New Formulation of Dispersion Relations for Potential Scattering

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(Received July 25, 1961)

The amplitude for Yukawa potential scattering is represented by a Watson-Sommerfeld integral over Legendre functions $P_{ip-\frac{1}{2}}$, p real. A dispersion relation and unitarity condition are given for the amplitudes appearing in this integral and it is shown that the resulting system for iterative calculation of the amplitude from the Born approximation is considerably simpler than in other formulations. It is also shown that these amplitudes behave similarly to the partial waves on the second energy sheet.

I. MANDELSTAM REPRESENTATION FOR CONICAL AMPLITUDES

It has been shown that for scattering by a potential which is a superposition of Yukawa potentials, the Mandelstam representation and unitarity suffice to determine the scattering amplitude by iteration from the first Born term.[1] For simplicity, ignoring bound states and subtractions,[2] the Mandelstam representation with the potential

$$V(r)=cr^{-1}\exp(-\mu r), \tag{1}$$

may be written

$$f(s,t)=c(\mu^2+t)^{-1}+\int_0^\infty\frac{ds'}{\pi}\int_0^\infty\frac{dt'}{\pi}\frac{\rho(s',t')}{(s'-s-i\epsilon)(t'+t)}, \tag{2}$$

where s and t are the energy and squared momentum transfer, respectively. The unitarity condition then implies a nonlinear integral equation for $\rho(s,t)$, namely:

$$\rho(s,t)=\rho_{\text{Born}}(s,t)+\mathrm{P}\int_0^\infty ds_1\int_0^\infty dt_1\frac{K_1(s,t,t_1)}{s_1-s}\rho(s,t_1)$$
$$+\int_0^\infty ds_1\int_0^\infty dt_1\int_0^\infty ds_2\int_0^\infty dt_2\frac{K_2(s,t,t_1,t_2)}{(s_1-s+i\epsilon)(s_2-s-i\epsilon)}\times\rho(s_1,t_1)\rho(s_2,t_2), \tag{3}$$

where K_1 and K_2 are complicated rational functions multiplied by step functions.[1] No particular simplification results from rewriting (2) in terms of partial wave amplitudes because $\rho(s,t)$ and hence (3) is required to specify the unphysical cut. The reason is that $\rho(s,t)$ cannot be represented in terms of the partial waves directly over its entire domain of definition (in particular outside the Lehmann ellipse).

The system (2)–(3) may, however, be somewhat simplified if one represents the scattering amplitude by the so-called Sommerfeld-Watson integral employed by Regge[3] in the proof of (2).[4] In this method one writes the partial wave series for fixed $s\geqslant 0$

$$\mathfrak{F}(s,z)\equiv f(s,\,2s(1-z))=\frac{1}{\sqrt{s}}\sum_{l=0}^{\infty}(2l+1)T_l(\sqrt{s})P_l(z), \tag{4}$$

as an integral in the complex angular momentum plane:

$$\frac{1}{i\sqrt{s}}\int_C\frac{\lambda d\lambda}{\cos\pi\lambda}T(\lambda,\sqrt{s})P_{\lambda-\frac{1}{2}}(-z),$$

where the contour C encloses the points $\lambda=l+\frac{1}{2}$, $l=0,\,1,\,2,\,\cdots$ and $T(\lambda,\sqrt{s})$ which interpolates the partial waves $T_l(\sqrt{s})$ at $\lambda=l+\frac{1}{2}$ must be analytic within C. It is shown in reference 3 that for a class of potentials including (1) such a function $T(\lambda,\sqrt{s})$ exists. It is further shown that $T(\lambda,\sqrt{s})$ is analytic in $\mathrm{Re}\lambda>0$ except for poles lying in $\mathrm{Im}\lambda>0$ and that its behavior as $|\lambda|\to\infty$ in $\mathrm{Re}\lambda>0$ permits one to deform the contour C into the line $\mathrm{Re}\lambda=0$, the resulting integral converging to an analytic function of z in the plane cut from $z=+1$ to $z=+\infty$. The residues at the poles encountered in the deformation contain Legendre functions of complex order which determine the asymptotic behavior of the amplitude in z, since the deformed integral vanishes at large z.[5] As we are omitting subtractions, we ignore these pole terms and write:

$$\mathfrak{F}(s,z)=\int_0^\infty\frac{pdp}{\cosh\pi p}\Delta(ip,s)P_{ip-\frac{1}{2}}(-z),\quad s\geqslant 0, \tag{5}$$

with

$$\Delta(ip,s)=\frac{i}{\sqrt{s}}[T(ip,\sqrt{s})-T(-ip,\,\sqrt{s})], \tag{6}$$

and we have used the evenness in p of the Legendre functions appearing in (4). Legendre functions of this type were first encountered in electrostatic problems involving conical surfaces and are known as conical functions[6]; we hence refer to the quantity $\Delta(ip,s)$ as the "pth conical amplitude." A table of some important properties of the conical functions is given in the

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† Supported in part by the U. S. Atomic Energy Commission.

[1] R. Blankenbecler, M. L. Goldberger, M. N. Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960).

[2] This assumption will be made throughout this note.

[3] T. Regge, Nuovo cimento **14**, 951 (1959).

[4] More precisely in the proof of analyticity in t.

[5] T. Regge (to be published).

[6] F. G. Mehler, Math. Ann. **18**, 161 (1881); C. Neumann, Math. Ann. **18**, 195 (1881).

Appendix. One may now write for $s\geqslant 0$ and physical t,

$$\mathrm{Im}f(s,t)=\int_0^\infty \frac{pdp}{\cosh\pi p}\,\mathrm{Im}\Delta(ip,s)\cdot P_{ip-\frac{1}{2}}(t/2s-1), \quad (7)$$

which may be analytically continued in t by means of (A1) as

$$\mathrm{Im}f(s,t)=\frac{1}{\pi}\int_0^\infty \frac{dt'}{t'+t}\int_0^\infty pdp\,\mathrm{Im}\Delta(ip,s)\cdot P_{ip-\frac{1}{2}}\left(1+\frac{t'}{2s}\right). \quad (8)$$

Hence, from (2),

$$\rho(s,t)=\int_0^\infty pdp\,\mathrm{Im}\Delta(ip,s)P_{ip-\frac{1}{2}}(1+t/2s), \quad (9)$$

and from (5), (A2), and (A3),

$$\Delta(ip,s)=\Delta_B(ip,s)+\frac{\tanh\pi p}{2s}\int_0^\infty\frac{ds'}{\pi}\int_0^\infty\frac{dt'}{\pi}\frac{\rho(s',t')}{s'-s-i\epsilon}\times P_{ip-\frac{1}{2}}(1+t'/2s), \quad (10)$$

with the Born term

$$\Delta_B(ip,s)=\frac{c\pi\tanh\pi p}{2s}P_{ip-\frac{1}{2}}\left(1+\frac{\mu^2}{2s}\right). \quad (11)$$

Equation (9) expresses the spectral function directly in terms of the conical amplitudes. Equation (10) defines the analytic continuation of $\Delta(ip,s)$ hitherto given only for $s\geqslant 0$. If one writes from (A1)

$$P_{ip-\frac{1}{2}}(1+t'/2s)=\frac{s\cosh\pi p}{\pi}\int_0^{t'/4}\frac{ds''}{s''(s''+s)}\times P_{ip-\frac{1}{2}}(t'/2s''-1),$$

and

$$(s'-s)^{-1}(s''+s)^{-1}=(s'+s'')^{-1}((s'-s)^{-1}+(s''+s)^{-1}),$$

in (10) it is clear that $\Delta(ip,s)$ is analytic in s except for cuts on $\mathrm{Im}s=0$. In fact,

$$\Delta(ip,s)=\Delta_B(ip,s)+\int_0^\infty\frac{\mathrm{Im}\Delta(ip,s')}{s'-s-i\epsilon}\frac{ds'}{\pi}-\int_0^\infty\frac{\mathrm{Im}\Delta(ip,-s')}{s'+s+i\epsilon}\frac{ds'}{\pi}, \quad (\mathrm{I})$$

with

$$\mathrm{Im}\Delta(ip,-s)=-\frac{\sinh\pi p}{2\pi}\int_0^\infty qdq\int_0^\infty\frac{du}{u+1}\times G(p,q,u)\,\mathrm{Im}\Delta(iq,su)\quad s\geqslant 0, \quad (\mathrm{IIa})$$

in which

$$G(p,q,u)=\int_1^\infty P_{ip-\frac{1}{2}}(y)P_{iq-\frac{1}{2}}\left(1+\frac{1+y}{u}\right)dy. \quad (\mathrm{IIb})$$

Since (IIa) gives the left-hand cut in terms of the right hand cut, (I) and (II) will determine $\Delta(ip,s)$ once unitarity has been introduced to obtain an expression for $\mathrm{Im}\Delta(ip,s)$, $s\geqslant 0$.

The unitarity condition for $\Delta(ip,s)$ may be obtained by substituting (7) directly into the unitarity condition for $f(s,t)$ and making use of addition theorems. It may be found more simply by means of the property

$$S^*(\lambda^*,\sqrt{s})=S^{-1}(\lambda,\sqrt{s}), \quad (12)$$

of the "S matrix"[7] defined by

$$S(\lambda,\sqrt{s})\equiv 1+2iT(\lambda,\sqrt{s}), \quad (13)$$

together with the known analytic properties of $\Delta(\lambda,s)$ in λ.[8] The result is

$$\mathrm{Im}\Delta(ip,s)=\sqrt{s}\,\mathrm{Re}\left\{\Delta^*(ip,s)\frac{\mathrm{P}}{\pi}\int_0^\infty\frac{2qdq}{q^2-p^2}\Delta(iq,s)\right\}. \quad (\mathrm{III})$$

Equations (I)–(III) define an iterative scheme for the computation of $\Delta(ip,s)$ from $\Delta_B(ip,s)$ which may be used in place of (2) and (3). While the major difficulty of nonlinearity has of course not been overcome, the new system has the advantage of containing only a one-variable dispersion relation with a direct relation between the left- and right-hand cuts.

It has been shown[9] that a Mandelstam representation holds for Yukawa potential scattering off the energy shell. In this case (I) and (II) continue to hold when $\Delta(ip,s)$ is replaced by $\Delta(ip,s;s')$, where s' is a fixed initial energy. The unitarity condition then becomes

$$\mathrm{Im}\Delta(ip,s;s')=\frac{\sqrt{s}}{2}\mathrm{Re}\left\{\Delta^*(ip,s:s')\frac{\mathrm{P}}{\pi}\int_0^\infty\frac{2qdq}{q^2-p^2}\Delta(iq,s)+\Delta^*(ip,s)\frac{\mathrm{P}}{\pi}\int_0^\infty\frac{2qdq}{q^2-p^2}\Delta(iq,s;s')\right\}, \quad (\mathrm{III}')$$

and (I′), (II′), (III′) constitute a linear system for the determination of $\Delta(ip,s,s')$ if $\Delta(ip,s)$ is given.

II. CONICAL AMPLITUDES ON UNPHYSICAL RIEMANN SHEETS

We shall now prove the remarkable fact that the conical amplitudes retain the simple properties of the partial wave amplitudes on the second energy sheet. As is well known the partial waves are meromorphic except for cuts on $\mathrm{Im}s=0$ in the second sheet, the poles corresponding to resonances. The full amplitude $f(s,t)$ may however have complex branch points on the second

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[7] This reduces to the familiar form at $\lambda=l+\frac{1}{2}$. See reference 3.

[8] $\Delta(\lambda,s)$ is the sum of a function analytic in $\mathrm{Re}\lambda>0$ and a function analytic in $\mathrm{Re}\lambda<0$ if the poles are ignored.

[9] D. Fivel (to be published).

s sheet.[10] Since $\Delta(ip,s)$ is defined for the continuous variable p, poles in $\Delta(ip,s)$ at complex s can give rise to such branch cuts in f.

$\Delta(ip,s)$ can be continued to the second s-sheet by the symmetry property of the "S matrix" (13):

$$S(\lambda, -\sqrt{s})=S^{-1}(\lambda,\sqrt{s}), \tag{14}$$

which reduces to the familiar form at $\lambda=l+\frac{1}{2}$. Thus one defines

$$\Delta^{\text{II}}(ip,s)=\Delta^{\text{I}}(ip,s)S^{-1}(ip,\sqrt{s})S^{-1}(-ip,\sqrt{s}), \tag{15}$$

with

$$\Delta^{\text{I}}(ip,s)\equiv\Delta(ip,s), \quad \text{Im}\sqrt{s}>0.$$

Since $\Delta^{\text{I}}(ip,s)$ is analytic in s except for cuts on $\text{Im}s=0$, it suffices in view of (6) and (13) to study the analytic properties of

$$\Sigma(ip,s)\equiv(2\sqrt{s})^{-1}[S(ip,\sqrt{s})+S(-ip,\sqrt{s})]. \tag{16}$$

From the known properties of $S(\lambda,\sqrt{s})$ as a function of λ for $s>0$, one readily obtains

$$\frac{i\text{P}}{\pi}\int_0^\infty \frac{2qdq}{q^2-p^2}\frac{\Delta^{\text{I}}(iq,s)}{\cosh q\pi} =\Sigma(ip,s)-\frac{1}{\sqrt{s}}+\frac{2i}{\sqrt{s}}\sum_{l=0}^{\infty}\frac{(2l+1)(-1)^l T_l(\sqrt{s})}{(l+\frac{1}{2})^2+p^2}. \tag{17}$$

The summation in (17) could have been avoided by omitting the $\cosh q\pi$ in the integral, but this factor is required for convergence as will be seen below. In any case the analytic properties of the sum are readily obtained by studying $T_l(\sqrt{s})$ for $l\rightarrow\infty$ by the methods of reference 3 using the following artifice. From the Schrödinger equation one sees that a problem with complex energy may be replaced by an equivalent one with a fixed real energy in which the range of the potential and the coupling constant are complex variables. As in reference 2 one then shows that $T_l(\sqrt{s})=0(e^{-\rho l})$, $l\rightarrow\infty$, where $\rho>0$ when $\text{Re}\sqrt{s}>0$ provided that the range μ^{-1} of (1) is finite. For $\text{Re}(\sqrt{s})<0$ one makes use of the extended unitarity $T_l(\sqrt{s})=T_l^*(\sqrt{s^*})$, with the result that the summation in (17) is an analytic function of s except for cuts on $\text{Im}s=0$. One may now substitute (I) and (IIa) into (17) and interchange orders of integration. The last step is justified by the uniform convergence of the integrals guaranteed by the known behavior of the functions for large p with $s>0$. Finally one concludes that $\Sigma(ip,s)$ and hence $S(\pm ip, s)$ are analytic in s except for cuts on $\text{Im}s=0$. Thus $\Delta^{\text{II}}(ip,s)$ has the same cuts as $\Delta^{\text{I}}(ip,s)$, but in addition it may have poles due to the zeros of $S(\pm ip, s)$ which may be called the resonances of the conical amplitudes.

[10] R. Blankenbecler, M. Goldberger, S. MacDowell, and S. Treiman, Phys. Rev. **123**, 692 (1961).

III. CONCLUSIONS

It has been shown that dispersion relations for Yukawa potential scattering may be formulated simply by a representation of the scattering amplitude in terms of conical function amplitudes. It has further been demonstrated that these amplitudes, while providing a complete description of the full amplitude throughout its domain of holomorphy, still retain much of the simplicity of the partial waves, in particular a similar behavior on the second energy sheet.

APPENDIX

The following properties of conical functions are required in the text[6,11]:

$$P_{ip-\frac{1}{2}}(-z)=\frac{\cosh\pi p}{\pi}\int_1^\infty \frac{dv}{v-z}P_{ip-\frac{1}{2}}(v), \tag{A1}$$

$$(y-z)^{-1}=\pi\int_0^\infty \frac{p\tanh\pi p}{\cosh\pi p}P_{ip-\frac{1}{2}}(y)P_{ip-\frac{1}{2}}(-z)dp, \tag{A2}$$

$$f(p)=p\tanh\pi p\int_1^\infty P_{ip-\frac{1}{2}}(x)g(x)dx, \quad g(x)=\int_0^\infty P_{ip-\frac{1}{2}}(x)f(p)dp, \tag{A3}$$

$$P_{ip-\frac{1}{2}}(\cos\theta)=1+\frac{4p^2+1^2}{2^2}[\sin(\theta/2)]^2+\frac{(4p^2+1^2)(4p^2+3^2)}{2^2\times4^2}\times[\sin(\theta/2)]^4+\cdots. \tag{A4}$$

ACKNOWLEDGMENTS

The author wishes to thank Professor A. Klein and Dr. B. Lee for many helpful conversations.

[11] H. Bateman, *The Higher Transcendental Functions. I.* (McGraw-Hill Book Company, Inc., New York, 1953), pp. 174–175.