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Can Massless Particles be Charged?*

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It is shown that for particles of spin one or larger, Lorentz invariance, masslessness, and conventional electromagnetic coupling are mutually incompatible.

A. INTRODUCTION

OF the many particles that exist in nature, only two, the neutrino and the photon, are massless. To these we must add the graviton, if it exists. All three are electrically neutral, and it is perhaps not an idle question to ask whether there is a deep reason for this. We shall show that for particles of spin one or larger, Lorentz invariance, masslessness, and "conventional" electromagnetic coupling,[1] are incompatible. Our argument does not apply to particles of spin zero and spin $\frac{1}{2}$.

B. NONINTERACTING MASSLESS PARTICLES

We shall lean heavily on the generally accepted definition of a massless particle as one whose possible states belong to an irreducible representation of the inhomogeneous Lorentz group. In particular, for discrete spin $s\neq 0$, we treat the massless particle states as belonging to the irreducible representation of the class O_s, in the notation of Bargmann and Wigner,[2] which is characterized by only two independent polarization states. This characterization is to be contrasted with the $(2s+1)$ polarization states possible for a particle with mass. If we use a representation in terms of fields $\phi_{\dots}(x)$ (we leave off the indices for the time being), then the free field equations of motion will be

$$\Box\phi_{\dots}(x)=0.$$

For $s\geqslant 1$, i.e., $2s+1>2$, these equations must be supplemented by subsidiary conditions expressing the constraint to two polarization states. Such constraints are generated by an additional invariance property called "gauge invariance of the second kind." For example, for the photon field, the equation of motion is

$$\Box A_\mu(x)=0. \tag{1}$$

The subsidiary conditions are

$$(\partial/\partial x_\mu)A_\mu(x)=0, \tag{2}$$

together with those generated by the requirement that all solutions of Eq. (1) obtained by the transformation

$$A_\mu(x)\rightarrow A_\mu'(x)=A_\mu(x)-\partial\Lambda(x)/\partial x_\mu, \tag{2'}$$

with

$$\Box\Lambda(x)=0, \tag{2''}$$

describe the same physical state. There appears to be an asymmetry in the two requirements that are necessary to eliminate the two unwanted polarizations, but this asymmetry is only apparent, as can be seen by the unified treatment of this problem for the case of spin 1, given in the Appendix. We assume that such a unified treatment can be given for all spins $s\geqslant 1$, and therefore include the divergence condition [Eq. (2)] and its generalization in our definition of "gauge invariance of the second kind."[3] For higher spins we can proceed in a manner analogous to that for spin 1. We shall use the following representations[4]:

1. Integral spin s. For integral spin s, the field is given by $\phi_{\alpha\beta\cdots\rho\sigma}(x)$, a traceless symmetric tensor of rank s, obeying the equation

$$\Box\phi_{\alpha\beta\cdots\sigma}(x)=0, \tag{3}$$

together with gauge invariance of the second kind, which includes the equation

$$\frac{\partial}{\partial x_\alpha}\phi_{\alpha\beta\cdots\sigma}(x)=0, \tag{4}$$

and the statement of the physical equivalence of all solutions of Eq. (3) generated by the transformation

$$\phi_{\alpha\beta\cdots\sigma}(x)\rightarrow\phi_{\alpha\beta\cdots\sigma}'(x)=\phi_{\alpha\beta\cdots\sigma}(x)+G_{\alpha\beta\cdots\sigma}(x), \tag{5}$$

where

$$G_{\alpha\beta\cdots\sigma}(x)=\frac{\partial}{\partial x_\alpha}\Lambda_{\beta\gamma\cdots\sigma}(x)+\frac{\partial}{\partial x_\beta}\Lambda_{\alpha\gamma\cdots\sigma}(x)+\cdots+\frac{\partial}{\partial x_\sigma}\Lambda_{\alpha\beta\cdots\rho}(x) \tag{6}$$

and $\Lambda_{\beta\gamma\cdots\sigma}(x)$ is a traceless symmetric tensor of rank

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[1] By conventional electromagnetic couplings we mean those generated by coordinate-dependent gauge transformations on the fields of the charged particles, as well as couplings due to an assignment of static electric or magnetic multipole moments to the particles, such as a Pauli magnetic moment coupling.

[2] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 34, 211 (1948).

[3] We do not want to give the impression that there is any question about the appearance of a divergence condition, but merely to point out that it can be considered part of a gauge invariance.

[4] See, for example, H. Umezawa, *Quantum Field Theory* (Interscience Publishers, Inc., New York, 1956).

$s-1$ obeying the equations

$$\Box \Lambda_{\beta\gamma\cdots\sigma}(x)=0, \qquad (7)$$

and

$$\frac{\partial}{\partial x_\beta}\Lambda_{\beta\gamma\cdots\sigma}(x)=0. \qquad (8)$$

2. *Odd half-integral spin* $(s+\frac{1}{2})$. For odd half-integral spin $(s+\frac{1}{2})$, the field is characterized by an additional spinor index A. The equation obeyed by $\phi_{\alpha\beta\cdots\sigma A}(x)$ is taken to be

$$\gamma_\mu{}^{BA}\frac{\partial}{\partial x_\mu}\phi_{\alpha\beta\cdots\sigma A}(x)=0, \qquad (9)$$

where the γ_μ are the usual Dirac matrices.

Repeated application of the operator $\gamma_\mu\partial/\partial x_\mu$ yields the Klein-Gordon equation

$$\Box\phi_{\alpha\beta\cdots\sigma A}(x)=0. \qquad (10)$$

The analog of Eq. (4) is the equation

$$\gamma_\alpha{}^{BA}\phi_{\alpha\beta\cdots\sigma A}(x)=0, \qquad (11)$$

which has as its consequence the equations

$$\frac{\partial}{\partial x_\alpha}\phi_{\alpha\beta\cdots\sigma A}(x)=0. \qquad (12)$$

Equation (11) is supplemented, as before, by a statement of equivalence of a certain class of solutions. The main results of this section, which we shall use in proving our assertion, are that (a) the fields obey the Klein-Gordon equation with no mass term, and (b) the fields obey a divergence condition [Eqs. (4), (12)]. These are necessary, though not sufficient, conditions for the characterization of a massless particle.

C. INTERACTION WITH THE ELECTROMAGNETIC FIELD

Invariance of the charged field under coordinate-dependent gauge transformations of the first kind, i.e., invariance of the equations of motion, when the field is transformed according to

$$\phi_{\alpha\beta\cdots}(x)\rightarrow\phi_{\alpha\beta\cdots}'(x)=\exp[ie\chi(x)]\phi_{\alpha\beta\cdots}(x), \qquad (13)$$

leads in well-known fashion to an equation of motion of the form

$$\left(\frac{\partial}{\partial x_\mu}-ieA_\mu(x)\right)^2\phi_{\alpha\beta\cdots}(x)=0. \qquad (14)$$

One would also expect the subsidiary conditions to be modified, but fortunately it turns out that it is not necessary to specify this modification, because the incompatibility between the equations of motion with interaction, and the *free-field subsidiary conditions* is sufficient to establish the result that massless particles of spin $s\geqslant 1$ cannot be charged. To see this in detail, let us consider the interaction of a massless particle with a very weak external electromagnetic field, $A_\mu{}^{\text{ext}}(x)$, which we take to obey the Lorentz condition

$$\frac{\partial}{\partial x_\mu}A_\mu{}^{\text{ext}}(x)=0.$$

The equation of motion is

$$\Box\phi_{\alpha\beta\cdots}(x)=-j_{\alpha\beta\cdots}(x), \qquad (15)$$

where

$$j_{\alpha\beta\cdots}(x)=-2ieA_\mu{}^{\text{ext}}(x)\frac{\partial}{\partial x_\mu}\phi_{\alpha\beta\cdots}(x). \qquad (16)$$

If "nonminimal" (i.e., arising from static moments) electromagnetic interactions are included, $j_{\alpha\beta\cdots}(x)$ will contain additional terms, but these will still have the property of being *linear* in the field $\phi_{\alpha\beta\cdots}(x)$. We now write the equation in integral form:

$$\phi_{\alpha\beta\cdots}(x)=\phi_{\alpha\beta\cdots}{}^{\text{in}}(x)+\int dx'\, D_R(x-x')j_{\alpha\beta\cdots}(x'), \qquad (17)$$

where $D_R(x-x')$ is the usual retarded Green's function for a massless field and $\phi_{\alpha\beta\cdots\alpha}{}^{\text{in}}(x)$ is a free field to which $\phi_{\alpha\beta\cdots}(x)$ reduces asymptotically as $x_0\rightarrow-\infty$. Although this form assumes an asymptotic condition which clearly cannot be satisfied when the interaction has infinite range (as is indeed the case for the electromagnetic field), there is no difficulty if we consider a weak external field which may, for example, be a screened Coulomb field. From Eq. (17) we may express the outgoing field in terms of the ingoing one by letting $x_0\rightarrow+\infty$:

$$\phi_{\alpha\beta\cdots}{}^{\text{out}}(x)=\phi_{\alpha\beta\cdots}{}^{\text{in}}(x)-\int dx'\, D(x-x')j_{\alpha\beta\cdots}(x'). \qquad (18)$$

It is now clear that if $\phi_{\alpha\beta\cdots}{}^{\text{in}}(x)$ represents a massless incoming particle and obeys the necessary condition

$$\frac{\partial}{\partial x_\alpha}\phi_{\alpha\beta\cdots}{}^{\text{in}}(x)=0,$$

then

$$\frac{\partial}{\partial x_\alpha}\phi_{\alpha\beta\cdots}{}^{\text{out}}(x)=-\frac{\partial}{\partial x_\alpha}\int dx'\, D(x-x')j_{\alpha\beta\cdots}(x')$$
$$=-\int dx'\, D(x-x')\frac{\partial}{\partial x_\alpha'}j_{\alpha\beta\cdots}(x'). \qquad (19)$$

Now one can see by inspection that

$$\frac{\partial}{\partial x_\alpha}j_{\alpha\beta\cdots}(x)\neq 0. \qquad (20)$$

The argument can be made more general: if

$(\partial/\partial x_\alpha)j_{\alpha\beta\cdots}(x)$ had vanished, it would be possible to construct a generalized "charge"

$$Q_{\beta\gamma\cdots}=-i\int d^3x\; j_{4\beta\gamma\cdots}(x),$$

which should be conserved. Since, however, $j_{\alpha\beta\cdots}(x)$ is linear in the field $\phi_{\alpha\beta\cdots}(x)$ no such conservation law can possibly hold.[5] Hence Eq. (20) is generally true, and therefore it follows that

$$\frac{\partial}{\partial x_\alpha}\phi_{\alpha\beta\cdots}{}^{\text{out}}(x)\neq 0. \tag{21}$$

Thus the outgoing field no longer satisfies gauge invariance of the second kind, as defined in the last section, and therefore the final state no longer has only two polarization states, which contradicts the requirements of Lorentz invariance.

It is instructive to compare this with the case of a massive particle. For simplicity we consider the vector meson, whose equation of motion is

$$\frac{\partial}{\partial x_\mu}f_{\mu\nu}-m^2\phi_\nu=0 \tag{22}$$

and

$$f_{\mu\nu}=\frac{\partial}{\partial x_\mu}\phi_\nu-\frac{\partial}{\partial x_\nu}\phi_\mu,$$

in the free field case, and

$$\frac{\partial}{\partial x_\mu}f_{\mu\nu}-m^2\phi_\nu=-j_\nu, \tag{23}$$

when electromagnetic couplings are introduced. The integral equation takes the form

$$\phi_\mu(x)=\phi_\mu{}^{\text{in}}(x)+\left(\delta_{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\times\int dx'\,\Delta_R(x-x',m)j_\nu(x'). \tag{24}$$

One can thus see that the condition

$$\frac{\partial}{\partial x_\mu}\phi_\mu(x)=0 \tag{22'}$$

that follows from Eq. (22) is much more complicated in the presence of an electromagnetic field. For the outgoing field, however, it follows from

$$\phi_\mu{}^{\text{out}}(x)=\phi_\mu{}^{\text{in}}(x)-\left(\delta_{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\times\int dx'\,\Delta(x-x',m)j_\nu(x'), \tag{25}$$

and

$$(\Box-m^2)\Delta(x-x',m)=0, \tag{26}$$

that

$$\frac{\partial}{\partial x_\mu}\phi_\mu{}^{\text{out}}(x)=0, \tag{27}$$

if

$$\frac{\partial}{\partial x_\mu}\phi_\mu{}^{\text{in}}(x)=0, \tag{28}$$

independently of the properties of the current. For the massless case, it is impossible to write the integral equation in a form analogous to Eq. (24) without introducing additional singularities into the Green's function: Changing the Green's function (in momentum space) from $\delta_{\mu\nu}/k^2$ to $(\delta_{\mu\nu}-k_\mu k_\nu/k^2)/k^2$, which would satisfy the divergence condition, automatically amounts to introducing an additional massless scalar field into the theory. This, however, violates the requirement that irreducible representations of the Lorentz group be used.

D. PHYSICAL INTERPRETATION

The argument that there is an incompatibility between masslessness, Lorentz invariance, and electromagnetic couplings, or in other words, between gauge invariance of the first kind and gauge invariance of the second kind, may be visualized physically if we consider the massless particle as a limiting case of a massive one. The $(2s+1)$ polarization states go over into two in a continuous manner as the mass goes to zero, and the mechanism is one by which $(2s-1)$ of the polarization states "decouple" from the remaining ones, with a factor proportional to m, the mass of the particle. An initial state that is transversely polarized remains so for admissible interactions. Our argument shows that the electromagnetic interaction is not admissible: The final state is not necessarily transversely polarized. This way of looking at our result shows why we can make the argument for spins $s\geq 1$: Only then is $2s+1>2$ and an incompatibility possible.

Explicit calculations support this interpretation. Consider for example the formulas for the differential cross section for the scattering of massless vector mesons by a Coulomb field (in the limit of vanishing

[5] We have been presenting the proof in purely classical terms, but the last argument is perhaps more easily made in terms of quanta of the field. Since the current is linear in the field operator, it has nonvanishing matrix elements between the one-quantum state, and states differing from it by an odd number of quanta. Hence it is impossible to get a condition like

$$\int dx'\,D^{(+)}(x-x')\frac{\partial}{\partial x_\alpha'}j_{\alpha\beta\cdots}(x')|q\rangle=0,$$

where $|q\rangle$ is a one-particle state vector.

screening): We have

$s=0$

$$\frac{d\sigma}{d\Omega}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}, \tag{29}$$

and

$s=\frac{1}{2}$

$$\frac{d\sigma}{d\Omega}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}[1+\sin^2(\theta/2)], \tag{30}$$

both of which are well behaved. However,[6] for

$s=1$

$$\frac{d\sigma}{d\Omega}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}\lim_{m=0}\left\{1+\frac{1}{6}\frac{(\omega^2-m^2)^2}{\omega^2m^2}\sin^2\theta\right\}, \tag{31}$$

which is infinite, so that there is a contradiction somewhere. The separation of this cross section into the following terms:

transverse-transverse spin transitions

$$\left(\frac{d\sigma}{d\Omega}\right)_{TT}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}(1+\cos^2\theta), \tag{32}$$

longitudinal-longitudinal spin transitions

$$\left(\frac{d\sigma}{d\Omega}\right)_{LL}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}\cos^2\theta, \tag{33}$$

and

transverse-longitudinal spin transitions

$$\left(\frac{d\sigma}{d\Omega}\right)_{TL}=\frac{(Z\alpha)^2}{4\omega^2}\frac{1}{\sin^4(\theta/2)}\lim_{m=0}\left\{\frac{(\omega^2+m^2)^2}{4\omega^2m^2}\sin^2\theta\right\}, \tag{34}$$

shows that the singular behavior occurs in just those transitions leading to a final state that violates Lorentz invariance for a massless vector meson.

In conclusion, we might point out that this argument can be used to forbid the coupling of massless particles with other interactions; the only condition for this is that the source of the field *not* be divergenceless. Another "application" of our conclusions has to do with the Yang-Mills field: Because two of its components are charged, they cannot be massless, and because of the charge symmetry among the three components, the same must hold for the third component, and it is not possible to identify the neutral one with the electromagnetic field.[7]

[6] Formulas (31) through (34) are taken from J. A. Young, Ph.D. thesis, Lawrence Radiation Laboratory Report UCRL-9563, 1961 (unpublished).

[7] We would like to thank Dr. S. Bludman for pointing this out to us.

APPENDIX

We sketch a treatment of subsidiary conditions for the case of spin 1, in which there is no artificial separation between the divergence condition and the remaining gauge invariance conditions.

Spin 1. In order to describe a relativistic particle of spin 1, we have a choice of using an antisymmetric tensor of rank 2, $(D^{(1,0)}+D^{(0,1)})$, or a four-vector, $(D^{(\frac{1}{2},\frac{1}{2})})$. We choose to describe the particle by the antisymmetric tensor $\Pi_{\mu\alpha}$ which obeys the equation

$$\Box\Pi_{\mu\alpha}=0. \tag{A1}$$

The field has six independent components. We may reduce these to two by requiring that the solutions of Eq. (A1) of the form

$$\Pi_{\mu\alpha}'=\Pi_{\mu\alpha}+\frac{\partial}{\partial x_\mu}W_\alpha-\frac{\partial}{\partial x_\alpha}W_\mu, \tag{A2}$$

where W_μ is an arbitrary four-vector obeying the wave equation

$$\Box W_\mu=0,$$

are physically indistinguishable.

The number of independent components is thus $6-4=2$. We can check that the remaining two components have indeed the correct transformation properties under the two-dimensional rotation group (the "little" group).[8]

$\Pi_{\mu\alpha}$. Under the homogeneous Lorentz group the tensor belongs to the representation $D^{(1,0)}+D^{(0,1)}$. There are, therefore, $2D^{(1)}$ representations of the 3-dimensional rotation group, and so the tensor splits up into the following representations of the little group: $2d^{(+1)}+2d^{(-1)}+2d^{(0)}$.

W_μ. The four-vector, belonging to $D^{(\frac{1}{2},\frac{1}{2})}$ transforms, under the 3-dimensional rotations group like $D^{(1)}+D^{(0)}$, i.e., like $d^{(+1)}+d^{(-1)}+2d^{(0)}$ under the two-dimensional rotation group. Thus the difference transforms like $d^{(+1)}+d^{(-1)}$, which is just what we want.

It is possible to construct a divergenceless field,

$$A_\mu=\frac{1}{3!}\epsilon_{\mu\alpha\rho\sigma}\frac{\partial}{\partial x_\alpha}\Pi_{\rho\sigma},$$

which satisfies the usual gauge-invariance conditions, so that the equivalence of the two methods is obvious, in this case, at least. We have carried out a similar treatment for a spin-2 field, but have not searched for a systematic way of unifying gauge invariance of the second kind, in general.

[8] E. P. Wigner, Ann. Math. 40, 149 (1939).