On the Uniqueness of Fock's Harmonic Coordinate Systems in the Presence of Static, Spherically Symmetric Sources*

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Fock has claimed that his "harmonic" coordinate systems in curved space flattening out toward spatial infinity are uniquely determined but for an arbitrary inhomogeneous Lorentz transformation. If this is so, introduction of Fock's harmonic coordinate conditions would provide a natural way of introducing a Lorentz subgroup of the general coordinate transformation group of Einstein's gravitational theory, and of defining a Minkowski metric besides the curved-space metric. This would open the way to close relations between Einstein's gravitational theory on the one hand, and Lorentz-covariant quantum field theory on the other hand. A general proof of the correctness of Fock's claim, for universes satisfying his boundary conditions, has never been given rigorously. Here we extend an earlier proof of this uniqueness for the Schwarzschild field around a single gravitational singularity, to the case of the static and spherically symmetric field generated in some coordinate system by an extended static and spherical distribution of energy and of stresses. The uniqueness (but for the zero point of time and for a spatial rotation) of the harmonic coordinate system, in which this field is spherical and at rest around the spatial origin, is here guaranteed by the condition that there must be a one-to-one correspondence between the points x, y, z, t of the harmonic coordinate system and the points in physical space.

1. FOCK'S HARMONIC CONDITIONS

LATELY, there has been an increased interest in Fock's claim that his "harmonic" coordinate systems form a Lorentz manifold.[1] One reason for this is that the existence of a unique Minkowski tensor in curved space would greatly simplify the unification of Einstein's gravitational theory with Lorentz-covariant quantum field theory.[2]

Fock's conditions for what he calls a harmonic coordinate system are, besides the validity of the De Donder coordinate condition

$$\mathfrak{g}^{\alpha\beta}{}_{,\beta}=0, \tag{1}$$

a number of boundary conditions imposed upon the field as well as on the coordinate systems singled out to describe the field. Crudely speaking,[3] these conditions amount to the following:

(A) Space shall flatten out sufficiently toward spatial infinity so that there exist world coordinate systems in which, at large distances r, the metric $g_{\mu\nu}$ or $\mathfrak{g}^{\mu\nu}$ approximates the Minkowski metric $\gamma_{\mu\nu}$ to within a difference of the order of $1/r$:

$$g_{\alpha\beta}\rightarrow\gamma_{\alpha\beta}+O(1/r),\quad \mathfrak{g}^{\alpha\beta}\rightarrow\gamma^{\alpha\beta}+O(1/r). \tag{2}$$

(B) Harmonic coordinate systems shall be such that Eqs. (2) are actually satisfied in them.

(C) If the $O(1/r)$ terms in (2) include gravitational waves, then *incoming* waves shall be *absent* from them.[3,4]

(D) The metric field shall everywhere in physical space be finite and differentiable.

(E) The metrical determinant g shall not vanish, and g_{00} shall have the correct sign to make the time direction timelike.

(F) One set of values of the coordinates in a harmonic coordinate system shall never determine more than one single point in space-time, and, conversely, there shall not be more than one set of harmonic coordinate values corresponding to a single point in space-time.

To this seemingly trivial condition (F), so far, little attention has been given. As we shall show below for a number of special cases, it is a great help in keeping the harmonic[5] coordinate system unique except for possible inhomogeneous Lorentz transformations.

2. TRANSFORMATIONS TO AND BETWEEN HARMONIC COORDINATE SYSTEMS

Suppose that an arbitrary coordinate system $x'^{\gamma'}$ is given,[6] and that we want to transform to a harmonic coordinate system x^{α}. The new $\mathfrak{g}^{\alpha\beta}$ is expressed in terms of the old $\mathfrak{g}^{\gamma'\delta'}$ by

$$\mathfrak{g}^{\alpha\beta}=J^{-1}(\partial x^{\alpha}/\partial x^{\gamma'})(\partial x^{\beta}/\partial x^{\delta'})\mathfrak{g}^{\gamma'\delta'}, \tag{3}$$

where

$$J=\det(\partial x^{\alpha}/\partial x^{\gamma'}). \tag{4}$$

We must satisfy the De Donder condition,

$$0=J\mathfrak{g}^{\alpha\beta}{}_{,\beta}=\mathfrak{g}^{\gamma'\delta'}{}_{,\delta'}(\partial x^{\alpha}/\partial x^{\gamma'})+\mathfrak{g}^{\gamma'\delta'}(\partial^2 x^{\alpha}/\partial x^{\gamma'}\partial x^{\delta'}), \tag{5}$$

as well as a number of boundary conditions. Therefore, we must solve Eq. (5), and then select the solutions which will satisfy the boundary conditions.

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[1] V. Fock, *The Theory of Space Time and Gravitation* (Pergamon Press, New York, 1959), pp. 342–352.

[2] J. C. Garrison, Ph.D. thesis, Purdue University, 1961 (unpublished), available as part of a National Science Foundation report on "The Interaction Picture in Gravitational Theory and Some Related Topics" (Purdue University, September 1961).

[3] For details, see reference 1.

[4] In a flat space with Minkowskian metric, the "radiation condition" (C) requires little explanation. Since Fock has "proved" his conjecture so far only in the approximation in which the metric appearing in Eq. (7) is Minkowskian as in Eq. (12), we here shall not venture any guess about the way in which this condition should be interpreted in a nonflat space, and we shall hold it as Fock's claim that *some* interpretation is possible that makes his "theorem" valid.

[5] We shall always use the word "harmonic" in Fock's sense, denoting the validity not only of Eq. (1), but also of conditions (A)–(F).

[6] From here on, we shall abbreviate $x'^{\gamma'}$ as $x^{\gamma'}$, etc.

If the $x^{\gamma'}$ coordinates were already harmonic themselves, then $\mathfrak{g}^{\gamma'\delta'}{}_{,\delta'}=0$, and Eq. (5) simplifies to

$$\mathfrak{g}^{\gamma'\delta'}(\partial^2 x^\alpha/\partial x^{\gamma'}\partial x^{\delta'})=0 \tag{6}$$

for the "harmonic" transformation from one harmonic coordinate system $x^{\gamma'}$ to a new harmonic coordinate system x^α.

3. DO HARMONIC COORDINATE TRANSFORMATIONS FORM A GROUP?

Let Σ_1, Σ_2, $\cdots$, Σ_n, $\cdots$ be harmonic coordinate systems. If the transformations $\Sigma_1\rightarrow\Sigma_2$, $\cdots$, $\Sigma_k\rightarrow\Sigma_l$, $\cdots$, $\Sigma_m\rightarrow\Sigma_n$, $\cdots$ are to form a group, we must define these transformations in such a form that also the "product" of the transformations $\Sigma_k\rightarrow\Sigma_l$ and $\Sigma_m\rightarrow\Sigma_n$ is defined, even if $\Sigma_l\neq\Sigma_m$. For this purpose, it is usual[7] to give the transformations as relations between the old and the new coordinates without reference to the coordinate system on which they are to be applied, so that the product of $x^\alpha=t^\alpha(x'^\gamma)$ and $x^\alpha=T^\alpha(x'^\gamma)$ is the transformations (Tt) given by $x^\alpha=T^\alpha(t^\gamma(x'^\epsilon))$.

Let us see what conclusions we can draw, *if* the harmonic coordinate transformations T *would* form a group. If T transforms every[8] (primed) harmonic coordinate system into another (unprimed) harmonic one, then Eq. (6) requires that

$$\mathfrak{g}^{\gamma'\delta'}(\partial^2 T^\alpha/\partial x^{\gamma'}\partial x^{\delta'})=0, \tag{7}$$

where $\mathfrak{g}^{\gamma'\delta'}$ represents the metric in the arbitrary initial harmonic coordinate system on which T is applied. Since the $\mathfrak{g}^{\gamma'\delta'}$ are quite different from one harmonic coordinate system to the next, the condition (7) can be satisfied for arbitrary initial harmonic coordinate systems only if

$$(\partial^2 T^\alpha/\partial x^{\gamma'}\partial x^{\delta'})=0, \tag{8}$$

that is, if the transformations T are linear transformations:

$$x^\alpha=T^\alpha(x'^{\gamma'})=A^\alpha{}_{\gamma'}x^{\gamma'}+B^\alpha, \tag{9}$$

with constant $A^\alpha{}_{\gamma'}$ and B^α. The transformation (9) transforms the metrical tensor $g_{\gamma'\delta'}$ into $g_{\alpha\beta}$ in such a way that

$$g_{\gamma'\delta'}=A^\alpha{}_{\gamma'}A^\beta{}_{\delta'}g_{\alpha\beta}. \tag{10}$$

Since the condition (2) is to be satisfied by $g_{\gamma'\delta'}$ as well as by $g_{\alpha\beta}$, it follows from (10) that

$$\gamma_{\gamma'\delta'}=A^\alpha{}_{\gamma'}A^\beta{}_{\delta'}\gamma_{\alpha\beta}. \tag{11}$$

By this condition, the linear transformations (9) are confined to inhomogeneous Lorentz transformations. We thus have proved that, *if the harmonic coordinate transformations form a group, they must form the inhomogeneous Lorentz group.*

Fock[1] claims that the set of boundary conditions (A)–(F) confines harmonic coordinate systems to a manifold of coordinate systems which all are related to each other by inhomogeneous Lorentz transformations only. If he is right, then the harmonic coordinate transformations between systems satisfying his conditions do form the inhomogeneous Lorentz group. But, if one can find one exception to Fock's claim [and if it is impossible to amend Fock's conditions (A)–(F) in some way to exclude this exception], then the harmonic transformations do not form (and cannot be redefined so as to form) any group at all.[9]

4. THE BASIC IDEA UNDERLYING FOCK'S CLAIM

Consider the transformation T of the preceding section, from some given harmonic coordinate system to an arbitrary one. Then, T is to satisfy Fock's boundary conditions in addition to the differential equation (7).

The latter equation shows similarity to its flat-space equivalent,

$$\Box' T^\alpha\equiv\gamma^{\gamma'\delta'}(\partial^2 T^\alpha/\partial x^{\gamma'}\partial x^{\delta'})=0. \tag{12}$$

In the right-hand member of this equation, no delta functions can be allowed.[10] Therefore, the solutions of Eq. (12) can all be written as the sum of a linear transformation (9) and a superposition of waves that run across space, coming in on one side, and going out on the other side. (There will be no waves originating from interior points.) Since, however, condition (C) prevents the waves from coming in,[11] they cannot go out either, and the only solution left is the linear transformation (9), which had to be an inhomogeneous Lorentz trans-

[7] See V. Fock, reference 1, p. 347, Eq. (93.09). We here follow Fock in not investigating the more general group theoretical question whether it is possible to define a harmonic transformation group in which the transformation functions, $x^\alpha=T^\alpha(x')$, are functions of the $g'_{\mu\nu}(x')$ field in space-time. The reason for this is that primarily we are interested merely in attempts at formulating "harmonic" coordinate conditions that allow only inhomogeneous Lorentz transformations. The latter certainly can be written in a form $(x^\alpha=A^\alpha{}_{\lambda'}x^{\lambda'}+B^\alpha$ with $A^\alpha{}_{\lambda'}\gamma^{\lambda'\mu'}A^\beta{}_{\mu'}=\gamma^{\alpha\beta})$ that does *not* consider the transformation functions to be functionals of some metric field. Therefore, we are not interested in the question, in case the harmonic coordinate transformations defined by us do *not* form the inhomogeneous Lorentz group, whether one could perhaps find a set of $g_{\mu\nu}$-field dependent transformations between harmonic coordinate systems that would form some *other* group.

[8] If T would transform not all but just some particular harmonic coordinate systems into other harmonic coordinate systems, one could not tell whether such a transformation T should be regarded a harmonic transformation or not, and it would be impossible to define unambiguously "the group of all harmonic transformations and of nothing else."

[9] P. G. Bergmann, Phys. Rev. 124, 274 (1961).

[10] Such singularities in the right-hand member of (12) would lead to singularities in $(\partial T^\alpha/\partial x^{\gamma'})=(\partial x^\alpha/\partial x^{\gamma'})$, which would transform the metrical tensor into an expression that would no longer satisfy Fock's condition, (D), of being everywhere finite.

[11] Condition (C) prevents gravitational waves from coming in. This forbids incoming waves in the T^α for the following reason. Suppose T^α contained a term representing such a wave. Then, there would be a similar wave in the transformation coefficient $A^\alpha{}_{\mu'}\equiv\partial T^\alpha/\partial x^{\mu'}$, and consequently also in the transformed metric $g_{\mu'\nu'}$ obtained by tensor transformation from the initial metric $g_{\alpha\beta}$ which was practically Minkowskian in the approximation considered in Eq. (12). The wave thus caused in $g_{\mu'\nu'}$ violates condition (C).

formation on account of the consequence (10) of condition (A).

This argument does not rigorously prove Fock's claim, because the difference between the differential equations (7) and (12) has not been taken into account. Even apart from waves, we have to verify that the original equation (7), like its simplification (12), does not allow static solutions different from (9) in absence of singularities in its right-hand member, when Fock's conditions are imposed.

5. HARMONIC COORDINATES IN THE FIELD OF A NONACCELERATED SPHERICAL SOURCE

In a given matter field, we define the "sources" of gravity as the tensor

$$S_{\mu\nu}\equiv(8\pi G/c^4)[\tfrac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda-T_{\mu\nu}], \tag{13}$$

so that Einstein's equations take the simple form

$$R_{\mu\nu}=S_{\mu\nu}. \tag{14}$$

We want to consider here the special case of sources which by a Lorentz transformation can be transformed into a static and spherically symmetric $S_{\mu\nu}$ field. We want to show that the harmonic coordinate system in which this field is static and spherical around the origin is uniquely[12] determined, without ambiguity in the radial coordinate to be used.

There is then automatically a corresponding uniqueness (but for an inhomogeneous Lorentz transformation) in the systems in which the sources are moving with an arbitrary constant velocity. This is seen as follows.

Let Σ'' and Σ''' be two harmonic coordinate systems at rest with respect to each other, and in which all sources would have the same constant velocity. Suppose the spatial transformation from Σ''' to Σ'' were not an inhomogeneous Lorentz transformation (that is, not just a translation and rotation of the axes). In that case, there must exist a corresponding lack of uniqueness in the harmonic coordinate systems in which the sources are at rest. This is immediately seen by considering the Lorentz transformations from Σ''' and Σ'' to coordinate systems Σ' and Σ in which the sources are at rest. The transformation from Σ' to Σ cannot be a Lorentz transformation, because it is the product of the three transformations $\Sigma'\to\Sigma'''$, $\Sigma'''\to\Sigma''$, and $\Sigma''\to\Sigma$, of which two are Lorentz transformations, but the one in the middle was not.

Therefore, it is not necessary to investigate separately the case in which the nonaccelerated sources are moving together at a fixed velocity, and we shall consider here merely the static case.

For that case, let us transform from some initially given (and possibly not yet harmonic) coordinate system x', y', z' to polar coordinates by

$$x'=r'\sin\theta'\cos\varphi',\quad y'=r'\sin\theta'\sin\varphi',\quad z'=r'\cos\theta', \tag{15}$$

where it is possible that r' is restricted to values $r'\geq r_0'$ or $r'>r_0'$ for satisfying condition (E). We shall assume that the initial coordinate system was one in which the sources were static and spherical around the origin. Then, after the transformation (15), the source components $S_{t't'}$, $S_{r'r'}$, and $S_{\theta'\theta'}=S_{\varphi'\varphi'}/\sin^2\theta'$ will depend merely on r'.

We now want to transform to a harmonic coordinate system x,y,z,t, in which the sources are still static and spherical around the origin. So, if this time we introduce polar coordinates by

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta, \tag{16}$$

then we want S_{tt}, S_{rr}, and $S_{\theta\theta}=S_{\varphi\varphi}/\sin^2\theta$ to be functions of r only.

The coordinate transformation which maintains this spherical symmetry and keeps the field static, and which does not alter the angles at $r=\infty$, is

$$t=at'+b,\quad r=r(r').\quad \varphi=\varphi',\quad \theta=\theta'. \tag{17}$$

We could, of course, obtain a more general solution by allowing a rotation of the coordinate axes (which at $r=\infty$ would change the angles to the axes), but this possibility is so trivial that we shall not further discuss it. By choosing already t' in such a way as to make $g_{t't'}=\gamma_{t't'}+O(1/r)$, we can make $a=1$, so that we might as well replace $t=at'+b$ by $t=t'$, as the possibility of a shift of the zero point of time again is too trivial to discuss.

Therefore, we introduce our harmonic coordinate system by merely introducing a new radial coordinate $r=r(r')$. The question of the uniqueness (except for trivial transformations that belong to the inhomogeneous Lorentz group anyhow) thus reduces to the question of the uniqueness of the harmonic radial coordinate $r(r')$.

For our nonharmonic initial choice of radial coordinate r', we shall from here on choose the Schwarzschild radial coordinate ρ, which is defined as $(1/2\pi)\times$ the circumference of a circle around the origin. The square of the invariant line element is then

$$ds^2=U^2d\rho^2+\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)-w^2c^2dt^2. \tag{18}$$

The dependence of U and of w on ρ depends on the spherical distribution of sources $S_{\mu\nu}$. We shall not specify it here, but we shall assume that U and w are neither zero nor infinite for any allowable value of ρ.

We now seek transformations from the Scharzschild coordinate system ρ,θ,φ to a harmonic coordinate system x,y,z by means of Eqs. (16)–(17). This transformation has been discussed before.[13] It was found that the functions u, v, and w in the new expression

$$\begin{aligned} ds^2&=(Ud\rho/dr)^2dr^2+\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)-w^2c^2dt^2\\ &\equiv u^2dr^2+v^2(d\theta^2+\sin^2\theta\, d\varphi^2)-w^2c^2dt^2 \end{aligned} \tag{19}$$

[12] Except for the trivial possibilities of a shift of the zero point of time, or of a spatial rotation of the coordinate axes around the origin.

[13] F. J. Belinfante, Phys. Rev. 98, 793 (1955).

would have to satisfy the condition[14]

$$d(u^{-1}v^2w)/dr = 2ruw \tag{20}$$

for ensuring the validity of Eq. (1) in the *xyzt* system. With $u = Ud\rho/dr$ and with $v = \rho$, this gives, after multiplication by $dr/d\rho$, the condition

$$\frac{d}{d\rho}\left[\frac{w}{U}\rho^2\frac{dr}{d\rho}\right] \tag{21}$$

imposed upon the function $r(\rho)$. The problem of uniqueness for the harmonic coordinate system in the static and spherically symmetric case now reduces to the uniqueness of the solution of Eq. (21) for $r = r(\rho)$ on account of the conditions imposed. We shall investigate this uniqueness first for some specific cases for which the functions $U(\rho)$ and $w(\rho)$ are known, and then more in general.

6. EXTERIOR SCHWARZSCHILD FIELD

For Schwarzschild's exterior solution [that is, the static and spherically symmetric solution of Eq. (14) for $S_{\mu\nu} = 0$], it has been found that Eq. (21) is solved by[15]

$$r = C_1\{1 + [(\rho - m)/2m]\ln[(\rho - 2m)/\rho]\} + C_2(\rho - m). \tag{22}$$

Since ρ for $r \to \infty$ asymptotically measures radial distances correctly where space flattens out, we must have $C_2 = 1$ in Eq. (22) because of condition (B).

In the past,[13] we rejected the term with the factor C_1 because it would make r become $-\infty$ for $\rho \to 2m$. [Since in this case $-g_{00} = w^2 = 1 - 2m/\rho$, values $\rho \leq 2m$ are not allowed by condition (E), but $\rho \to 2m+0$ is possible.]

However, this objection against the term with C_1 is superfluous, as we have a much stronger objection against this term on account of our condition (F). For, if C_1 were positive, r would become negative already for values of ρ larger than $2m$, and one could find for ρ pairs of values ρ_1 and ρ_2 such that $\rho_1 > \rho_2 > 2m$ and that $r_2 = -r_1$, if $r_1 = r(\rho_1)$ and $r_2 = r(\rho_2)$. Then, for $\theta_2 = \pi - \theta_1$ and $\varphi_2 = \pi + \varphi_1$, the two different points in physical space with Schwarzschild coordinates ρ_1, θ_1, φ_1 and ρ_2, θ_2, φ_2 would by (16) have the same set of harmonic coordinates x, y, z, in violation of condition (F).[16]

If, on the other hand, C_1 were negative, then there would exist a point with $\rho = \rho_m$, at which $r(\rho)$ would reach a minimum value. We then could find pairs of values, $\rho_1 > \rho_m$ and $\rho_2 < \rho_m$, such that $r_1 = r_2$. This time, the different points ρ_1, θ_1, φ_1 and ρ_2, θ_2, φ_2 with $\theta_1 = \theta_2$ and $\varphi_1 = \varphi_2$ would be represented by the same sets of harmonic coordinates x, y, z, in violation of condition (F).[16]

Therefore, the only possibility left for a harmonic coordinate system x, y, z in which the Schwarzschild exterior field is static and spherically symmetric around the origin, is given by Eqs. (16), (17), and (22) with

$$C_1 = 0 \quad \text{and} \quad C_2 = 1, \tag{23}$$

or is obtained from this by a trivial spatial rotation, or shift of the zero point of time.

7. FIELD OF AN ELECTROSTATIC POINT CHARGE

The case considered in the previous section was special because of the occurrence of a "Swiss cheese" hole (world tube) in space,[13] that is, a region $\rho < 2m$ (or $r < m$) which was considered unphysical because $-g_{00} = w^2 = 1 - (2m/\rho)$ would become negative in it. In reality, such holes may never occur, since the presence of the material sources $S_{\mu\nu}$ in the gravitational equations (14) has a tendency to fill up these holes.

For instance, if the $S_{\mu\nu}$ field is due to the energy and stress tensor of an electrostatic field with potential $V = \epsilon/\rho$, then the solution of Einstein's and Maxwell's equations can be expressed in Schwarzschild coordinates with the square of the invariant line element given by Eq. (18) with[17]

$$w^2 = 1/U^2 = 1 - 2m/\rho + \eta^2/\rho^2, \tag{24}$$

where $m = GM/c^2$ and where $\eta^2 = G\epsilon^2/c^4$. We now must distinguish two cases, depending on the charge-to-mass ratio ϵ/M.

Small Charge $\epsilon < M\sqrt{G}$

A Swiss-cheese hole is left if $\eta^2 < m^2$, that is, if $\epsilon^2 < GM^2$. The "radius" of the hole is now given by

$$\rho_0 = m + \delta, \tag{25}$$

where

$$\delta = +(m^2 - \eta^2)^{\frac{1}{2}}. \tag{26}$$

The general solution from (21) for the De Donder radial coordinate is now found to be

$$r = C_1\left[1 + \frac{\rho - m}{2\delta}\ln\frac{\rho - m - \delta}{\rho - m + \delta}\right] + C_2(\rho - m), \tag{27}$$

where $C_2 = 1$ and $C_1 = 0$ for the same reasons as before, if r is to be the harmonic radial coordinate for which the conditions (B) and (F) are valid. This simplifies (27) to

$$r = \rho - m, \tag{28}$$

as in absence of charge. Since we must require

[14] Reference 13, Eq. (22).

[15] Compare Eqs. (30) and (34) of reference 13.

[16] The fact that we can invent functions $r(\rho)$ that become negative below some point $\rho = \rho_n$, or that have a minimum at some point $\rho = \rho_m$, is not a physical reason that would allow us to exclude points with $\rho < \rho_n$ or with $\rho < \rho_m$ from physical space.

[17] See, for instance, H. Weyl, *Space-Time-Matter* (Dover Publications, New York, 1950), bottom of pp. 252 and 256 together with top of p. 261; or W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958), p. 171, Eq. (435), in which κ is smaller by a factor 8π than it is in Eq. (409) on p. 163.

$\rho > \rho_0 \equiv m+\delta$, the harmonic radius r this time is restricted by

$$r > \delta (>0). \tag{29}$$

For finding the metric in the harmonic coordinate system, we use Eqs. (19) and (24) and obtain

$$ds^2 = U^2 dr^2 + (r+m)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) - w^2 c^2 dt^2, \tag{30}$$

where, on account of (24) with (26),

$$w^2 = 1/U^2 = \left(1-\frac{m}{\rho}\right)^2 - \frac{\delta^2}{\rho^2} = \frac{(r^2-\delta^2)}{(r+m)^2}. \tag{31}$$

Thence,

$$ds^2 = \left(\frac{r+m}{r}\right)^2 \left[(dx^2+dy^2+dz^2) + \left(\frac{1}{r^2-\delta^2} - \frac{1}{r^2}\right)(xdx+ydy+zdz)^2 \right] - \frac{(r^2-\delta^2)}{(r+m)^2} c^2 dt^2. \tag{32}$$

This gives for the metric in the harmonic $xyzt$ system

$$g_{kl} = \left(1+\frac{m}{r}\right)^2 \left[\delta_{kl} + \left(\frac{\delta^2}{r^2-\delta^2}\right)\frac{x^k x^l}{r^2}\right],$$

$$g_{00} = -\left(1+\frac{m}{r}\right)^{-2}\left[1-\frac{\delta^2}{r^2}\right]; \quad (-g)^{\frac{1}{2}} = \left(1+\frac{m}{r}\right)^2; \tag{33}$$

$$\mathfrak{g}^{kl} = \delta^{kl} - \frac{\delta^2}{r^2}\frac{x^k x^l}{r^2}, \quad \mathfrak{g}^{00} = -\frac{\left(1+\frac{m}{r}\right)^4}{\left[1-\frac{\delta^2}{r^2}\right]}.$$

These $\mathfrak{g}^{\mu\nu}$ (not the ones computed in polar coordinates) satisfy Eq. (1). We can use them to illustrate the use of Eq. (6). Transformations from the above harmonic $xyzt$ system to a new coordinate system $x'y'z't'$ that, too, satisfies the De Donder condition (1), will have to satisfy the differential equations (6) with the $xyzt$ and $x'y'z't'$ interchanged. By (33), this is

$$\left[\nabla^2 - \frac{\delta^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{(1+m/r)^4}{[1-\delta^2/r^2]}\frac{\partial^2}{c^2\partial t^2}\right] x^{\alpha'} = 0. \tag{34}$$

For keeping a static and spherically symmetric solution, we want to solve (34), in accordance with Eqs. (15)–(17), by

$$x^{0'} = x^0 \equiv ct, \quad x^{k'} = f(r) x^k. \tag{35}$$

Then, Eq. (34) yields

$$[f'' + (2/r)f'][1-(\delta^2/r^2)] + (2/r)f' = 0. \tag{36}$$

Solving this equation for $f(r)$, and then putting $r' = f(r)r$, we obtain

$$r' = C_1\left[1+\frac{r}{2\delta}\ln\frac{r-\delta}{r+\delta}\right] + C_2 r. \tag{37}$$

This agrees with the right-hand member of (27) with (28) substituted. Conditions (B) and (F) then yield again $C_2=1$ and $C_1=0$, if $x'y'z't'$ is to be a harmonic system. This gives

$$r' = r, \tag{38}$$

showing the uniqueness of this system.

Large Charge $\varepsilon > M\sqrt{G}$

The situation is completely changed for $\eta^2 > m^2$, that is, if $\epsilon^2 > GM^2$, as it is for all charged elementary particles. In this case, $-g_{00} \equiv w^2$ never becomes zero, and ρ can take all possible values down to $\rho=0$. Putting in this case

$$\Delta = +(\eta^2 - m^2)^{\frac{1}{2}}, \tag{39}$$

we find, instead of (27),

$$r = C_1\left[1 - \frac{\rho-m}{\Delta}\arctan^*\frac{\Delta}{\rho-m}\right] + C_2(\rho-m), \tag{40}$$

where the meaning of the asterisk following the arc tan symbol is the following.

In principle, one could use in (40) any branch of the arc tan function. These branches differ by constant (positive or negative) multiples of π from the conventional branch, that is, the one bounded between $-\pi/2$ and $+\pi/2$. We notice at once that use of a different branch merely alters the value of C_2 by a multiple of $\pi C_1/\Delta$.

Therefore, for $\rho > m$ we can without loss of generality pick the conventional branch of the arc tan function [between 0 and $\pi/2$ for positive $(\rho - m)$ in Eq. (40)]. Then, condition (B) will provide as in the previous case that $C_2 = 1$.

The derivative of Eq. (40) is

$$\frac{dr}{d\rho} = C_1\left[\frac{\rho-m}{(\rho-m)^2+\Delta^2} - \frac{1}{\Delta}\arctan^*\frac{\Delta}{\rho-m}\right] + C_2. \tag{41}$$

For a solution of Eq. (21), that is, in our case, of

$$\frac{d}{d\rho}\left\{[(\rho-m)^2+\Delta^2]\frac{dr}{d\rho}\right\} = 2r, \tag{42}$$

it is necessary that $dr/d\rho$ is continuous and differentiable for all positive values of ρ, including the point $\rho = m$. At this point, the conventional branch of $\arctan[\Delta/(\rho-m)]$ has a discontinuity. To enforce continuity of $dr/d\rho$ at this point, we have to go over from the conventional branch (which becomes negative for $\rho < m$) to the branch which lies a distance π higher. Therefore, we define

$$\begin{aligned} \arctan^* x &= \arctan x + \pi \quad \text{for} \quad x<0, \\ & \arctan x \quad \text{for} \quad x \geq 0, \end{aligned} \tag{43}$$

where arc tan represents the conventional branch between $-\pi/2$ and $+\pi/2$, so that $0 \leq \arctan^* x < \pi$. This definition does not alter our convention for $\rho > m$, which led to $C_2 = 1$.

As before, we shall use condition (F) for fixing the value of C_1, but this time we shall use the second half of this condition. We notice that for $\rho = 0$ there is only one single point in physical space, independent of the values of θ or φ. Therefore, we must have $r=0$ at $\rho=0$, for otherwise Eqs. (16) would lead to an entire sphere of points x, y, z all corresponding to the one point at $\rho=0$, in violation of the second half of condition (F). For $\rho=0$, Eq. (40) with (43) now gives

$$0 = r(0) = C_1\{1+(m/\Delta)[\pi - \arctan(\Delta/m)]\} - C_2 m. \tag{44}$$

Thence, with $C_2=1$, Eq. (40) becomes

$$r=\rho-m+\frac{\Delta-(\rho-m)\arctan^*[\Delta/(\rho-m)]}{\pi+(\Delta/m)-\arctan(\Delta/m)}\quad [>(\rho-m)]. \tag{45}$$

(Notice that we use arc tan* in the numerator, but arc tan in the denominator.)

The complication of this formula makes it hard to express ρ explicitly in terms of r, and, therefore, to give explicit expressions for u^2, v^2. and w^2 in Eq. (19) in terms of r, or for the $\mathfrak{g}^{\mu\nu}$ of the harmonic coordinate system in terms of the coordinates x, y, z. In any case, the expression (45) is unique, which is all we wanted to show here.

That the expression (45) is indeed a positive, monotonic increasing function of ρ, so that it cannot take twice the same value for different points ρ, is proved in the next section.

8. ARBITRARY EXTENDED STATIC SPHERICAL SOURCE

We consider here the field generated by arbitrary static, spherically symmetric sources $S_{\mu\nu}$. We only will assume that the sources prevent the occurrence of a "Swiss-cheese hole" in space-time, so that the Schwarzschild radial coordinate occurring in the expression (18) for ds^2 can take all positive values down to $\rho=0$.

The fields U and w in (18) are solutions of Einstein's gravitational equations (14), which in Schwarzschild coordinates take the explicit form[18]

$$-\frac{U^2}{ww'}S_{00}=\frac{w''}{w'}-\frac{U'}{U}+\frac{2}{\rho}=\frac{d}{d\rho}\left(\ln\frac{\rho^2w'}{U}\right), \tag{46-0}$$

$$-\frac{\rho}{2}\left[S_{rr}+\frac{U^2}{w^2}S_{00}\right]=\frac{U'}{U}+\frac{w'}{w}=\frac{d}{d\rho}(\ln Uw), \tag{46-1}$$

$$\frac{U^2}{\rho}(1+\delta_{\theta\theta})=\frac{U^2}{\rho}\left(1+\frac{S_{\varphi\varphi}}{\sin^2\theta}\right)=\frac{w'}{w}-\frac{U'}{U}+\frac{1}{\rho}=\frac{d}{d\rho}\left(\ln\frac{\rho w}{U}\right). \tag{46-2}$$

We assume that the left-hand members are free of singularities for $\rho>0$. Therefore, $\ln(Uw)$, $\ln(\rho w/U)$, and $\ln(\rho^2w'/U)$ will be finite for finite nonvanishing ρ, so that U, w, and w' will be neither zero nor infinite. By Eqs. (46-1) and (46-2), also U'/U will stay finite for finite nonvanishing ρ. If, therefore, we rewrite Eq. (21) as

$$(\rho^2w/U)d^2r/d\rho^2+Fdr/d\rho-2Uwr=0, \tag{47}$$

then

$$F\equiv(2\rho w/U)+(\rho^2w'/U)-(\rho^2wU'/U^2) \tag{48}$$

will stay finite for finite nonvanishing ρ. Consequently, Eq. (47) permits, for finite nonvanishing ρ,

$$dr/d\rho=0\quad\text{for}\quad d^2r/d\rho^2=(2U^2/\rho^2)r\quad\text{only.} \tag{49}$$

As we assume that there is a single physical point with $\rho=0$, the second half of condition (F) tells us again, as in the large-charge case, that the harmonic radial coordinate r has to vanish at $\rho=0$.

The general solution of the linear homogeneous second-order differential equation (47) will be of the form

$$r(\rho)=C_1f_1(\rho)+C_2f_2(\rho). \tag{50}$$

Therefore, the condition (F) tells us that

$$0=C_1f_1(0)+C_2f_2(0). \tag{51}$$

Here, $f_1(0)$ and $f_2(0)$ cannot both vanish, because Eq. (47) must have solutions (50) which do not vanish at $\rho=0$. Therefore, Eq. (51) determines the ratio C_1/C_2, and we obtain

$$r=Cf(\rho), \tag{52}$$

with

$$f(\rho)=f_1(0)f_2(\rho)-f_2(0)f_1(\rho). \tag{53}$$

We shall now show that r is a monotonic function of ρ. Because of the condition (B),

$$dr/d\rho\to1\quad\text{for}\quad\rho\to\infty, \tag{54}$$

so that we must reject the solution $C_1=C_2=0$ that would be characterized by the initial conditions $r(0)=(dr/d\rho)_0=0$ at $\rho=0$. So, $(dr/d\rho)_0\neq0$.

Now, consider the value of $dr/d\rho$ from $\rho=0$ on up. If $dr/d\rho$ at first is positive, we find for small positive ρ positive values of r. Then, (49) tells us that the only local extremum of r possible is a local minimum.[19] In other words, *if r starts to rise, it cannot reach a maximum for finite ρ.*

Similarly, if $dr/d\rho$ had been negative to start with, r would have become negative, and would have been forced to keep decreasing for increasing ρ, because in this case (49) would have made any minimum of $r(\rho)$ impossible.

In either case, *r is a monotonic function of ρ, and therefore will not twice take the same value for different values of ρ.* Points with $dr/d\rho=0$ will by (49) with $r(0)=0$ never occur.

We now use the constant C left in (52) for satisfying the condition (54):

$$1=(dr/d\rho)_\infty=Cf'(\infty). \tag{55}$$

Thence,

$$r=\frac{f_1(0)f_2(\rho)-f_2(0)f_1(\rho)}{f_1(0)f_2'(\infty)-f_2(0)f_1'(\infty)}. \tag{56}$$

This determines the harmonic radial coordinate r uniquely. It is easily verified that this result is not al-

[18] See Eqs. (27) and (34) of reference 13.

[19] Maxima of r at sharp peaks of the $r(\rho)$ curve with $dr/d\rho\neq0$ are impossible, because $dr/d\rho$ must be continuous for being differentiable as in Eq. (21).

tered, if the special solutions $f_1(\rho)$ and $f_2(\rho)$ of Eq. (21) are replaced by any two independent linear combinations of them.

In practice, we may expect that for large values of ρ the solutions will asymptotically become rather similar to the solutions found in Eq. (22) in the absence of sources $S_{\mu\nu}$ that reach as far as a Coulomb field does, or to the solutions found in Eqs. (27) or (40), if a Coulomb field is included in the source. In the case without Swiss-cheese holes in space, the particular form of the actual solutions of Eq. (21) or (47) did not have to be used for proving the uniqueness of the harmonic radial coordinate.

In the case, however, of a field with excluded positive values of ρ, we cannot use Eq. (51), as $\rho=0$ is forbidden, and we would have to investigate whether there is only one ratio C_1/C_2 in (50), for which $r(\rho)$ will stay positive for all allowed values of ρ. This case is in reality a rather hypothetical one, and therefore we have not considered it any further than in the extreme case treated in Sec. 6, which is often used as a model for the field around a sun of which the radius is negligible.

9. CONCLUSION

The examples discussed by us in Secs. 5–8 show the importance of the condition (F) that coordinate transformations shall be one-to-one correspondences between the old and the new coordinates. Without this condition, Fock's claim would be violated already in the static spherically symmetric case.

With the condition (F) imposed, we have not been able to find a case violating Fock's claim. This increases our hope that Fock's claim may be correct, as we would like to believe, because of the help which Fock's theorem would provide in establishing a relation between Einstein's gravitational field theory and Lorentz-covariant quantum field theory.[2]

However, so far a rigorous proof of Fock's theorem or conjecture is still lacking, so that further investigations about its validity or invalidity remain desirable.

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