section computed, one obtains the generalized Rosenbluth expression[26]

$$d\sigma/d\Omega=\sigma'\{F_1^2+(q_\mu^2/4M^2)\times[2(F_1+KF_2)^2\tan^2(\theta/2)+K^2F_2^2]\}. \quad (A11)$$

To obtain the approximation appropriate to a nonrelativistic nucleon, we note that, for a given θ, the nucleon's recoil velocity $v=q/M$, appears explicitly only in q_μ^2/M^2, which to second order in v is q^2/M^2, since $\omega=q^2/2M$. Hence (A11) and (A9) agree through second order, as they should. In contrast, we note that, e.g., the diagonal terms of Eq. (19) of reference 5, which should in the same way agree through second order with Eq. (A11), fail to do so because the Darwin-Foldy term was neglected in the interaction.[27]

[26] M. N. Rosenbluth, Phys. Rev. **79**, 619 (1950).

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[27] The neglect of the Darwin-Foldy term in other calculations by the Stanford group has been pointed out by L. Durand, III, Phys. Rev. **115**, 1020 (1959).

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Non-Abelian Gauge Fields. Commutation Relations*

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The question is raised for non-Abelian vector gauge fields whether gauge invariance necessarily implies a massless physical particle. As a preliminary to studying this problem, the action principle is used to discover the independent dynamical variables of such gauge fields and construct their commutation relations.

INTRODUCTION

IT is well known that gauge invariance intimately ties the electromagnetic field $A_\mu(x)$, $F_{\mu\nu}(x)$ to the set of all fields $\chi(x)$ that bear electrical charge. This internal property is described by a finite imaginary Hermitian matrix q with integer eigenvalues. A gauge transformation involves an arbitrary numerical function $\lambda(x)$. It is a linear homogeneous transformation for the charged fields $\chi(x)$, but an inhomogeneous one for the gauge field $A_\mu(x)$,

$$\chi(x)\to e^{iq\lambda(x)}\chi(x),\quad A_\mu(x)\to A_\mu(x)+\partial_\mu\lambda(x),$$
$$F_{\mu\nu}(x)\to F_{\mu\nu}(x).$$

Such transformations form an Abelian group, in which the gauge function,

$$\lambda(x)=\lambda^{(1)}(x)+\lambda^{(2)}(x),$$

describes the superposition of two individual transformations. The integer spectrum of charge is related to the compact structure of this group, which has the topology of the circle. Gauge invariance implies that local conservation of charge is not just a consequence of the equations of motion of the charge bearing fields but appears as an identity characteristic of the gauge field differential equations.

In this familiar situation the gauge field does not carry the internal property to which it is coupled. A different example is furnished by the gravitational field, for this couples with energy and momentum, to which all physical systems must contribute. In other respects, however, the requirement of general coordinate invariance is quite analogous to that of gauge invariance. There is an intermediate possibility in which the gauge field is coupled to, and also carries, internal rather than space-time properties. Then the gauge field retains the space-time transformation properties of the electromagnetic field. This is indicated by the tensor notation $\phi_{\mu a}$, $G_{\mu\nu a}$, where the index $a=1\cdots n$ refers to the internal space. For the gravitational field the latter is also a coordinate index, which requires fields of more complicated space-time transformation properties.

The gauge transformations of a field $\chi(x)$ that supports a number of internal properties, as represented by finite linearly independent matrices T_a, $a=1\cdots n$, can generally be stated explicitly only for infinitesimal transformations,

$$\chi(x)\to[1+i\sum_{a=1}^{n}T_a\delta\lambda_a(x)]\chi(x).$$

If these are to generate a transformation group, two

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successive infinitesimal transformations, performed in opposite order, must be connected by another such transformation. This implies the commutation relations,

$$[T_b,T_c]=\sum_{a=1}^{n} T_a t_{abc},$$

and the constants,

$$t_{abc}=-t_{acb},$$

characterize the structure of the group.

The statement that the gauge field also carries these internal properties is conveyed by the infinitesimal gauge transformation,

$$\phi_\mu(x)\to[1+i\sum_{a=1}^{n} t_a\delta\lambda_a(x)]\phi_\mu(x)+\partial_\mu\delta\lambda(x),$$

$$G_{\mu\nu}(x)\to[1+i\sum t_a\delta\lambda_a(x)]G_{\mu\nu}(x),$$

which uses a matrix notation for the n-dimensional internal space. The homogeneous transformation of $G_{\mu\nu}$ implies that the matrices t_a obey the group commutation relations,

$$[t_b,t_c]=\sum_{a=1}^{n} t_a t_{abc}.$$

But the inhomogeneous transformations of ϕ_μ must also represent the group structure. The corresponding condition is

$$\sum_{bc}[(t_b)_{ac}\delta\lambda_b^{(1)}\partial_\mu\delta\lambda_c^{(2)}-(t_b)_{ac}\delta\lambda_b^{(2)}\partial_\mu\delta\lambda_c^{(1)}]=\partial_\mu[\sum_{bc} t_{abc}\delta\lambda_b^{(1)}\delta\lambda_c^{(2)}],$$

which asserts that

$$(t_b)_{ac}=t_{abc}.$$

Thus the matrices t_a are derived from the structure constants of the group. To verify that these matrices do obey the group commutation relations, we write the latter in the matrix form

$$[T_b,T]=Tt_b.$$

Then

$$[T_b,[T_c,T]]-[T_c,[T_b,T]]=T[t_b,t_c],$$

which also equals

$$[[T_b,T_c],T]=T\sum_a t_a t_{abc},$$

and the desired result follows from the linear independence of the T matrices.

A general intuition about the space of internal properties can be formulated as the requirement that its symmetry group be compact, in contrast to the open Lorentz group. It is then possible to make all matrix representations be unitary, so that the matrices T_a are Hermitian. This includes the matrices t_a, which generate an n-dimensional representation. But we should also note that the structure constants associated with the Hermitian T matrices are imaginary, and thus the imaginary Hermitian t matrices must be antisymmetrical, or

$$(t_b)_{ac}=t_{abc}=-t_{cba}.$$

This property, in conjunction with

$$t_{abc}=-t_{acb},$$

expresses the total antisymmetry of the set of n^3 numbers t_{abc}. In order to construct nonzero t matrices it is necessary that $n\geq 3$, and for $n=3$ the structure of a non-Abelian group is uniquely that of the three-dimensional Euclidean rotation group.[1]

The concept of an internal symmetry group has long been considered a possible basis for describing the non-space-time properties of physical particles. To relate such a group to gauge transformations of vector fields is an attractive idea, but one which seems to run into difficulty immediately if it is accepted that a gauge field implies a corresponding massless particle. Only the photon is known as an example of this class of physical particle. It is hard to agree that the objection is overcome by destroying completely[2] the gauge invariance which is the entire motivation of the gauge fields. But there may be an escape from this dilemma. The author has remarked that gauge-invariant systems of the electromagnetic or, more generally expressed, Abelian type need not have an accompanying massless particle if the coupling is sufficiently strong.[3] The question is whether a similar possibility exists for systems with non-Abelian gauge groups. To discuss this problem requires at least a full knowledge of the operator properties of the gauge field, treated as a physical quantum-mechanical system without reference to weak coupling approximations. These commutation relations are not known. And it is not a trivial query whether a consistent quantum field theory is possible at all for a system that admits a non-Abelian gauge group. But the latter can hardly be answered until a set of commutation relations has been displayed, for, without these, the nature of the operator description, with its necessary attribute of completeness, remains unknown. It is the purpose of this paper to produce such commutation relations, but we shall leave untouched the more difficult question of consistency.

THE ACTION PRINCIPLE

In order to construct an invariant Lagrange function in the standard first order differential form we must combine the antisymmetrical tensor of Hermitian operators $G_{\mu\nu}$ with a similarly transforming differential construct of the Hermitian operators ϕ_μ. Unlike the electromagnetic situation, the antisymmetrical gradient or curl will not suffice, since its infinitesimal gauge

[1] It is this context that non-Abelian gauge groups were first discussed, C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).
[2] See, for example, J. Sakurai, Ann. Phys. **11**, 1 (1960).
[3] J. Schwinger, Phys. Rev. **125**, 397 (1962).

transformation is

$$\partial_\mu\phi_\nu-\partial_\nu\phi_\mu\rightarrow[1+i\sum_a t_a\delta\lambda_a](\partial_\mu\phi_\nu-\partial_\nu\phi_\mu)+i\sum_a t_a(\phi_\nu\partial_\mu\delta\lambda_a-\phi_\mu\partial_\nu\delta\lambda_a),$$

and the last term has no counterpart in the $G_{\mu\nu}$ transformation law. We are thus led to consider the compensating gauge transformation of the expression

$$i(\phi_\mu . t\phi_\nu),$$

which employs a notation for a vector in the n-dimensional internal space. The components are

$$i(\phi_\mu . t_b\phi_\nu)=\sum_{ac}\phi_{\mu a}.it_{abc}\phi_{\nu c}.$$

In addition, the dot symbolizes symmetrized multiplication of the operators,

$$\phi_{\mu a}.\phi_{\nu c}=\tfrac{1}{2}\{\phi_{\mu a},\phi_{\nu c}\},$$

so that the whole structure is a Hermitian operator. In view of the complete antisymmetry of t_{abc}, the same vector can be written alternatively as

$$i(\phi_\mu . t\phi_\nu)=-i`t\phi_\mu'.\phi_\nu=-i\phi_\mu.`t\phi_\nu'=-i(\phi_\nu . t\phi_\mu).$$

Here we have introduced a notation for a matrix,

$$`t\phi'=\sum_a t_a\phi_a,$$

which has the elements,

$$(t\phi)_{bc}=\sum_a t_{bac}\phi_a.$$

On transcribing the gauge transformation of the curl into this notation, it reads

$$\partial_\mu\phi_\nu-\partial_\nu\phi_\mu\rightarrow(1+i`t\delta\lambda')(\partial_\mu\phi_\nu-\partial_\nu\phi_\mu)+i(\phi_\nu t\partial_\mu\delta\lambda)-i(\phi_\mu t\partial_\nu\delta\lambda),$$

which is to be compared with

$$i(\phi_\mu . t\phi_\nu)\rightarrow i(\phi_\mu . t\phi_\nu)-(\phi_\mu .[t,`t\delta\lambda']\phi_\nu)-i(\phi_\nu t\partial_\mu\delta\lambda)+i(\phi_\mu t\partial_\nu\delta\lambda).$$

The commutation properties of the t-matrices are expressed by

$$(\phi_\mu .[t,`t\delta\lambda']\phi_\nu)=`t\delta\lambda'(\phi_\mu . t\phi_\nu),$$

and we have the desired result

$$\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu . t\phi_\nu)\rightarrow(1+i`t\delta\lambda')[\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu . t\phi_\nu)].$$

A possible Lagrange function is given by

$$\mathcal{L}=-\tfrac{1}{2}G^{\mu\nu}.[\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu . t\phi_\nu)]+\tfrac{1}{4}f^2G^{\mu\nu}G_{\mu\nu}+k^\mu.\phi_\mu+\mathcal{L}(\chi),$$

where scalar products of vectors in the internal space are understood. The contributory Lagrange function,

$$\mathcal{L}(\chi)=\tfrac{1}{4}(\chi A^\mu\partial_\mu\chi-\partial_\mu\chi A^\mu\chi)-\mathcal{H}(\chi),$$

is that of the systems carrying the properties represented by the matrices T_a. The latter, incidentally, are imaginary and antisymmetrical if all fields are chosen to be Hermitian. The flux of these properties is described by the Hermitian current vector

$$k_a{}^\mu(x)=-\tfrac{1}{2}i\chi(x)A^\mu T_a\chi(x),$$

the structure of which follows from the requirement of gauge invariance. If $\mathcal{H}(\chi)$ is a gauge scalar, the response of $\mathcal{L}(\chi)$ to an infinitesimal gauge transformation is

$$\mathcal{L}(\chi)\rightarrow\mathcal{L}(\chi)-k^\mu\partial_\mu\delta\lambda,$$

under the assumption that the kinematical matrices A^μ operate entirely in space-time, or that

$$[A^\mu,T_a]=0.$$

A compensating term is produced by $k^\mu.\phi_\mu$, on taking into account the homogeneous gauge transformation of the current,

$$k^\mu\rightarrow(1+i`t\delta\lambda')k^\mu,$$

which follows from the commutation properties of the T matrices, displayed as

$$[T,`T\delta\lambda']=`t\delta\lambda'T.$$

The dimensionless number f^2 appears as an arbitrary coupling constant. We shall see that it must be a positive quantity.

Until the commutation properties of the fields are known, a Lagrange function $\mathcal{L}$ or action operator

$$W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\mathcal{L},$$

constructed from formal invariance arguments, has only a heuristic significance, leading through the principle of stationary action,

$$\delta W_{12}=G_1-G_2,$$

to the tentative statement of covariant field equations and infinitesimal transformation generators. Let us consider first the infinitesimal gauge transformations of the χ field alone,

$$\delta W_{12}=\int(dx)[-k^\mu\partial_\mu\delta\lambda+\phi_\mu . i`t\delta\lambda'k^\mu],$$

which imply the extended conservation law

$$\partial_\mu k^\mu-i`t\phi_\mu'.k^\mu=0,$$

and the infinitesimal generator

$$G_\lambda=\int d\sigma_\mu(-k^\mu\delta\lambda)=\int(d\mathbf{x})(-k^0\delta\lambda).$$

The significance of the latter is expressed by

$$(1/i)[\chi,G_\lambda]=\delta\chi=i`T\delta\lambda'\chi,$$

which gives the commutation rule

$$x^0=x^{0\prime}:\quad [\chi(x),k_a^0(x')]=\delta(\mathbf{x}-\mathbf{x}')T_a\chi(x).$$

The corresponding integral form is

$$[\chi(x),K_a]=T_a\chi(x),$$

where

$$K_a=\int d\sigma_\mu\, k_a^\mu=\int (d\mathbf{x})k_a^0.$$

These Hermitian operators obey the group commutation relations,

$$[K_b,K_c]=\sum_a K_a t_{abc}.$$

They are not constants of the motion, however, since k^μ is not governed by a true conservation equation.

The differential equations of the gauge field implied by the action principle are

$$\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu . t\phi_\nu)=f^2G_{\mu\nu},$$

and

$$\partial_\nu G^{\mu\nu}-i`t\phi_\nu' . G^{\mu\nu}=k^\mu.$$

These equations are also conveyed by the following matrix statements:

$$[\partial_\mu-i`t\phi_\mu',\ \partial_\nu-i`t\phi_\nu']=-if^2`tG_{\mu\nu}',$$
$$[\partial_\nu-i`t\phi_\nu',\ `tG^{\mu\nu}']=`tk^\mu',$$

in which it must be clearly understood that the commutators refer only to the coordinate and matrix indices; the operators, on the contrary, are to be symmetrically multiplied. In this manner of displaying the operators, an infinitesimal gauge transformation is generated by orthogonal transformation with the matrix $1+i`t\delta\lambda'$.

According to the antisymmetry of $G^{\mu\nu}$, the vector

$$j^\mu=k^\mu-i(\phi_\nu . tG^{\mu\nu})$$

is divergenceless,

$$\partial_\mu j^\mu=0,$$

and thus it is the Hermitian operators,

$$\mathbf{T}_a=\int d\sigma_\mu\, j_a^\mu$$
$$=K_a-i\int (d\mathbf{x})\ \phi_k . t_aG^{0k},$$

that are constants of the motion. It is natural to expect that these operators also obey the group commutation relations, but the verification must await the specification of the gauge field's operator properties.

The field equations can be decomposed into apparent equations of motion,

$$\partial_0\phi_k=\partial_k\phi_0-i`t\phi_k' . \phi_0+f^2G_{0k},$$
$$-\partial_0G^{0k}=-\partial_lG^{kl}+i`t\phi_l' . G^{kl}-i`t\phi_0' . G^{0k}+k^k,$$

and equations of constraint,

$$f^2G_{kl}=\partial_k\phi_l-\partial_l\phi_k+i(\phi_k . t\phi_l),$$
$$\partial_kG^{0k}=i`t\phi_k' . G^{0k}+k^0.$$

The latter show that neither G_{kl} nor the longitudinal part of the three-dimensional vector G^{0k} are independent dynamical variables. It will be useful, then, to write

$$G^{0k}=G^{0kT}+G^{0kL},$$

where

$$\partial_kG^{0kT}=0,$$

and

$$G^{0kL}=-\partial^k\psi.$$

In an ordinary three-dimensional notation, the equation to determine the Hermitian operator $\psi(x)$ is

$$-\nabla^2\psi+i`t\boldsymbol{\phi}'\cdot\nabla . \psi=i`t\boldsymbol{\phi}' : \mathbf{G}^T+k^0.$$

This information can be utilized in the equation of motion for ϕ^k by taking the divergence,

$$\partial_0\nabla\cdot\boldsymbol{\phi}-i`t\phi_0' . \nabla\cdot\boldsymbol{\phi}=\nabla^2\phi_0-i`t\boldsymbol{\phi}'\cdot\nabla . \phi_0-f^2\nabla\cdot\mathbf{G}.$$

But we must still reckon with the freedom of gauge transformation, which shows the impossibility of a complete specification of ϕ^k and ϕ^0 by the field equations. In order to obtain a definite set of operators we adopt a specific gauge, and the naturally indicated choice is the three-dimensional transverse or radiation gauge,

$$\partial_k\phi^k=\nabla\cdot\boldsymbol{\phi}=0,$$

for this extracts from the apparent equations of motion another equation of constraint,

$$-\nabla^2\phi^0+i`t\boldsymbol{\phi}'\cdot\nabla . \phi^0=-f^2\nabla^2\psi,$$

which must serve to determine ϕ^0.

It has now become clear that the independent dynamical variables of the gauge field are the three-dimensional transverse vectors ϕ_a^k and G_a^{0kT}, in close analogy with electrodynamics. If the variations of ϕ^k are performed within the radiation gauge,

$$\partial_k\delta\phi^k=0,$$

only the transverse part of G^{0k} appears in the generator,

$$G_\phi=-\int (d\mathbf{x})\ G^{0kT}\delta\phi_k,$$

the interpretation of which is given by

$$(1/i)[\phi_k(x),G_\phi]=\delta\phi_k(x),\quad (1/i)[G^{0kT}(x),G_\phi]=0.$$

When these statements are combined with the similar properties of the alternative generator

$$G_{G^T}=\int (d\mathbf{x})\ \phi_k\delta G^{0kT},$$

we obtain the full set of commutation relations for the fundamental dynamical variables of the gauge field:

$$x^0=x^{0\prime}:\quad [\phi_{ka}(x),\phi_{lb}(x')]=0,$$
$$[G_a{}^{0k}(x)^T,G_b{}^{0l}(x')^T]=0,$$
$$i[\phi_{ka}(x),G_b{}^{0l}(x')^T]=\delta_{ab}(\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}'))^T.$$

Apart from the multiplicity of the internal space, these are identical with the electromagnetic field commutation relations.

The ability to display the fundamental commutation relations is of the greatest importance, for it provides the assurance that the operators designated as the basic dynamical variables do constitute the generators of a complete operator basis, which is their primary role as operators. Otherwise stated, it makes explicit the infinitesimal transformations of the quantum transformation group which, together with the coordinate and other invariance groups, largely characterize the physical system.

We can now confirm a previous expectation. The contribution of the gauge field to the operators $\mathbf{T}_a$ involves only the transverse components of G^{0k}, since ϕ^k is divergenceless. The canonical structure of the commutation relations implies that

$$\left[-i\int(d\mathbf{x})\ \boldsymbol{\phi}\cdot t_a\mathbf{G}^T,\ -i\int(d\mathbf{x})\ \boldsymbol{\phi}\cdot t_b\mathbf{G}^T\right]=-i\int(d\mathbf{x})\ \boldsymbol{\phi}\cdot[t_a,t_b]\mathbf{G}^T,$$

which, together with statements of kinematical independence between the gauge field and the other systems,

$$x^0=x^{0\prime}:\quad [\phi_a{}^k(x),k_b{}^0(x')]=[G_a{}^{0k}(x)^T,k_b{}^0(x')]=0$$

supplies the verification that the group commutation relations are obeyed by the conserved $\mathbf{T}$-operators,

$$[\mathbf{T}_b,\mathbf{T}_c]=\textstyle\sum_a \mathbf{T}_a t_{abc}.$$

To complete this part of our study we must give the explicit operator construction of the longitudinal part of G^{0k} and of ϕ^0. Consider then $(x^0=x^{0\prime})$

$$(-\nabla^2+i^t t\boldsymbol{\phi}(x)'\cdot\nabla).[\psi(x),\boldsymbol{\phi}(x')]=i^t t\boldsymbol{\phi}(x)'\cdot[\mathbf{G}^T(x),\boldsymbol{\phi}(x')],$$

and note that the solution of this equation requires no reference to operator properties since it is concerned only with the completely commutative components of $\boldsymbol{\phi}(x)$ at a common time. The relevant Green's function is defined by the matrix differential equation

$$[-\nabla^2+i^t t\boldsymbol{\phi}(x)'\cdot\nabla]\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'),$$

in which the indicated functional dependence on $\boldsymbol{\phi}$ also produces a time dependence. The self-adjointness of the defining equation implies the symmetry of this real function,

$$\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')_{ab}=\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})_{ba}.$$

Now return to the ψ equation and remove the symmetrization of $\boldsymbol{\phi}$ with ψ, which gives

$$[-\nabla^2+i^t t\boldsymbol{\phi}'\cdot\nabla]\psi=i^t t\boldsymbol{\phi}':\mathbf{G}^T+k^0+ih(\boldsymbol{\phi}),$$

where $h(\boldsymbol{\phi})$ is a Hermitian function of $\boldsymbol{\phi}$. On solving this equation the last term will contribute a skew-Hermitian function of $\boldsymbol{\phi}$, which disappears on combining ψ with its identical Hermitian conjugate operator. The result is the symmetrized form

$$\psi(x)=\int(d\mathbf{x}')\ \mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').[i^t t\phi_k(x')'.G^{0k}(x')^T+k^0(x')],$$

or, more symbolically,

$$\psi=\mathfrak{D}_\phi.[i^t t\boldsymbol{\phi}':\mathbf{G}^T+k^0].$$

It is immaterial whether the symmetrization of $\boldsymbol{\phi}$ with $\mathbf{G}^T$ is independently performed, as indicated, or accompanies the symmetrization of the $\mathfrak{D}_\phi$ product. That is,

$$A.(B.C)-(A.B).C=\tfrac{1}{4}[[A,C],B],$$

which vanishes when A, B, C are $\mathfrak{D}_\phi$, $\boldsymbol{\phi}$, and $\mathbf{G}^T$, respectively.

The form of the complete operator G^{0k}, as it has now been obtained, is given symbolically by

$$\mathbf{G}=(1-\nabla\mathfrak{D}_\phi i^t t\boldsymbol{\phi}'):\mathbf{G}^T-\nabla\mathfrak{D}_\phi k^0,$$

or, equally well, by the versions

$$\mathbf{G}=[1+\nabla\mathfrak{D}_\phi(\nabla-i^t t\boldsymbol{\phi}')]:\mathbf{G}^T-\nabla\mathfrak{D}_\phi k^0$$
$$=\mathbf{G}^T:[1+(\nabla-i^t t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla]-\nabla\mathfrak{D}_\phi k^0,$$

which are related through the symmetry of $\mathfrak{D}_\phi$. The significance of the bracketed structures is indicated by

$$\nabla\cdot[1+(\nabla-i^t t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla]=0,$$

and

$$(\nabla-i^t t\boldsymbol{\phi}')\cdot[1+\nabla\mathfrak{D}_\phi(\nabla-i^t t\boldsymbol{\phi}')]=0.$$

The equal-time commutator of $\boldsymbol{\phi}$ with $\mathbf{G}$ is immediately evaluated as

$$i[\phi_{ka}(x),G_b{}^{0l}(x')]=[\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}')-(\partial_k-i^t t\phi_k(x)')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l]_{ab},$$

where the last gradient acts on the function to its left.

The construction of $\phi_a{}^0(x)$ proceeds from the equation

$$(-\nabla^2+i^t t\boldsymbol{\phi}'\cdot\nabla).\phi^0=f^2\nabla\cdot\mathbf{G},$$

and the solution is

$$\phi^0=f^2\mathfrak{D}_\phi.\nabla\cdot\mathbf{G},$$

or alternatively

$$\phi^0=f^2\mathfrak{D}_\phi i^t t\boldsymbol{\phi}'\cdot[1+\nabla\mathfrak{D}_\phi(\nabla-i^t t\boldsymbol{\phi}')]:\mathbf{G}^T+f^2(\mathfrak{D}_\phi-\mathfrak{D}_\phi i^t t\boldsymbol{\phi}'\cdot\nabla\mathfrak{D}_\phi)k^0.$$

It follows directly that

$$(1/i)[\phi^0,\boldsymbol{\phi}]=f^2\mathfrak{D}_\phi i^t t\boldsymbol{\phi}'\cdot[1+\nabla\mathfrak{D}_\phi(\nabla-i^t t\boldsymbol{\phi}')],$$

which symbolic equation has the following explicit meaning as an equal-time commutator:

$$(1/i)[\phi_a{}^0(x),\phi_b{}^k(x')]=f^2\bigg[\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')i`t\phi^k(x')' + \int (d\mathbf{x}'')\, \mathfrak{D}_\phi(\mathbf{x},\mathbf{x}'')i`t\phi(x'')' \cdot\nabla''\mathfrak{D}_\phi(\mathbf{x}'',\mathbf{x}')(-\partial'^k - i`t\phi^k(x')')\bigg]_{ab}.$$

If the expression for ϕ^0 is substituted in the equation for $\partial_0\boldsymbol{\phi}$, one obtains a fundamental transverse equation of motion,

$$-\partial_0\boldsymbol{\phi}=f^2[1+(\nabla-i`t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla]:\mathbf{G},$$

or, equivalently,

$$-\partial_0\boldsymbol{\phi}=f^2\{[1+(\nabla-i`t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla] \cdot[1+\nabla\mathfrak{D}_\phi(\nabla-i`t\boldsymbol{\phi}')]\}:\mathbf{G}^T - f^2[1+(\nabla-i`t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla]\cdot i`t\boldsymbol{\phi}'\mathfrak{D}_\phi k^0.$$

One implication of this equation of motion is the equal-time commutator

$$[i\partial_0\boldsymbol{\phi},\boldsymbol{\phi}]=f^2[1+(\nabla-i`t\boldsymbol{\phi}')\mathfrak{D}_\phi\nabla]\cdot[1+\nabla\mathfrak{D}_\phi(\nabla-i`t\boldsymbol{\phi}')].$$

If we introduce an arbitrary real numerical transverse vector function $\mathbf{a}_a(\mathbf{x})$ and define the Hermitian operator

$$A(x^0)=\int (d\mathbf{x})\ \mathbf{a}\cdot\boldsymbol{\phi},$$

this commutator becomes

$$[i\partial_0 A,A]=f^2\int (d\mathbf{x})\ \mathbf{b}\cdot\mathbf{b},$$

where the Hermitian vector $\mathbf{b}$ is

$$\mathbf{b}=[1+\nabla\mathfrak{D}_\phi(\nabla-i`t\boldsymbol{\phi}')]\cdot\mathbf{a}.$$

But the vacuum expectation value of such a commutator can never be negative,

$$\langle[[A,P^0],A]\rangle=2\langle AP^0A\rangle>0,$$

and therefore

$$f^2>0.$$

Finally, we state the equal-time commutator

$$i[G_a{}^{0k}(x),G_b{}^{0l}(x')]=[\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\cdot i`tG^{0l}(x')' + i`tG^{0k}(x)'\cdot\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l]_{ab},$$

and leave the proof to the reader.

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Strange Particle Production in Proton-Proton Collisions*

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An estimate is made for strange particle production in p-p collisions based on the single pion exchange model. For the three-particle final state (KYN), fair agreement with experiment is achieved in both the total cross sections and the momentum and angular distributions of the final particles. For the four-particle final state ($KYN\pi$), general qualitative agreement is achieved for all total cross sections but one. In this, the case of $\pi^0+p+K^++\Lambda^0(\Sigma^0)$ production, it is suggested that perhaps Y^* production in p-p collisions plays an important role.

I. INTRODUCTION

RECENTLY there has been much interest in the single boson exchange mechanism in high-energy collisions.[1] Ferrari has made a calculation for associated production in proton-proton collisions at incoming energies of 2 to 3 Bev in the lab system.[2] Of the two models he considered (single pion exchange and single K-meson exchange) the pion exchange model seems to fit the experimental data rather well.[3] This result naturally suggests that perhaps single pion exchange plays a dominant role even at the relatively low energies of a few Bev for many other nucleon-nucleon collision processes. In this note we shall concern ourselves only with strange particle productions with or without an additional pion in the final states.

In Sec. II the three-particle final states of the kind $p+p\rightarrow Y+K+N$ are considered. The treatment is essentially that of Ferrari's with some modifications.

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[1] F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1960). I. M. Dremin and D. S. Chernavskii, Soviet Phys.—JETP **11**, 167 (1960); V. I. Veksler, *Proceedings of the Tenth Annual International Rochester Conference on High-Energy Physics, 1960* (Interscience Publishers, Inc., New York, 1960).

[2] E. Ferrari, Nuovo cimento **15**, 652 (1960); Phys. Rev. **120**, 988 (1960).

[3] R. I. Louttit, T. W. Morris, D. C. Rahm, R. R. Rau, A. M. Thorndike, and W. J. Willis, *Proceedings of the Tenth International Rochester Conference on High-Energy Physics, 1960* (Interscience Publishers, Inc., New York, 1960).