(m/w)\bar{u}_fTu_i, \qquad (3.1)$$

where W is the total energy in the c.m. system. $\mathfrak{F}$ is to be inserted between Pauli spinors for the initial and final nucleons in that c.m. system. Explicitly,[6]

$$\mathfrak{F}=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}F_1+\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}\frac{F_2}{qk}$$

$$+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{F_3}{qk}+i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{F_4}{q^2}, \qquad (3.2)$$

where F_1, F_2, F_3, F_4 are functions of s and t (energy and angle in the c.m. system). Expressed in terms of $\mathfrak{F}$, the differential cross section $d\sigma_{(\gamma+N\to\pi+N)}$, (2.2), for $\gamma+N\to N+\pi$ is written

$$d\sigma_{(\gamma+N\to\pi+N)}=\frac{1}{(4\pi)^2}|\mathfrak{F}|^2\frac{q}{k}d\Omega_q. \qquad (3.3)$$

The matrix element $\langle\pi,\mathbf{q}|\mathfrak{F}_d|\gamma,\mathbf{k}\rangle$ for the coherent photoproduction of a pion from deuterium will have the form

$$\langle\pi,\mathbf{q}|\mathfrak{F}_d|\gamma,\mathbf{k}\rangle=\int d^3(\mathbf{r})\Psi_d(\mathbf{r})[\mathfrak{F}_1+\mathfrak{F}_2]\Psi_d(\mathbf{r}), \qquad (3.4)$$

where ψ_d is the deuteron wave function with its spin indices etc. suppressed, and $\mathbf{Q}=\mathbf{k}-\mathbf{q}$ is the deuteron recoil.

The single-nucleon pion photoproduction amplitudes $\mathfrak{F}_{(1,2)}$ (1, 2 are labels for nucleons 1, 2 here) would in general depend on $\mathbf{r}$, $\boldsymbol{\nabla}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$. In impulse approximation $\mathfrak{F}_{1,2}$ will be replaced by the free nucleon operators (3.1) and average $\mathfrak{F}$ between inner momentum $\mathbf{p}=0$ and $\mathbf{p}=-\mathbf{Q}/2$ as mentioned in the introduction.

$\mathfrak{F}$ is further decomposed into isospin projection operators[6]

$$\mathfrak{F}_\alpha=\mathfrak{F}^{(+)}\mathcal{G}_\alpha{}^{(+)}+\mathfrak{F}^{(-)}\mathcal{G}_\alpha{}^{(-)}+\mathfrak{F}^{(0)}\mathcal{G}_\alpha{}^{(0)}, \tag{3.5}$$

where

$$\mathcal{G}_\alpha{}^{(+)}=\tfrac{1}{2}[\tau_\alpha,\tau_3]_+=\delta_{\alpha 3},\qquad \mathcal{G}^{(-)}=\tfrac{1}{2}[\tau_\alpha,\tau_3]_-,\quad \mathcal{G}_\alpha{}^{(0)}=\tau_\alpha. \tag{3.6}$$

α is the isotopic index for the π to be produced. In this case, photoproducing π^0, $\alpha=3$, $\mathcal{G}_3{}^{(-)}=0$, $\mathcal{G}_3{}^{(+)}=1$. $\mathcal{G}^{(0)}$ will not contribute since the deuteron is isosinglet. Therefore

$$\langle\pi^0,\mathbf{q}|\mathfrak{F}_d|\gamma,\mathbf{k}\rangle=\int d^3r\,\exp(i\mathbf{Q}\cdot\mathbf{r}/2)\Psi_d(r)\times[\langle\mathfrak{F}_1{}^{(+)}+\mathfrak{F}_2{}^{(+)}\rangle_{\mathrm{av}}]_{\alpha=3}\Psi_d(r). \tag{3.7}$$

For ready reference we give[6] $\mathfrak{F}^{(+)}$ explicitly in Appendix A.

Because of the awkwardness of carrying along the complete $\mathfrak{F}^{(+)}$ expression, we shall substitute for $\mathfrak{F}^{(+)}$ the familiar spin-flip and spin-nonflip notation used in the phenomenological description. We write

$$\frac{1}{4\pi}\mathfrak{F}_d=\mathbf{K}^{(+)}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+(L_1{}^{(+)}+L_2{}^{(+)})\cong(\mathbf{K}_p+\mathbf{K}_n)\frac{(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)}{2}+(L_p+L_n), \tag{3.8}$$

with $\simeq$ indicating that on the right-hand side we should not include the $\mathcal{G}^0$ part since the ground state of deuterium is isotopic singlet.

In terms of $\mathfrak{F}_d$ the differential cross section for the elastic photoproduction of π^0 from deuterium will be

$$d\sigma_{(\gamma+d\to\pi^0+d)}=\frac{1}{(4\pi)^2}\left[\left\langle\frac{w^2}{\epsilon_i\epsilon_f}\right\rangle_{\mathrm{av}}\frac{E_iE_f}{W^2}\right]|\mathfrak{F}_d{}^{(+)}|^2\frac{q}{k}d\Omega_q, \tag{3.9}$$

where w is the total energy of the one nucleon+pion in their own c.m. system. ϵ_i, ϵ_f are initial and final energies of the same one nucleon. The quantity in square brackets is practically equal to 1. Therefore

$$d\sigma_{(\gamma+d\to\pi^0+d)}\simeq\frac{1}{(4\pi)^2}|\mathfrak{F}_d{}^{(+)}|^2\frac{q}{k}d\Omega_q. \tag{3.10}$$

Numerical values of the quantities (averaged) are tabulated in Table I.

IV. FORM FACTORS AND SPIN SUMS: NOTATION

We represent the static deuteron wave function in the form

$$\Psi_d(\mathbf{r})=\frac{1}{(4\pi)^{\frac{1}{2}}}\left[(1-\eta^2)^{\frac{1}{2}}\frac{u(r)}{r}+\frac{\eta}{\sqrt{8}}\frac{w(r)}{r}S_{12}(\hat{r})\right]\chi_t, \tag{4.1}$$

where $(1-\eta^2)^{\frac{1}{2}}[u(r)/r]\chi_t$ is the s wave part, $(\eta/\sqrt{8})\times[w(r)/r]S_{12}(\hat{r})\chi_t$ the D-wave part, $S_{12}(\hat{r})=3\boldsymbol{\sigma}_1\cdot\hat{r}\boldsymbol{\sigma}_2\cdot\hat{r}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ the familiar tensor operator, χ_t the triplet spin state, and η^2 is the D-state percentage explicitly as a parameter. Using this wave function and averaging over unpolarized incident deuterons, and summing over all final spin states, we have the trace

$$\begin{aligned}\frac{\text{average}}{\text{initial}}\sum_{\text{final}}|\mathfrak{F}_d|^2=\tfrac{1}{3}\,\mathrm{Tr}\Bigg\{&\int\frac{d^3r}{4\pi}\exp(i\mathbf{Q}\cdot\mathbf{r}/2)\left((1-\eta^2)^{\frac{1}{2}}\frac{u(r)}{r}+\frac{\eta}{\sqrt{8}}\frac{w(r)}{r}S_{12}(\hat{r})\right)[\mathbf{K}^{(+)}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+(L_1{}^{(+)}+L_2{}^{(+)})]\\&\times\left((1-\eta^2)^{\frac{1}{2}}\frac{u(r)}{r}+\frac{\eta}{\sqrt{8}}\frac{w(r)}{r}\right)\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}\int\frac{d^3\mathbf{r}'}{4\pi}\exp(-i\mathbf{Q}\cdot\mathbf{r}')/2\left((1-\eta^2)^{\frac{1}{2}}\frac{u(r')}{r'}+\frac{\eta}{\sqrt{8}}\frac{w(r')}{r'}S_{12}(\hat{r}')\right)\\&\times[K^{(+)*}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+(L_1{}^{(+)*}+L_2{}^{(+)*})]\left((1-\eta^2)^{\frac{1}{2}}\frac{u(r')}{r'}+\frac{\eta}{\sqrt{8}}\frac{w(r')}{r'}S_{12}\right)\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}\Bigg\},\end{aligned} \tag{4.2}$$

where $(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)/4$ is the spin triplet projection operator. To calculate the above trace we find it convenient to introduce the following notation. We define

$$F_{fq}(Q)\equiv\frac{1}{4\pi}\int\exp(i\mathbf{Q}\cdot\mathbf{r}/2)\frac{f(r)}{r}\frac{g(r)}{r}d^3\mathbf{r}=\int_0^\infty j_0(Qr/2)fgdr, \tag{4.3}$$

where $j_0(x)=(\sin x)/x$ is the zero-order spherical Bessel function. For $f=g(r)$, $F_{ff}(q)$ defines the form factor $F_0(q)$, of the s-wave function $(4\pi)^{-\frac{1}{2}}f(r)/r$. Also, through the identity

$$\int\exp(i\mathbf{Q}\cdot\mathbf{r}/2)S_{12}(\hat{r})\frac{f(r)}{r}\frac{g(r)}{r}d^3\mathbf{r}=-4\pi S_{12}(\mathbf{Q}/Q)\int_0^\infty j_2(Qr/2)fgdr \tag{4.4}$$

we shall use the definition

$$G_{fg}(Q)\equiv\int_0^\infty j_2(Qr/2)f(r)g(r)dr, \qquad (4.5)$$

where

$$j_2(x)=\left[-\frac{3}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)-\frac{\sin x}{x}\right]$$

is the spherical Bessel function of second order. $G_{fg}(q)$ is related to what is called F_2 in the deuteron wave functions.[7] With a deuteron wave function written there in the form

$$\Psi_d=\frac{1}{(4\pi)^{\frac{1}{2}}}\left(\frac{u'(r)}{r}+\frac{1}{\sqrt{8}}\frac{w'}{r}S_{12}(\hat{r})\right)\chi_t, \qquad (4.6)$$

$F_2(q)$ is defined by

$$F_2(Q)=\int_0^\infty 2w'\left(u'-\frac{1}{\sqrt{8}}w'\right)j_2(Qr/2)dr. \qquad (4.7)$$

In our notation, $u'=(1-\eta^2)^{\frac{1}{2}}u$, $w'=\eta w$,

$$F_2(Q)=2\eta(1-\eta^2)^{\frac{1}{2}}G_{uw}(Q) -(\eta^2/\sqrt{2})G_{ww}(Q)\simeq(2\eta-\eta^2/\sqrt{2})G_{uw}, \qquad (4.8)$$

where the approximation in (4.8) is good since ϵ is going to be a small number and also $j_2(qr)$ keeps u and w away from the central region where they are different.

Since the formulas are rather lengthy (on account of the tensor operator) we will give here only the final result.

$$\begin{aligned}(d\sigma/d\Omega)_{(\gamma+d\to\pi^0+d)}=&[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][(1-\eta^2)F_{uu}{}^2+\eta^2F_{ww}{}^2]+[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][4\eta^2(1-\eta^2)G_{uw}{}^2+\tfrac{1}{2}\eta^4G_{ww}{}^2]\\ &-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2][\sqrt{2}\eta(1-\eta^2)^{\frac{3}{2}}F_{uu}G_{wu}-\tfrac{1}{2}\eta^2(1-\eta^2)F_{uu}G_{ww}\\ &+\eta^2(1-\eta^2)G_{wu}{}^2+\tfrac{1}{4}\sqrt{2}\eta^3(1-\eta^2)^{\frac{1}{2}}F_{ww}G_{wu}-\tfrac{1}{2}\sqrt{2}\eta^3(1-\eta^2)^{\frac{1}{2}}G_{ww}G_{wu}\\ &-\tfrac{1}{2}\eta^4F_{ww}G_{ww}+\tfrac{1}{8}\eta^4G_{ww}{}^2]-\tfrac{4}{3}\sqrt{2}\eta^3(1-\eta^2)^{\frac{1}{2}}|\mathbf{K}^{(+)}|^2G_{ww}G_{wu}, \qquad (4.9)\end{aligned}$$

where for completeness we give all terms irrespective of their importance. We shall express formula (4.9) in familiar quantities and in doing so give it in a much simpler form. The first of the four terms in formula (4.9) is nothing else but the usual $[\frac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2]F_0{}^2(Q)$. The three latter are our new terms. Their importance is a function of the D-state percentage η^2, and of Q. They come from the mixing of s and D states, through the transition matrix [except the negligible $\frac{1}{2}\eta^4G_{ww}(F_{ww}-\frac{1}{4}G_{ww}{}^2)$ which obviously involves only D state], and vanish for a pure s-state deuteron wave function. Since all correction terms contain G they vanish for zero recoil and $(d\sigma/d\Omega_q)(Q\to 0)=\frac{2}{3}|K^{(+)}|^2+|L^{(+)}|^2$ as it should, since at zero recoil we see only a point deuteron. As Q increases, first, the linear term in G would tend to make the correction negative, and the positive term proportional to G^2 would dominate very soon as Q increases. Since for small Q the new terms tend to be very small, actually we do not expect to see the negative term at all. To further simplify (4.9) we shall replace G_{ww} by G_{uw} in the small terms (also F_{ww} with F_{uu}) and leave out higher order terms. The cross section then takes the form

$$\begin{aligned}d\sigma/d\Omega_q=&[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0{}^2+4\eta^2(1-\tfrac{7}{8}\eta^2)G_{ww}{}^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\\ &\times\{[\sqrt{2}\eta(1-\eta^2)^{\frac{3}{2}}-\tfrac{1}{4}\sqrt{2}(1-\eta^2)+\tfrac{1}{4}\eta^2(1-\eta^2)^{\frac{1}{2}}-\tfrac{1}{4}\sqrt{2}\eta^2]F_{uu}G_{uw}+\eta^2[(1-\eta^2)-\tfrac{1}{2}\sqrt{2}\eta(1-\eta^2)^{\frac{1}{2}}+\tfrac{1}{8}\eta^2]G_{uw}{}^2\}\\ &-\tfrac{4}{3}\sqrt{2}\eta^3|\mathbf{K}^{(+)}|^2G_{wu}{}^2\cong[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0{}^2+4\eta^2G_{uw}{}^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\\ &\times[\sqrt{2}\eta(1-\tfrac{1}{4}\sqrt{2})F_{uu}G_{uw}+\eta^2G_{uw}{}^2]-\tfrac{4}{3}\sqrt{2}\eta^3|\mathbf{K}^{(+)}|^2G_{wu}{}^2. \qquad (4.10)\end{aligned}$$

The formula (4.10) gives the differential cross section for photoproduction of neutral pions from deuterium with the percentage η^2 of D state in the ground-state deuteron wave function, explicitly as a parameter. To compare different models we introduce numerical values.

For Yukawa model 1, Y_1 (reference 7), with $\eta^2=0.03$ D-state admixture, formula (4.10) gives

$$\begin{aligned}(d\sigma/d\Omega_q)_{(\gamma+d\to\pi^0+d)}=&[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0{}^2+0.12G_{uw}{}^2][2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\\ &\times[0.08F_0G_{uw}+0.03G_{uw}{}^2]-0.01|\mathbf{K}^{(+)}|^2G_{uw}{}^2.\end{aligned}$$

For Yukawa model 2, Y_2, the formula becomes

$$\begin{aligned}(d\sigma/d\Omega_q)_{(\gamma+d\to\gamma^0+d)}=&[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0{}^2+0.16G_{uw}{}^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\\ &\times[0.098F_0G_{uw}+0.046G_{uw}{}^2]-0.015|\mathbf{K}^{(+)}|^2G_{uw}{}^2.\end{aligned}$$

Replacing the form factor G_{uw} in (4.11) and (4.12) by the more familiar F_2 through formula (4.8), we put the above formulas in the alternative forms.

$$(d\sigma/d\Omega_q)_{(\gamma+d\to\pi^0+d)}=[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0^2+0.87F_2^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\times[0.216F_2F_0+0.22F_2^2]-0.7|\mathbf{K}^{(+)}|^2F_2^2, \quad (4.11a)$$

$$(d\sigma/d\Omega_q)_{(\gamma+d\to\pi^0+d)}=[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0^2+1.5F_2^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\times[0.3F_2F_0+0.43F_2^2]-0.14|\mathbf{K}^{(+)}|^2F_2^2, \quad (4.12a)$$

where according to (4.8) we have put $G_{uw}\simeq 3.08F_2$ for the Y_1 model and $C_{uw}\simeq 2.7F_2$ for the Y_2. For both previous models Y_1 and Y_2, the new terms are numerically small. Thus for the highest momentum transfers considered here (2.7 f^{-1}) they contribute less than 10%, and for all practical purposes the cross section is well approximated in the usual formula

$$(d\sigma/d\Omega_q)_{(\gamma+d\to\pi^0+d)}=[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2]F_0^2(Q) \quad (4.13)$$

derived for purely s-wave ground-state deuteron wave functions.

We come now to the interesting case of the repulsive core potential with the accompanying higher percentage of D state. In the model considered here with a 7% D state the cross section becomes

$$(d\sigma/d\Omega_q)_{(\gamma+d\to\pi^0+d)}\cong[\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2][F_0^2+1.2F_2^2]-[2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\tfrac{2}{3}|\mathbf{K}^{(+)}|^2]\times[0.64F_{uu}F_2+0.24F_2^2]-0.14|\mathbf{K}^{(+)}|^2F_2^2. \quad (4.13a)$$

The larger percentage of D state here increases the new terms substantially: Compare (4.13) with (4.11) and (4.12). At the same time $F_0(Q)$, the s-wave form factor in the repulsive core model as seen in Fig. 1, falls more rapidly at high momentum transfers and the relative importance of the new terms is further enhanced at high Q.

The new terms contribute as much as 40% of the cross section for our highest momentum transfers and the form factors with or without the D state show quite different behavior as we will see in the following section.

V. EFFECTIVE S-WAVE FORM FACTOR COMPARISON WITH EXPERIMENT

To compare with experiment one finds it convenient to define an "effective s-wave" form factor $(F_0)_{\mathrm{eff}}(Q)$,

$$(F_0)_{\mathrm{eff}}\equiv(d\sigma/d\Omega_q)/(\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2), \quad (5.1)$$

and compare with

$$(d\sigma/d\Omega_q)_{\mathrm{exp}}/(\tfrac{2}{3}|\mathbf{K}^{(+)}|^2+|L^{(+)}|^2),$$

where $(d\sigma/d\Omega)_{\mathrm{exp}}$ is the experimental $\gamma+d\to\pi^0+d$ cross section. Calculated quantities have been tabulated in Table I. In this table $2|\mathbf{K}^{(+)}\cdot\mathbf{Q}/Q|^2-\frac{2}{3}|\mathbf{K}^{(+)}|^2$ is expressed in the form $(2\cos^2\gamma-\frac{2}{3})|\mathbf{K}^{(+)}|^2$. Explicit calculations of $\cos\gamma$, $|\mathbf{K}^{(+)}|^2$, $|L^{(+)}|^2$, and other kinematic quantities have been omitted as too lengthy and awkward to present here.

Figure 1 shows some of the form factors plotted as functions of the deuteron momentum transfer. The scale is logarithmic. Curves 1 and 2 refer to the two Yukawa models.[7] Curve 3 shows $F_0(Q)$ for the repulsive core model. Because of the repulsive core it falls faster at high Q. Curve 4 shows the effective form factor we calculate in this repulsive core model, including the 7% D state. The change of the curvature of the effective form factor as a function of Q in the various models is sharply distinct. The curvature even changes sign. Thus, while for the s-wave part of the repulsive-core model as well as in Y_1 and Y_2 (which for practical purposes behave as pure s waves) the curvature is always downwards, for the repulsive core model 4 the curvature of the effective form factor changes sign. Curve 5 contains the experimental points.

We observe a characteristic similarity between curves 4 and 5. A few more points for higher Q's would be very interesting here. We are already in the region where the form factors change rapidly.

The experimental points lie below our curves. This is attributed to the fact that our calculation is in impulse approximation; the actual cross sections are smaller due to multiple scattering. In this paper however, more attention is given to the slope of the form factors rather than to absolute values.

We think that this experiment is in favor of the repulsive core picture with an appreciable percentage of D-state admixture. Perhaps with more refinement of the theory as to potential, multiple scattering corrections, and relativistic effects, one might be able to decide between models. We need not stress the importance of this not only for the deuteron itself but also to other investigations, e.g., study of the neutron structure, since the deuteron offers a natural target for neutrons.

We remark also that this experiment is formally related to Compton scattering from deuterium. A comparison is worthwhile though at present our knowledge of Compton scattering from single nucleons and deuterium is not good enough to permit drawing conclusions.

ACKNOWLEDGMENTS

I wish to thank J. Friedman and H. W. Kendall whose experiments and interest stimulated this work to a great extent. Many of the calculations were done with J. Friedman and the numerical results were compared. I also wish to thank S. D. Drell for his constant advice throughout this work.

APPENDIX

We give here the full expression for $\mathfrak{F}^{(+)}$, taken from reference 6:

$$\mathfrak{F}^{(+)}/ef=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}[\tfrac{2}{3}i(\delta_{\frac{1}{2}S}-\delta_{\frac{3}{2}S})F_S+\omega N^{(+)}]+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}[-\lambda h^{(+-)}-\tfrac{2}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q-\tfrac{1}{3}F_M)]$$
$$+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}[\lambda h^{(+-)}-\tfrac{2}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q+\tfrac{1}{3}F_M)]+\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})$$
$$\times[\lambda h^{(++)}+(4/9)ie^{i\delta_{33}}\sin\delta_{33}F_M]+i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}(1/2m\omega), \quad \text{(A.1)}$$

where

$$h^{(++)}=\tfrac{1}{3}(h_{11}+2h_{13}+2h_{31}+4h_{33}),$$
$$h^{(+-)}=\tfrac{1}{3}(h_{11}-h_{13}+2h_{31}-2h_{33}),$$
$$h_{Ij}=(e^{i\delta_{Ij}}\sin\delta_{Ij})/q^3,$$
$$F_S=1-\tfrac{1}{2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right),$$
$$F_Q=\frac{1}{\omega^2}\left[1-\frac{3}{4v^2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right)\right],$$
$$F_M=\frac{3}{4q^2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right),$$

$\omega=$ Total c.m. energy $-m=(2mk_{\text{lab}}+m^2)^{\frac{1}{2}}-m$,

$v=q/\omega_q$, the c.m. velocity of the pion,

$\lambda=(g_p-g_n)/4mf^2, \quad e^2/4\pi=1/137, \quad f^2/4\pi=0.08.$

A detailed discussion of the physical interpretation of the various terms, in terms of electric and magnetic multipoles and the pion-nucleon phase shifts, is given in reference 4. In the s-wave spin flip term the term proportional to the difference of the small s-wave phase shifts $(\delta_{\frac{1}{2}s}-\delta_{\frac{3}{2}s})$ results from charge exchange. Charge exchange also gives rise to the terms involving F_Q, F_M. The following two spin-flip terms are mainly $M1$ and $E2$. The third term, spin-nonflip, is mainly $M1$. The term $i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}(1/2m\omega)$ accounts for nuclear recoil.

CALCULATION OF THE SPIN-FLIP $|K^{(+)}|^2$ AND SPIN-NONFLIP $|L^{(+)}|^2$ AMPLITUDES

In the evaluation of the cross section we shall need $|\mathbf{K}^{(+)}|^2$ and $|L^{(+)}|^2$. In order to simplify the calculation, in squaring the amplitude $\mathfrak{F}^{(+)}$ (A.1) we shall keep only the terms which lead in magnitude plus terms from interference of large and small amplitudes when appreciable. Thus we obtain

$$|L_{(1,2)}{}^{(+)}|^2=\frac{e^2}{4\pi}\frac{f^2}{4\pi}|\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}|^2\left(\frac{4\lambda}{3q^3}\right)^2[\sin^2\delta_{33}+\sin\delta_{33}\cos\delta_{33}\sin(2\delta_{31})],$$

$$|\mathbf{K}^{(+)}|^2=\frac{e^2}{4\pi}\frac{f^2}{4\pi}\Bigg\{(\mathbf{q}\cdot\mathbf{k})^2\left(\frac{\lambda}{3q^3}\right)^2[4\sin\delta_{33}(\sin\delta_{33}-\sin\delta_{13}-\sin\delta_{11})]$$
$$+(8/3)(\mathbf{k}\cdot\mathbf{q})\frac{\lambda}{3q^3}F_S(\delta_{\frac{1}{2}S}-\delta_{\frac{3}{2}S})\sin^2\delta_{33}+[\tfrac{2}{3}(\delta_{\frac{1}{2}S}-\delta_{\frac{3}{2}S})F_S]^2 \quad \text{(A.2)}$$
$$+k^2(\mathbf{q}\cdot\boldsymbol{\varepsilon})^2\left[\left(\frac{\lambda}{3q^3}\right)^2 4\sin\delta_{33}(\sin\delta_{33}-\sin\delta_{13}-\sin\delta_{11})+(4/9)(F_Q+F_M)^2\sin^2\delta_{33}\right]$$
$$+(8/3)\frac{\lambda}{3q^3}(F_Q+\tfrac{1}{3}F_M)\sin^2\delta_{33}(\sin\delta_{11}+\sin\delta_{13})\Bigg\}.$$

The phase shifts are to be calculated from the energy in the π^0 one-nucleon c.m. system and q is the momentum of the π^0 in the one-nucleon $+\pi^0$ c.m. system. Numerical values of the quantities (averaged) are tabulated in Table I.