Inelastic Electron-Nucleus Scattering and Nucleon-Nucleon Correlations*

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A constant-q and -θ sum rule is constructed in Born approximation for inelastic scattering of electrons on nuclei by integrating over the outgoing electron energy spectrum at constant 3-momentum transfer, q, and scattering angle θ. A consistent treatment of the nucleon current interaction through second order in q/M, the recoil nucleon velocity, gives a result reliable to 5–10% out to $q\sim 2.5$ f^{-1}, and shows that the often neglected Darwin-Foldy term contributes substantially. It is well known that the q dependence of the sum can be directly related by closure to the Fourier transform of two-nucleon correlation functions in the ground state of the nucleus. This q dependence is studied for various nuclear models of O^{16}; the effect of the Pauli exclusion principle on the sum is found to be large for $q\sim 1$ f^{-1}, but the effect of a hard-core force of radius $\leqslant 0.4$ f appears to be small for all values of q.

I. INTRODUCTION

Sum rules for the inelastic scattering of x rays on atoms and of slow neutrons on solids and liquids have provided useful information on two-particle spatial correlations in these systems.[1] The technique is based on the fact that waves scattered with momentum transfer $\mathbf{q}$ from particles at positions $\mathbf{r}_j$ and $\mathbf{r}_k$ in the target system will interfere with a phase difference of $\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)$, so that the $\mathbf{q}$ dependence of the scattering bears a direct relation to $P_2(|\mathbf{r}_j-\mathbf{r}_k|)$, the pair distribution function in the target system.

It has also long been recognized[2] that the inelastic scattering of high-energy electrons by nuclei should, by analogy, be a means of investigating nucleon-nucleon correlations, provided that momentum transfers of a few hundred Mev/c can be attained.[3] Since the necessary experimental techniques are rapidly becoming available[4] our purpose is to construct a sum rule which can be used in the analysis of such experiments, and to investigate its relation to realistic nucleon pair correlation functions. In particular, we wish to estimate the sensitivity of the sum rule to Pauli and hard-core correlations. Similar analyses have been published recently by Drell and Schwartz,[5] Fowler and Watson,[6] and Drummond[7]; since the points of distinction between their work and ours are rather involved, we shall postpone discussion of them to the appropriate places in the development of our formalism.

In Sec. II and Appendix A we present the formalism necessary for a Born approximation analysis of the scattering, taking especial care to retain all terms in the interaction of second order in (q/M), the recoil velocity of a nucleon. In Sec. III the sum rule is constructed for the simple case of Coulomb scattering alone, which bears the closest relation to the work of reference 1. This is extended in Sec. IV to include the current interaction terms, and the resulting sum rule is evaluated in Sec. V for specific models of the nuclear ground state.

II. BORN APPROXIMATION FORMALISM

Our analysis will be based on the first Born approximation, and is consequently limited to light nuclei; explicit calculations will be performed for O^{16}. Further, because of our total lack of information on a relativistic wave function for the target nucleus, we shall restrict ourselves to events in which the nucleons interact like nonrelativistic Pauli particles. This is accomplished by expanding the electron-nucleon interaction in powers of $1/M$, the inverse nucleon mass, and retaining terms through order M^{-2}. The range of validity of this approximation is determined by the 3-momentum, q, transferred to the target; an estimate of the leading correction term indicates that it should not exceed 5–10% of the terms retained at $q/M=\frac{1}{2}$, or $q\sim 2.5$ f^{-1}, so we take this as the maximum q at which the M^{-2} approximation is useful. Since the hard core radius of the nuclear force is assumed to be about 0.4 f, this range of q ought to be just about adequate to detect hard-core effects on the scattering if they are large enough to be seen at all.

Our treatment of the nucleon-electron interaction in-

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[1] See, e.g., A. Guinier, G. Fournat, C. Walker, and K. Yudowitch, *Small-Angle Scattering of X Rays* (John Wiley & Sons, Inc., New York, 1955); and L. Van Hove, Phys. Rev. **95**, 249 (1954).

[2] References to the literature previous to 1957 are given by R. Hofstadter, Ann. Rev. Nuclear Sci. VII, 231 (1957), see especially p. 305.

[3] We note that 200 Mev/c=1 f^{-1}=10^{13} cm^{-1}. This is a convenient unit for momentum and indicates why momentum transfers of this order of magnitude are necessary for investigations on a nuclear scale.

[4] See in particular, D. B. Isabelle and G. R. Bishop, Rapport du Laboratoire de l'Accélérateur Linéaire 1017, Faculté des Sciences de Paris, Orsay, France (unpublished).

[5] S. D. Drell and C. L. Schwartz, Phys. Rev. **112**, 568 (1958).

[6] T. K. Fowler and K. M. Watson, Nuclear Phys. **13**, 549 (1959).

[7] W. E. Drummond, Phys. Rev. **116**, 183 (1959).

cludes the nucleon form factors, but it neglects the possibility of pion electroproduction. The analysis consequently rests on the assumption that the theory of the "elastic" (nonmesonic) events is self-contained, and is not seriously affected by the presence of an inelastic channel. In particular, the sum rule considered will include only nonmesonic final states and so presumes the possibility of carrying out the nonmesonic sum experimentally. As is explained below, it seems likely that this can be done reliably over the momentum range $q<2.5\ \mathrm{f}^{-1}$ in which we are interested.

The covariant interaction between electrons and relativistic nucleons is of course well known, but its reduction to two-component form for the nucleon is not entirely trivial and has often been done inconsistently in the past (in particular in references 5 and 7). One way of performing the reduction consistently is via the Foldy-Wouthuysen transformation, as is sketched in Appendix A.

The conclusion of this appendix is that, if the electron is described by plane waves, then in the (two-component) *nucleon* space the Hamiltonian operator which describes the electron-nucleon interaction, correct through order q^2/M^2, is

$$H'=\frac{4\pi e^2}{q_\mu^2}\Big\langle u_2\Big|F_1e^{iq_\mu x_\mu}-\frac{F_1}{2M}[(\mathbf{p}\cdot\boldsymbol{\alpha})e^{iq_\mu x_\mu}+e^{iq_\mu x_\mu}(\mathbf{p}\cdot\boldsymbol{\alpha})]$$
$$-\frac{(F_1+KF_2)}{2M}i\boldsymbol{\sigma}\cdot(\mathbf{q}\times\boldsymbol{\alpha})e^{iq_\mu x_\mu}$$
$$-\frac{q_\mu^2}{8M^2}(F_1+2KF_2)e^{iq_\mu x_\mu}+\frac{(F_1+2KF_2)}{8M^2}i\boldsymbol{\sigma}$$
$$\cdot\{\mathbf{p}\times(\omega\boldsymbol{\alpha}-\mathbf{q})e^{iq_\mu x_\mu}-e^{iq_\mu x_\mu}(\omega\boldsymbol{\alpha}-\mathbf{q})\times\mathbf{p}\}\Big|u_1\Big\rangle. \quad (1)$$

The notation used here is as follows: the electron momenta in the initial and final states are $\mathbf{k}_1$ and $\mathbf{k}_2$, with energies $\mathcal{E}_1=(k_1^2+m^2)^{\frac{1}{2}}$ and $\mathcal{E}_2=(k_2^2+m^2)^{\frac{1}{2}}$. In terms of them, the momentum and energy transferred to the target are defined as

$$\mathbf{q}=\mathbf{k}_1-\mathbf{k}_2 \quad\text{and}\quad \omega=\mathcal{E}_1-\mathcal{E}_2, \quad (2)$$

so that they are (with correct signs) the momentum and energy of the recoiling target particle, assumed initially at rest; $q_\mu^2=q^2-\omega^2$ is then the square of the 4-momentum transfer. $\boldsymbol{\alpha}$ is the electron's Dirac operator, which operates on the free-electron spinors $|u_1\rangle$ and $|u_2\rangle$, $\mathbf{p}$ and $\boldsymbol{\sigma}$ are the momentum and Pauli spin operators in the nucleon's space, $F_1(q_\mu^2)$ and $F_2(q_\mu^2)$ are the nucleon's charge and magnetic momentum form factors, and K is its static anomalous magnetic moment, in nuclear magnetons.

The first three terms of (1) describe the usual static Coulomb, convection current, and spin current interactions, and are of zeroth and first order in M^{-1}. The last two expressions, of order M^{-2}, are the Darwin-Foldy and spin-orbit terms. Because the square of the matrix element contains a cross term between them and the Coulomb term which is of order M^{-2}, they must be included in a consistent calculation of the cross section through order M^{-2}. For unpolarized nucleons, the spin-orbit term does not contribute to this M^{-2} cross term, so if higher-order contributions are to be neglected, we can drop it at once. The Darwin-Foldy term, though, must be kept for an M^{-2}-order calculation of the cross section; inspection of the Rosenbluth formula for the single-proton cross section shows its contribution to be quite substantial.

Equation (1) describes the scattering by a single nucleon. To handle the many-nucleon system we make the usual assumption that the nucleons in a nucleus do not "distort" one another, so that the form factors F_1 and F_2 are the same inside a nucleus as out. Then since the operators for distinct nucleons commute, we can apply a Foldy-Wouthuysen transformation for *each* nucleon to the relativistic many-body equation, to see that the electron-nucleus interaction Hamiltonian is simply Eq. (1) summed over all the nucleons present.

The form factors $F_1(q_\mu^2)$ and $F_2(q_\mu^2)$ are known quite accurately for the proton[8] and with fair accuracy for the neutron[9] over the range of q we need. They are consistent with the relations $F_{1p}(q_\mu^2)=F_{2p}(q_\mu^2)=f(q_\mu^2)$ for the proton and $F_{1n}(q_\mu^2)=0$, $F_{2n}(q_\mu^2)=f(q_\mu^2)$ for the neutron, which is the approximation we shall use.

When the time dependence of the nuclear states is included, the x_4 integration produces the energy delta function. The Born approximation matrix element (with respect to nuclear as well as electron coordinates) of the electron-nucleus interaction Hamiltonian, correct through order M^{-2}, can then be written in the compact form

$$M_{0n}=\delta(\mathcal{E}_1-\mathcal{E}_2-q^2/2AM-E_n+E_0)(4\pi e^2/q_\mu^2)f(q_\mu^2)$$
$$\times\{(u_2|u_1)Q_{n0}-(u_2|\boldsymbol{\alpha}|u_1)\cdot\mathbf{J}_{n0}\}, \quad (3)$$

in which E_0 and E_n are the initial and final state energy levels of the system of A nucleons. The nuclear matrix elements between these states are[10]

$$Q_{n0}(\mathbf{q})=\Big\langle n'\Big|\sum_{j=1}^{A}\Big[e_j+\frac{q^2}{8M^2}(e_j-2\mu_j)\Big]e^{i\mathbf{q}\cdot\mathbf{r}_j}\Big|0\Big\rangle,$$

$$\mathbf{J}_{n0}(\mathbf{q})=\Big\langle n'\Big|\sum_{j=1}^{A}\Big\{\frac{e_j}{2M}(\mathbf{p}_je^{i\mathbf{q}\cdot\mathbf{r}_j}+e^{i\mathbf{q}\cdot\mathbf{r}_j}\mathbf{p}_j) \quad (4)$$
$$+\frac{\mu_j}{2M}i\boldsymbol{\sigma}_j\times\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}_j}\Big\}\Big|0\Big\rangle.$$

[8] R. Hofstadter, F. Bumiller, and M. Croissiaux, Phys. Rev. Letters **5**, 263 (1960).

[9] R. Hofstadter and R. Herman, Phys. Rev. Letters **6**, 293 (1961). The conclusions of this reference suggest small deviations from our approximations for $q>2.5\ \mathrm{f}^{-1}$, but a re-analysis by L. Durand, III, Phys. Rev. Letters **6**, 631 (1961), largely removes them again.

[10] The factor q_μ^2/M^2 occurs in the Darwin-Foldy term. Because $\omega\lesssim q^2/2M$ in all practical cases for the present problem, we take $q_\mu^2/M^2\approx q^2/M^2$ throughout.

$\mathbf{r}_j$ and $\mathbf{p}_j$ are the position and momentum operators for the jth nucleon and $\boldsymbol{\sigma}_j$ is its Pauli spin operator; the nuclear states $|0\rangle$ and $|n'\rangle$ are now two-component nonrelativistic states. In terms of isospin operators $e_j=\frac{1}{2}(1+\tau_{3j})$ is the charge or proton projection operator, and we are using the approximation

$$\mu_j=\tfrac{1}{2}(1+\tau_{3j})(1+K)-\tfrac{1}{2}(1-\tau_{3j})K=\tfrac{1}{2}(1+\tau_{3j})+\tau_{3j}K$$

as the magnetic moment projection operator, with $K\approx 1.85$. In writing down the expression for $\mathbf{J}$ we have used the result found by Drell and Schwartz,[5] that the presence of an exchange potential in the nuclear Hamiltonian produces only a negligible effect on the (non-energy-weighted) sum rule which we wish to discuss, so we have left out the exchange currents from the beginning.

We have written the nuclear states in Eq. (4) with a prime, $|n'\rangle$, to indicate that they are written in the laboratory system, where the recoiling nucleus has momentum $\mathbf{q}$. The transformation to the center-of-mass system of the nucleus, where they become $|n\rangle=e^{-i\mathbf{q}\cdot\mathbf{R}}|n'\rangle$, is accomplished in symmetric fashion by employing the transformation of Gartenhaus and Schwartz.[11] The matrix elements Q_{n0} and $\mathbf{J}_{n0}$ are left by this transformation in exactly the form of Eq. (4) provided the replacements $|n'\rangle\rightarrow|n\rangle$, $\mathbf{r}_j\rightarrow\boldsymbol{\rho}_j=\mathbf{r}_j-\mathbf{R}$, and $\mathbf{p}_j\rightarrow\boldsymbol{\pi}_j=\mathbf{p}_j-A^{-1}\mathbf{P}$ are made;

$$\mathbf{R}=A^{-1}\sum_{k=1}^{A}\mathbf{r}_k \quad\text{and}\quad \mathbf{P}=\sum_{k=1}^{A}\mathbf{p}_k.$$

Also, the matrix elements are understood to be zero unless the final nuclear state has momentum $\mathbf{q}$ relative to the laboratory system.

For the remainder of the paper we shall use the extreme relativistic approximation for the electrons, in which $\mathcal{E}_1=k_1$ and $\mathcal{E}_2=k_2$. Squaring M_{n0} and summing it over initial and final electron spin states gives in this approximation

$$\tfrac{1}{2}\sum_{\substack{\text{electron}\\ \text{spins}}}\sum|M_{n0}|^2 =2\delta(\omega-q^2/2AM-E_n+E_0)(4\pi e^2/q_\mu^2)f^2(q_\mu^2)W,$$

$$W=\{|Q_{n0}|^2(1+\cos\theta)-(\hat{k}_1+\hat{k}_2)\cdot(Q_{n0}^*\mathbf{J}_{n0}+\mathbf{J}_{n0}^*Q_{n0}) +|J_{n0}|^2(1-\cos\theta)+(\mathbf{J}_{n0}^*\cdot\hat{k}_2)(\mathbf{J}_{n0}\cdot\hat{k}_1) +(\mathbf{J}_{n0}^*\cdot\hat{k}_1)(\mathbf{J}_{n0}\cdot\hat{k}_2)\},\quad(5)$$

with the abbreviation $\hat{a}=a^{-1}\mathbf{a}$ for any vector $\mathbf{a}$.

III. SUM RULE FOR COULOMB SCATTERING

Since Eq. (5) is algebraically complicated we shall, before evaluating it for a specific nuclear model, briefly discuss the Coulomb scattering terms alone. These are the terms of Eq. (5) which are of zeroth order in $(1/M)$,

[11] S. Gartenhaus and C. L. Schwartz, Phys. Rev. **108**, 482 (1957).

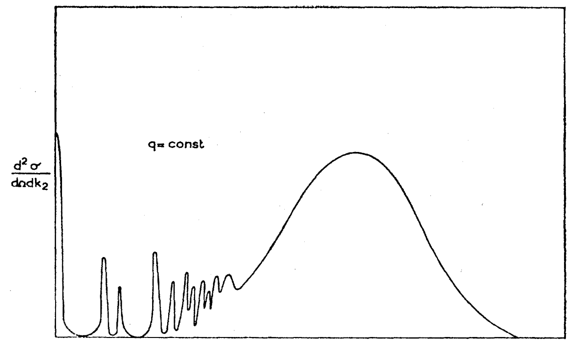

FIG. 1. Schematic energy spectrum of electrons scattered with a given momentum loss q, indicating resolved discrete nuclear levels, unresolved discrete levels, and a continuum corresponding mainly to single ejected nucleons ("quasi-elastic scattering"). The latter peaks at roughly $\omega=q^2/2M$, and is Doppler-broadened by the momentum distribution of the target nucleons.

and they give for the Coulomb cross section

$$\frac{d^2\sigma_C}{d\Omega_2 dk_2}=\frac{d\sigma_0}{d\Omega}C(q,\omega),\quad(6)$$

with

$$\frac{d\sigma_0}{d\Omega}=|f(q_\mu^2)|^2\frac{e^4}{4k_1^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)},\quad(7)$$

the free-proton cross section, and

$$C(q,\omega)=\sum_n|\langle n|\sum_{j=1}^{A}e_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_j)|0\rangle|^2 \times\delta(\omega-q^2/2AM-E_n+E_0),\quad(8)$$

a function containing only properties of the target nucleus. At fixed q, the ω dependence of $C(q,\omega)$ gives the energy spectrum of electrons scattered with a given momentum transfer, which looks something like Fig. 1.[12] The sum rule we wish to discuss is obtained by integrating over this spectrum at fixed q; constant q rather than constant θ is of course chosen to enable us to use closure. At small q, the elastic scattering becomes overwhelmingly large (note that by orthogonality it is the only nonzero term of the sum at $q=0$) and masks the more interesting q dependence of the inelastic terms, so we shall eliminate it from the sum rule and define a function $C(q)$ by the inelastic integral

$$\begin{aligned}C(q)&=Z^{-1}\int_{E_1}^{\infty}C(q,\omega)d\omega\\ &=Z^{-1}\sum_{n=0}^{\infty}|\langle n|\sum_{j=1}^{A}e_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_j)|0\rangle|^2 -Z^{-1}|\langle 0|\sum_{j=1}^{A}e_j\exp(i\mathbf{q}\cdot\boldsymbol{\rho}_j)|0\rangle|^2\\ &=Z^{-1}\langle 0|\sum_j\sum_k e_je_k\exp[i\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)]|0\rangle -Z|F(q)|^2,\end{aligned}\quad(9)$$

[12] See, e.g., J. H. Fregeau and R. Hofstadter, Phys. Rev. **99**, 1503 (1955); G. R. Burleson (to be published); and reference 4.

where

$$F(q)=\langle 0|e_j\exp(i\mathbf{q}\cdot\boldsymbol{\varrho}_j)|0\rangle \qquad (10)$$

is the elastic scattering form factor, which we assume known over the necessary range of q.[13]

The "diagonal terms" ($j=k$) in the double sum give

$$\langle 0|\sum_{j=1}^{A}e_j^2|0\rangle=Z,$$

and for the off-diagonal terms it is useful to define the proton-proton correlation function (probability density) in the ground state

$$P_{pp}(\mathbf{r}',\mathbf{r}'')=[Z(Z-1)]^{-1}\times\langle 0|\sum_{j\neq k}^{A}\sum^{A}e_je_k\delta(\mathbf{r}-\mathbf{r}_j)\delta(\mathbf{r}''-\mathbf{r}_k)|0\rangle, \qquad (11)$$

which is normalized to 1. Then

$$C(q)=1+(Z-1)\int\int d^3r'd^3r''\,e^{i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r}'')}P_{pp}(\mathbf{r}',\mathbf{r}'')-Z|F(q)|^2, \qquad (12)$$

from which the dependence of the Coulomb sum rule on the proton-proton spatial correlation function is clear.

$C(q)$ has a simple and important asymptotic behavior: because of oscillating integrands the last two terms of Eq. (12) vanish as $q\rightarrow\infty$, so that $C(q)\rightarrow 1$. The vanishing of the off-diagonal or correlation terms as $q\rightarrow\infty$ means that ZC reduces to a sum of free-proton cross sections, without interference, a process often called impulse approximation or quasi-elastic scattering. Since $C(0)=0$, $C(q)$ is a function which increases from zero to 1 as q increases, in a manner determined by $F(q)$ and the Fourier transform of $P_{pp}(\mathbf{r}_1,\mathbf{r}_2)$.

We mention in passing that $C(q)$ has an especially simple form in the special case of a shell-model approximation for a proton-magic nucleus. In this case the correlation function, in terms of the single-particle functions ψ_α, is the sum of a direct and an exchange term,

$$\begin{aligned}Z(Z-1)P_{pp}(\mathbf{r}_1,\mathbf{r}_2)&=\sum_{\alpha=1}^{Z}\sum_{\beta=1}^{Z}\{|\psi_\alpha(1)|^2|\psi_\beta(2)|^2-\psi_\alpha^*(1)\psi_\beta(1)\psi_\beta^*(2)\psi_\alpha(2)\}\\&=Z^2P_1(r_1)P_1(r_2)-ZP_e(\mathbf{r}_1,\mathbf{r}_2),\end{aligned} \qquad (13)$$

$P_1(r)$ is the one-particle probability density and P_e is defined as the exchange sum, normalized to 1. If we now approximate $F(q)$ by substituting $\mathbf{r}_j$ for $\boldsymbol{\varrho}_j=\mathbf{r}_j-A^{-1}\sum\mathbf{r}_k$ in Eq. (10) (thus making an error of order A^{-1}), we get the simple result

$$C(q)=1-\int\int d^3r_1d^3r_2\,e^{i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)}P_e(\mathbf{r}_1,\mathbf{r}_2). \qquad (14)$$

In this case, $C(q)$ is determined entirely by the exchange term of the correlation function, which indicates that the Pauli principle can be expected to exert an important influence on the shape of $C(q)$. {It definitely does in the case of a large Fermi gas with periodic boundary conditions, for which $C(q)=\frac{3}{4}(q/k_F)[1-\frac{1}{12}(q/k_F)^2]$ for $q<2k_F$ and $C(q)=1$ for $q>2k_F$; $1-C(q)$ is just the fraction of the volume of the Fermi sphere excluded by the Pauli principle for a scattering with momentum transfer $\mathbf{q}$.}

Before going on to a discussion of the current interaction terms, we note two important experimental restrictions on the construction of this sum rule. The first is that the entire range of ω from 0 to ∞ is of course never available. The accessible range can be seen from the kinematic relation $\omega^2=q^2-4k_1k_2\sin^2(\theta/2)$, from which the restriction $\omega\leqslant q$ follows. It is only the fact (as is indicated in Fig. 1) that the contributions to the sum from very large ω's are insignificant which makes $C(q)$ accessible experimentally at all; the use of closure is reliable only when essentially the entire area under the $C(q,\omega)$ curve is found to the left of $\omega_{\max}=q$.

For q sufficiently large, the scattering is expected to be predominantly quasi-elastic. In this case the impulse

[13] Two important comments must be made with regard to Eqs. (9) and (10). First, in the closure term only $\boldsymbol{\varrho}_j-\boldsymbol{\varrho}_k=\mathbf{r}_j-\mathbf{r}_k$ appears, so the Gartenhaus-Schwartz transformation has no effect on this term. Secondly, we emphasize the fact that the "density of final states" factor is unity because $C(q)$ is defined as the integral of $C(q,\omega)$ over ω (or k_2 if k_1 remains fixed) at constant q. That is, since $\mathcal{E}_2\equiv k_2+q^2/2MA+E_n$ appears in the argument of the energy delta function, the integral over k_2 introduces the factor $(\partial\mathcal{E}_2/\partial k_2)^{-1}$. At constant q this is 1, but experimentally the "constant q" sum is actually performed in small steps at constant θ. In the continuous part of the spectrum, the delta function is actually removed by the sum Σ_n (which includes an integral over E_n, the energy of relative motion of the parts of the recoiling nucleus). Then $C(q,\omega)$ can be presumed nearly constant over a small interval Δk_2; $\int_{\Delta k_2}C(q,\omega)d\omega\approx C(q,\omega)\Delta k_2$, and summing such increments at constant q we get
$$\sum_m C(q,\omega_m)\Delta k_2^{(m)}\approx\int_{q=\text{const}}C(q,\omega)d\omega=\sum_n|\langle n|\sum_j e_j\exp(i\mathbf{q}\cdot\boldsymbol{\varrho}_j)|0\rangle|^2.$$
Provided that $C(q,\omega)$ varies little over the Δk_2 intervals, the result of adding small constant-θ increments in this way is the same as a true constant-q integral. In the discrete spectrum, on the other hand, the delta-function dependence of $C(q,\omega)$ on k_2 *does* introduce the factor $(\partial\mathcal{E}_2/\partial k_2)_\theta^{-1}=[1+(k_2-k_1\cos\theta)/AM]^{-1}$, when $C(q,\omega)$ is integrated over k_2 in constant-θ steps. This will only be true provided the energy level spacings ΔE_n are large compared to Δk_2; if $\Delta E_n\ll\Delta k_2$, enough levels will be included to make the experimental $C(q,\omega)$ effectively continuous, so the first argument applies. In summary, the prescription for obtaining $C(q)$ experimentally is to multiply the contribution of an interval Δk_2 by $[1+(k_2-k_1\cos\theta)/AM]^{-1}$ if $\Delta E_n>\Delta k_2$, but *not* if $\Delta E_n<\Delta k_2$. This prescription also applies to the construction of the function $T(q,\theta)$, Eq. (20).

approximation provides a simple estimate of $C(q,\omega)$,

$$C_{\rm im}(q,\omega)=\int N(p)\delta[(\mathbf{p}+\mathbf{q})^2/2M-p^2/2M-\omega]d^3p, \quad (15)$$

where $N(p)$ is the momentum distribution of the struck nucleons. The approximation $N(p)=N_0\exp(-p^2/p_0^2)$, with $p_0\sim 0.9\ \mathrm{f}^{-1}$, is often used for light nuclei,[14] and gives

$$C_{\rm im}(q,\omega)=\mathrm{const}\,\exp[-(\omega-q^2/2M)^2/(p_0q/M)^2], \quad (16)$$

i.e., a Gaussian function of ω, peaked at $\bar{\omega}=q^2/2M$, with a half-width of $(p_0/\sqrt{2}M)q\sim q/7$.

If we use this impulse approximation, the condition that the entire area under $C(q,\omega)$ be accessible experimentally becomes $\bar{\omega}=q^2/2M\ll q$, i.e., $q\ll 2M\sim 10\ \mathrm{f}^{-1}$.

The neglect of mesonic events from the closure sum imposes a more severe restriction on the range of q. For $\omega>m_\pi$, pion electroproduction can occur. If only the electrons are observed, the non-mesonic events cannot be distinguished from the mesonic ones, so the number of non-mesonic events can only be estimated by extrapolating under the meson tail. For $C(q)$, only the area under $C(q,\omega)$ is needed, and for this purpose one can probably use some large-ω model such as Eq. (16), provided that enough of the single-particle peak occurs for $\omega<m_\pi$ to allow this model curve to be fitted to the experimental one. Requiring, e.g., that half the single-particle peak occur for $\omega<m_\pi$ gives $\bar{\omega}=q^2/2M<m_\pi$, or $q<2.5\ \mathrm{f}^{-1}$. Since this is also about the value of q at which corrections to our nonrelativistic approximation Hamiltonian begin to become important, we conclude that $0<q<2.5\ \mathrm{f}^{-1}$ is the maximum range over which the present analysis can be trusted.

IV. SUM RULE INCLUDING THE CURRENT INTERACTION TERMS

Writing the expression W of Eq. (5) in the form

$$\begin{aligned}W=&(1+\cos\theta)Q_{n0}^*Q_{n0}-(\hat{k}_1+\hat{k}_2)\cdot(\mathbf{J}_{n0}^*Q_{n0}+Q_{n0}^*\mathbf{J}_{n0})\\&+(1-\tfrac{1}{3}\cos\theta)\mathbf{J}_{n0}^*\cdot\mathbf{J}_{n0}+(\mathbf{J}_{n0}^*\cdot\hat{k}_1)(\mathbf{J}_{n0}\cdot\hat{k}_2)\\&-\tfrac{1}{3}(\hat{k}_1\cdot\hat{k}_2)(\mathbf{J}_{n0}^*\cdot\mathbf{J}_{n0})+(\mathbf{J}_{n0}\cdot\hat{k}_1)(\mathbf{J}_{n0}^*\cdot\hat{k}_2)\\&-\tfrac{1}{3}(\hat{k}_1\cdot\hat{k}_2)(\mathbf{J}_{n0}^*\cdot\mathbf{J}_{n0}), \quad (17)\end{aligned}$$

we see that $\mathbf{J}$ appears linearly as $\mathbf{J}\cdot\hat{k}$ and quadratically either as $\mathbf{J}\cdot\mathbf{J}$ or in the product of two second-rank tensors. We shall be interested only in this expression summed over the final states n. Since this sum includes an average over the orientations of the spin and momentum vectors in $|n\rangle$, $\mathbf{q}$ is the only vector in $\mathbf{J}$ which survives the sum. Consequently for the purpose of the n sum we can make the replacements

$$\begin{aligned}&\mathbf{J}\to\mathbf{q}(\mathbf{q}\cdot\mathbf{J})/q^2,\\&\mathbf{JJ}-\tfrac{1}{3}(\mathbf{J}\cdot\mathbf{J})\to[\mathbf{qq}-\tfrac{1}{3}(\mathbf{q}\cdot\mathbf{q})]\\&\qquad\times[(\mathbf{q}\cdot\mathbf{J})(\mathbf{q}\cdot\mathbf{J})-\tfrac{1}{3}q^2(\mathbf{J}\cdot\mathbf{J})]/\tfrac{2}{3}q^4,\end{aligned} \quad (18)$$

in which case W becomes

$$\begin{aligned}&(1+\cos\theta)^{-1}W\\&=Q_{n0}^*Q_{n0}-(\omega/q)[(\hat{q}\cdot\mathbf{J}_{n0}^*)Q_{n0}+Q_{n0}^*(\hat{q}\cdot\mathbf{J}_{n0})]\\&\quad+\tfrac{1}{3}[2\tan^2(\theta/2)+1]\mathbf{J}_{n0}^*\cdot\mathbf{J}_{n0}\\&\quad-\tfrac{1}{2}[2\tan^2(\theta/2)+1-3\omega^2/q^2]\\&\quad\times[(\hat{q}\cdot\mathbf{J}_{n0}^*)(\hat{q}\cdot\mathbf{J}_{n0})-\tfrac{1}{3}(\mathbf{J}_{n0}^*\cdot\mathbf{J}_{n0})]. \quad (19)\end{aligned}$$

In contrast to the Q^2 term alone, which only depends on q^2, this complete expression also contains the familiar dependence of the current terms on the scattering angle θ. This unfortunately imposes the additional experimental restriction that, in order to permit the sum over n to be evaluated by closure, it must be performed experimentally at constant q *and* constant θ—i.e., the various parts of the electron spectrum must be reached by varying both the initial and final electron energies k_1 and k_2, keeping q and θ fixed, rather than by varying θ and k_2 with k_1 and q fixed.

Furthermore, Eq. (19) also shows that the current terms contain an explicit dependence on ω, the energy transfer. This means that, unless we know the correct averages $\bar{\omega}$ and $\langle\omega^2\rangle_{\rm av}$ to employ, exact sum rules cannot be constructed without explicit introduction of the Hamiltonian. However, using what we believe to be reasonable estimates for $\bar{\omega}$ and $\langle\omega^2\rangle_{\rm av}$, we shall find below that the evaluation of the sum in the shell-model approximation shows these ω-dependent terms to be very small relative to the others for $q<2\ \mathrm{f}^{-1}$ at $\theta=90°$. Consequently we believe that there is a substantial region in which such a "closure approximation" can be used reliably.

Dividing the cross section derived from W by the cross section (7), we define a function analogous to the $C(q)$ of Eq. (9) in terms of an integral over the electron spectrum at constant q and θ

$$T(q,\theta)\equiv\int_0^\infty d\omega\frac{d^2\sigma}{d\Omega dk_2}\left[Z\frac{d\sigma_0}{d\Omega}\right]^{-1}-Z|F(q)|^2, \quad (20)$$

with $d\sigma_0/d\Omega=|f(q_\mu^2)|^2(e^2/4k_1^2)\cos^2(\theta/2)/\sin^4(\theta/2)$, as before. Within the q range considered here, the elastic contribution is $Z^2|F(q)|^2$, so $T(q,\theta)$ is just the integral over the inelastic spectrum. Note that it is normalized, somewhat arbitrarily, per proton. Using closure and replacing ω, ω^2 by their averages, $ZT(q,\theta)+Z^2|F|^2$ is simply the ground-state expectation value of $(1+\cos\theta)^{-1}W(q,\bar{\omega},\langle\omega^2\rangle_{\rm av},\theta)$, with Q_{n0} and $\mathbf{J}_{n0}$ replaced by the operators Q and $\mathbf{J}$ given in Eq. (4). Substituting these expressions from Eq. (4), $T(q,\theta)$ can be manipu-

[14] See, e.g., A. Wattenberg, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 40, p. 452. Also, A. G. Sitenko and V. N. Gur'ev, Soviet Phys.—JETP **12**, 1228 (1961).

lated into the form[15]

$$Z^2|F(q)|^2+ZT(q,\theta)$$

$$=Z\left(1-\frac{\bar{\omega}}{M}\right)+\frac{q^2}{4M^2}\left\{(Z\mu_p^2+N\mu_n^2)\left[2\tan^2(\theta/2)+1-\frac{\langle\omega^2\rangle_{\rm av}}{q^2}\right]-Z(1+2K)\right\}+\tfrac{1}{3}Z\frac{\langle p^2\rangle}{M^2}[2\tan^2(\theta/2)+1]+Z\frac{\langle\omega^2\rangle_{\rm av}}{4M^2}$$

$$+\left\langle 0\left|\sum_{j\neq k}^{A}\sum^{A} e^{i\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)}\left\{e_je_k+\mu_j\mu_k\frac{1}{8M^2}[q^2\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k-(\mathbf{q}\cdot\boldsymbol{\sigma}_j)(\mathbf{q}\cdot\boldsymbol{\sigma}_k)]\left[2\tan^2(\theta/2)+1-\frac{\langle\omega^2\rangle_{\rm av}}{q^2}\right]\right.\right.\right.$$

$$\left.\left.\left.+\frac{q^2}{4M^2}e_j(e_k-2\mu_k)+e_je_k\frac{p_{xj}p_{xk}}{M^2}[2\tan^2(\theta/2)+1]+\frac{\langle\omega^2\rangle_{\rm av}}{q^2}e_je_k\left[\frac{p_{zj}p_{zk}-p_{xj}p_{xk}+q^2/4}{M^2}\right]\right\}\right|0\right\rangle. \quad (21)$$

Here $p_{zj}=\mathbf{p}_j\cdot\hat{\mathbf{q}}$ and p_{xj} is the component of $\mathbf{p}_j$ in a direction normal to $\mathbf{q}$. This expression is considerably more complicated than the analogous Eq. (19) of Drell and Schwartz,[5] in which many terms, not all of them small, were neglected; the Darwin-Foldy terms in particular are found to make a substantial contribution. The terms for which the expectation value has been explicitly evaluated are the "diagonal" or $j=k$ terms; we note that, if we take $\bar{\omega}=q^2/2M$, as for a free nucleon, and neglect the $\langle p^2\rangle$ terms (=zero for a free nucleon) and the $\langle\omega^2\rangle_{\rm av}/q^2$ terms (of order M^{-4} for a free nucleon), these diagonal terms give

$$Z+(q^2/4M^2)\{2(Z\mu_p^2+N\mu_n^2)\tan^2(\theta/2)+(N+Z)K^2-2Z\},$$

exactly as they should for the sum of free proton and neutron cross sections, through order q^2/M^2.[16] [See Eq. (A9).]

V. EVALUATION OF THE SUM FOR SPECIFIC NUCLEAR MODELS

The function $C(q)$ of Eq. (9) is defined in terms of the ground-state proton-proton correlation function and so is directly related to the spatial correlations in the wave function. Similarly $T(q,\theta)$, which is the function accessible experimentally, is dependent upon the spatial, spin, and momentum correlations in the ground state. Our purpose in the present section is to determine how sensitive $T(q,\theta)$ (i.e., its q dependence) is to the differences between various nuclear models for the ground state.

We begin with the shell model, for which the obvious choice of nucleus is O^{16}, where there is no ambiguity about the coupling scheme. It has the additional advantage that it is one of the p-shell nuclei, for which the harmonic oscillator shell model is known to fit the elastic scattering form factor well[2]; for the parameter α in the Gaussian factor $\exp(-\frac{1}{2}r^2\alpha^2)$ of the oscillator wave function, the elastic data (corrected for the finite proton size) give $\alpha=0.6$ f^{-1}. In order to evaluate Eq. (21) we must also have values for $\bar{\omega}$ and $\langle\omega^2\rangle_{\rm av}$. Since these terms contribute significantly only for q large enough so that the discrete part of the spectrum has vanished, we use $\bar{\omega}=q^2/2M$ and $\langle\omega^2\rangle_{\rm av}=1.5\bar{\omega}^2$, which are the values given by the quasi-elastic scattering spectrum Eq. (16).

The evaluation of Eq. (21) for the Slater-determinant function corresponding to doubly-filled s and p shells, with oscillator functions, then gives

$$\begin{aligned}T(q,\theta)=1&+(q^2/4M^2)\{-2+(\mu_p^2+\mu_n^2)\\&\times[2\tan^2(\theta/2)+1-3q^2/8M^2]-(1+2K)\}\\&+(3\alpha^2/4M^2)[2\tan^2(\theta/2)+1]+3q^4/32M^4\\&+8(1-\tfrac{1}{4}x)^2e^{-x}(1-e^{x/16})\\&-(1+\tfrac{1}{4}x^2)e^{-x}-(q^2/8M^2)e^{-x}\{\tfrac{1}{2}(\mu_p^2+\mu_n^2)\\&\times[2\tan^2(\theta/2)+1-3q^2/8M^2](1+\tfrac{1}{4}x^2)\\&+[14-8x+\tfrac{1}{2}x^2-K(4+x^2)]\\&+[1+(1/x)][2\tan^2(\theta/2)+1]-3q^2/8M^2\},\end{aligned}\quad (22)$$

with $x=q^2/2\alpha^2$; the terms have been written in the same order as in Eq. (22).

It is interesting to compare this shell-model result, which contains the effect of the Pauli principle, with the corresponding result for a "classical perfect gas" model of the nucleus. In this model the off-diagonal ($j\neq k$) current terms all vanish, and for the $Q^\dagger Q$ terms [see Eq. (12)] we use

$$P_2(\mathbf{r}_1,\mathbf{r}_2)=P_1(\mathbf{r}_1)P_1(\mathbf{r}_2), \quad (23)$$

P_2 and P_1 being the two-particle and one-particle probability densities, respectively.

Using the same harmonic oscillator wave functions, the off-diagonal terms of $T(q,\theta)$ for O^{16} are

$$8(1-\tfrac{1}{4}x)^2(1-e^{x/16})e^{-x}-(1-\tfrac{1}{4}x)^2e^{-x}(1+7q^2/4M^2);\quad (24)$$

the diagonal terms are of course the same as in Eq. (22).

The difference between the inelastic scattering given by these two models is shown in Fig. 2, where the q dependence of Eqs. (22) and (24) is shown for the case $\theta=90°$. T_{SM} (total shell-model scattering) is the $T(q,90°)$ function of Eq. (22); for comparison we have

[15] The only place the effects of the Gartenhaus-Schwartz transformation survive are in the terms containing momentum operators. Since these are found for the shell-model function used here to be small, we have dropped these A^{-1} corrections altogether.

[16] The final term, $-2Z(q^2/4M^2)$, does not appear in the free-nucleon cross section, Eq. (A9). It *does* appear in the square of the free-proton matrix element, and is cancelled by a term from the phase-space factor. As explained in reference 13), doing the sum at constant q instead of constant θ produces a different phase-space factor, and the cancellation no longer occurs.

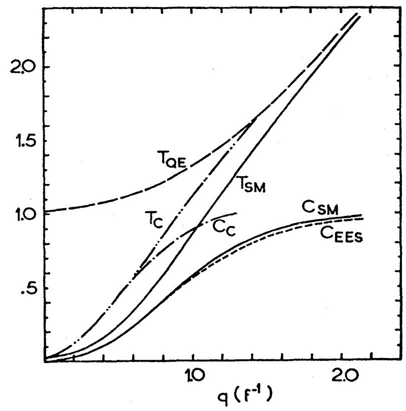

FIG. 2. Calculations of the constant q and θ inelastic scattering sum rule for various models of O^{16}, at $\theta=90°$. T_C is the total scattering (i.e., both Coulomb and current) given by the classical perfect gas model (shell model without antisymmetrization), and C_C is its Coulomb part. T_{SM} and C_{SM} are the same functions for the harmonic oscillator shell model, and C_{EES} is the Coulomb scattering computed from the two-nucleon correlation function of Eden, Emery, and Sampanthar, which includes the effect of a hard core of radius 0.4 f^{-1}. Lack of sufficient information on the ground-state wave function prevents a detailed calculation of the hard-core effect on T, but it is estimated not to exceed $C_{SM}-C_{\text{EES}}$. The Coulomb scattering curves approach 1 asymptotically, while the T curves approach T_{QE} (quasi-elastic), which is given by the "diagonal" terms of Eq. (21).

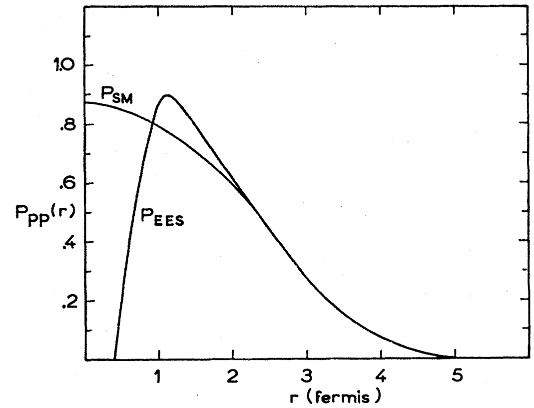

FIG. 3. The two-nucleon correlation function calculated for O^{16} on the basis of the Gammel-Thaler force (hard-core radius 0.4 f) by Eden, Emery, and Sampanthar. The corresponding shell-model curve is included for comparison; the rapid convergence of the two curves is what Gomez, Walecka, and Weisskopf refer to as a short healing distance.

also plotted C_{SM}, the Coulomb scattering part of this function, which shows clearly that the current interactions play a significant role over this q range. T_C is the total scattering for the classical perfect gas model, C_C is its Coulomb part, and the curve labeled T_{QE} (total quasi-elastic scattering) is given by the diagonal terms of Eq. (22). $T(q,\theta)$ converges to this curve at large q, where the correlation effects disappear.

The fact that $T_{SM}<T_C$ shows the inhibiting influence of the Pauli principle on the inelastic scattering. T_C-T_{SM} has a broad maximum peaked at about $q=0.9$ f^{-1}, where it is about a 30% effect. In this sense we can say that the "Pauli correlations" have a substantial influence on the inelastic scattering.[17]

This calculation also shows that, even though the "current" contributions (including the Foldy-Darwin term) to the diagonal or quasi-elastic terms are large, the off-diagonal "current" terms are less than 10% of the off-diagonal Coulomb terms over this range of q. In other words the correlation effects for these models are predominantly spatial correlation effects and are mainly in the Coulomb scattering for $q<2$ f^{-1}. In terms of the figures, we have $T_{QE}-T_{SM}\cong 1-C_{SM}$. In addition it is found that the contributions of the $\bar{\omega}$ and $\langle\omega^2\rangle_{\text{av}}$ terms, which increase with increasing q, are together only about 6% of $T(q,90°)$ at $q=2$ f^{-1} for these models.

One of the most interesting correlation effects to look for is that due to the supposed hard-core part of the nucleon-nucleon interaction. Its effect on the Coulomb scattering can be estimated with some confidence on the basis of the work of Eden, Emery, and Sampanthar[18] (EES), who have calculated for O^{16} the two-particle correlation function

$$P_2(r)=\int P_2(\mathbf{R}+\tfrac{1}{2}\mathbf{r},\ \mathbf{R}-\tfrac{1}{2}\mathbf{r})d^3\mathbf{R}, \tag{25}$$

using a Brueckner-type approximation and a Gammel-Thaler force with hard-core radius 0.4 f. Their result is reproduced in Fig. 3, where it is compared with the corresponding function for the oscillator shell model. The rapid convergence of these functions to each other is what Gomez *et al.*[19] refer to as a short "healing distance."

The Coulomb contribution to the sum rule $C(q)$ is determined by the Fourier transform of $P_2(r)$, according to Eq. (12). The result for the EES correlation function, $C_{\text{EES}}(q)$, is readily calculable, and is included in Fig. 2. Remarkably enough, it does not deviate from $C_{SM}(q)$ by more than 5% at any value of q, from which we conclude that the effect of hard cores (of this size) on the Coulomb scattering is extremely small.[20] One way of understanding this is to note that

$$\int [C_{SM}(q)-C_{\text{EES}}(q)]d^3q=(2\pi)^3(Z-1)P_{SM}(0),$$

[17] This agrees with the result found by Drummond,[7] for large nuclei, but distinctly contradicts the estimate of Fowler and Watson,[6] that the Pauli correlation effect should be negligible for $A=16$.

[18] R. J. Eden, J. Emery, and S. Sampanthar, Proc. Roy. Soc. (London) **A253**, 186 (1959).

[19] L. C. Gomez, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. **3**, 241 (1958).

[20] This contradicts Fowler and Watson, reference 6, who seem to find a large effect attributable to hard cores. We believe this to be due in part to their neglect of the Pauli correlation effect, so that they in effect attributed the entire difference C_C-C_{EES} to hard-core correlations. Since $C_C-C_{\text{EES}}\approx C_C-C_{SM}\approx T_C-T_{SM}$ from Fig. 2, this means that most of what they called hard-core effect was probably due to Pauli correlations.

i.e., the momentum space volume integral of the difference is a fixed number, independent of the core radius. Decreasing the core radius r_c has the effect of spreading this function $\Delta C(q)$ over a larger range of q, thus decreasing its magnitude. At $r_c=0.4$ f, $\Delta C_{\max}<0.05$, but since it is readily seen to depend on r_c^3 (actually on the ratio of the core volume to the volume per particle), it would become significant if r_c were much larger.

Since the off-diagonal magnetic terms of Eq. (21) also depend on the Fourier transform of a P_2-like function, we estimate that they, too, will be nearly unaffected by hard-core correlations. The $\mathbf{p}_j\cdot\mathbf{p}_k$ terms, which depend on the derivative of the wave function, are less easy to estimate. The diagonal derivative term of Eq. (21),

$$T_{p^2}=(\langle p^2\rangle/3M^2)[2\tan^2(\theta/2)+1],$$

can be estimated from what is known experimentally about the one-nucleon momentum distribution in light nuclei.[14] This information gives $\langle p^2\rangle/M^2\approx 0.054$; since $\langle p^2\rangle/M^2=9\alpha^2/4M^2=0.036$ for the shell-model function of Eq. (22), $\Delta T_{p^2}=0.018[2\tan^2(\theta/2)+1]/3$ gives the deviation of this term from the shell-model prediction.

The hard-core effect on the $\mathbf{p}_j\cdot\mathbf{p}_k$ terms can be evaluated only on the basis of a specific model. One such is given by a Jastrow-type ground-state wave function, $\psi_0=\prod_{j>k}f(r_{jk})\varphi_{SM}$, with φ_{SM} the shell-model function used above; $f^2(r)$ is approximately the ratio $P_{\mathrm{EES}}(r)/P_{SM}(r)$ of the functions of Fig. 3, and so rises from zero at $r=0.4$ f to 1 at $r\cong 1$ f. Within the binary collision or "independent pair" approximation, one finds

$$\langle\psi_0|\mathbf{p}_1\cdot\mathbf{p}_2|\psi_0\rangle\approx-\langle f'(r_{12})\varphi_{SM}|f'(r_{12})\varphi_{SM}\rangle+\langle\varphi_{SM}|\mathbf{p}_1\cdot\mathbf{p}_2|\varphi_{SM}\rangle,$$

in terms of which the off-diagonal $\mathbf{p}_k\cdot\mathbf{p}_j$ contribution to Eq. (21) at $q=0$ is given by

$$T_{\mathbf{p}_k\cdot\mathbf{p}_j}=(Z-1)(\langle\psi_0|\mathbf{p}_1\cdot\mathbf{p}_2|\psi_0\rangle/3M^2)[2\tan^2(\theta/2)+1].$$

Evaluating this with the above choice of $f(r)$, we summarize in Table I the diagonal as well as off-diagonal contributions of the derivative terms to T at $\theta=90°$ and $q=0$, for the shell model and for this hard-core model. Cancellations appear to make the total contribution to T for this hard-core model very nearly equal to that of the shell model. Whether or not this cancellation is to be taken seriously, it seems clear that the net effect of hard cores on the derivative terms of T is certainly less than, say, 0.05 at $q=0$ and $\theta=90°$.

Since it can also be inferred from the binary collision approximation that this effect does not depend strongly on q, our conclusion is that the net effect of hard-core correlations on T is of the same order of magnitude as the difference $(C_{\mathrm{EES}}-C_{SM})$ indicated in Fig. 2, and consequently is probably too small to be observable.

Finally it should be noted that the long-range pair correlations associated with collective motion to be expected in deformed nuclei have been neglected in the models considered here. It would seem quite possible for their effect on the sum rule to be substantial at small q (say, $q<0.5$ f^{-1}), but it should decrease at larger q's, and in any case ought to be at a minimum in doubly-magic nuclei.

TABLE I. The estimated contribution of the derivative (or current interaction) terms to T for two models. The diagonal, off-diagonal, and total contributions are listed. The relation of T to the differential cross section is given in Eq. (20).

	T_{p^2}	$T_{p_j\cdot p_k}$	Total
Shell model	+0.036	−0.012	+0.024
Hard-core model	+0.054	−0.037	+0.017

VI. CONCLUSION

Via the constant-q and -θ sum rule, inelastic electron scattering provides one of the most direct means of access to two-nucleon correlation functions. For small q and moderate θ (e.g., $q<1$ f^{-1} at $\theta=90°$), where Coulomb scattering predominates, the sum can be done at constant q and k_1, and is sensitive mainly to proton-proton correlations. At larger q and θ the current interactions predominate; the sum must then be done at constant q and θ, and depends on pp, nn, and np correlations.

Although one cannot hope to extract the correlation functions directly from the experimental data, specific nuclear models can be tested by the use of Eq. (21). For the specific case of O^{16}, it is found that Pauli correlations depress the summed scattering appreciably for 0.5 f$^{-1}<q<1.5$ f^{-1}. However, the effect of a hard-core force of radius $r_c\lesssim 0.4$ f^{-1} appears to alter it by only a few percent for all q.

APPENDIX A

Derivation of the Interaction Hamiltonian for Nonrelativistic Nucleons

The fact that spatial correlations of the nucleons play an important role in the problem at hand suggests that a coordinate space treatment of the scattering will be more advantageous than the usual momentum space diagram technique. For Born approximation scattering this is called the Møller-potential approach,[21] and requires that we first find the nonrelativistic interaction of the nucleon with an arbitrary external field. This seems to be most readily accomplished, to a given order in inverse powers of the nucleon mass, by means of the Foldy-Wouthuysen transformation scheme.[22]

We start from the generalized Dirac equation for a spin-$\frac{1}{2}$ particle in a given electromagnetic field, including coupling terms only to first order in the field,[23]

$$(\gamma_\mu\partial_\mu+M)\psi-i\{(\sum_{n=0}^{\infty}\epsilon_n\Box^n A_\mu)\gamma_\mu+i(\sum_{n=0}^{\infty}\mu_n\Box^n\partial_\nu A_\mu)\sigma_{\mu\nu}\}\psi=0.\quad\text{(A1)}$$

[21] C. Møller, Z. Physik **70**, 786 (1931).
[22] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[23] L. L. Foldy, Phys. Rev. **87**, 688 (1952).

The notation being used is $x_\mu=(x,it)$ for 4-vectors, $\partial_\mu=\partial/\partial x_\mu$, $\Box=\partial_\mu\partial_\mu$, $\gamma_\mu=(-i\beta\boldsymbol{\alpha},\beta)$, and $\sigma_{\mu\nu}=(-i/2)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. The field $A_\mu(x_\nu)$ is to be evaluated at the same argument as ψ. ϵ_n and μ_n are the unspecified constants which determine the interaction; ϵ_0 and μ_0 are of course the charge and anomalous magnetic moment of the particle, its total magnetic moment being $\{(e/2M)+\mu_0\}\boldsymbol{\sigma}$, with $\sigma_x=\sigma_{23}$, etc. We shall also define K by $\mu_0=(e/2M)K$. Define the differential operators

$$eF_1(\Box)=\sum\epsilon_n\Box^n,\quad \mu_0F_2(\Box)=\sum\mu_n\Box^n. \qquad (A2)$$

In momentum variables q they are the familiar electromagnetic form factors $F_1(q_\mu^2)$ and $F_2(q_\mu^2)$ of the nucleon, normalized to $F_1(0)=F_2(0)=1$ for the proton and $F_1(0)=0$, $F_2(0)=1$ for the neutron. Equation (A2) then becomes

$$(\gamma_\mu\partial_\mu+M)\psi-ie[(F_1A_\mu)\gamma_\mu+(iK/2M)(F_2\partial_\nu A_\mu)\sigma_{\mu\nu}]\psi=0. \qquad (A3)$$

Following Foldy,[23] we apply the Foldy-Wouthuysen transformation[22] to Eq. (A3) to obtain the corresponding two-component, positive-energy equation for the nucleon. We retain only terms through order M^{-2}, and in their selection we consider K to be independent of M, i.e., μ_0 to be of order M^{-1}. We further neglect all terms of order higher than 1 in the electromagnetic potentials. The result is

$$\frac{p^2}{2M}\Psi+e(F_1\varphi)\Psi-\frac{e}{2M}[\mathbf{p}\cdot(F_1\mathbf{A})+(F_1\mathbf{A})\cdot\mathbf{p}]\Psi-\frac{e}{2M}[\boldsymbol{\sigma}\cdot(F_4\mathbf{H})]\Psi+\frac{e}{8M^2}\{\boldsymbol{\sigma}\cdot[\mathbf{p}\times(F_3\mathbf{E})-(F_3\mathbf{E})\times\mathbf{p}]\}\Psi-\frac{e}{8M^2}\operatorname{div}(F_3\mathbf{E})\Psi=i\frac{\partial\Psi}{\partial t}, \qquad (A4)$$

with the following notation,

$$\varphi=-iA_4,\quad \mathbf{A}=(A_1,A_2,A_3),\quad \mathbf{E}=-\operatorname{grad}\varphi-\partial\mathbf{A}/\partial t,\quad \mathbf{H}=\operatorname{curl}\mathbf{A},\quad F_3=F_1+2KF_2,\quad F_4=F_1+KF_2. \qquad (A5)$$

Ψ is the two-component nucleon wave function and $\boldsymbol{\sigma}$ is the Pauli spin operator.[24] The electromagnetic potentials and fields in (A4) must be taken at the same arguments as Ψ.

To apply this Hamiltonian to the calculation of electron-nucleus scattering, we use the semiclassical Møller-potential method,[21] according to which the $\mathbf{A}$ and A_4 of Eq. (A5) are the fields generated by the "transition current" $(j_\mu)_{21}$ of the passing electron. In noncovariant notation this current is

$$\rho_{21}(\mathbf{r},t)=eu_2^\dagger u_1e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)},\quad \mathbf{j}_{21}(\mathbf{r},t)=eu_2^\dagger\boldsymbol{\alpha}u_1e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}, \qquad (A6)$$

the u's being free-electron (or muon) spinors, and $\mathbf{q}$ and ω the momentum and energy transfers defined in Eq. (2) of the main text.

In the Lorentz gauge being employed, $\Box A_\mu=-4\pi j_\mu$, so the fields generated by the current (A6) are

$$\varphi(\mathbf{r},t)=\frac{4\pi eu_2^\dagger u_1}{q^2-\omega^2}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)},\quad \mathbf{A}(\mathbf{r},t)=\frac{4\pi eu_2^\dagger\boldsymbol{\alpha}u_1}{q^2-\omega^2}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}. \qquad (A7)$$

Substituting them into Eq. (A4) we obtain our desired result, the Born-approximation electron-nucleon interaction Hamiltonian in the nucleon space, correct through order M^{-2},

$$H'=\frac{4\pi e^2}{q_\mu^2}\Big\langle u_2\Big|F_1e^{iq_\mu x_\mu}-\frac{F_1}{2M}(\mathbf{p}\cdot\boldsymbol{\alpha}e^{iq_\mu x_\mu}+e^{iq_\mu x_\mu}\mathbf{p}\cdot\boldsymbol{\alpha})-\frac{(F_1+KF_2)}{2M}i\boldsymbol{\sigma}\cdot(\mathbf{q}\times\boldsymbol{\alpha})e^{iq_\mu x_\mu}-\frac{q_\mu^2}{8M^2}(F_1+2KF_2)e^{iq_\mu x_\mu}+\frac{(F_1+2KF_2)}{8M^2}i\boldsymbol{\sigma}\cdot\{\mathbf{p}\times(\omega\boldsymbol{\alpha}-\mathbf{q})e^{iq_\mu x_\mu}-e^{iq_\mu x_\mu}(\omega\boldsymbol{\alpha}-\mathbf{q})\times\mathbf{p}\}\Big|u_1\Big\rangle, \qquad (A8)$$

in which F_1 and F_2 are to be taken as the series (A2) with $\Box$ replaced by $q_\mu^2=q^2-\omega^2$.

The simplest application of this Hamiltonian is to the calculation of the scattering of electrons on free "Foldy-Wouthuysen" particles. The cross section obtained, correct through order q^2/M^2, can be written in the form[25]

$$d\sigma/d\Omega=\sigma'\{F_1^2+(q^2/4M^2)\times[2(F_1+KF_2)^2\tan^2(\theta/2)+K^2F_2^2]\}, \qquad (A9)$$

with

$$\sigma'=\left(\frac{e}{2k}\right)^2\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\left[1+\frac{2k_1}{M}\sin^2(\theta/2)\right]^{-1}. \qquad (A10)$$

(θ is the electron's scattering angle.)

For comparison, if the Møller-potentials Eq. (A7) are substituted into the equation for a relativistic nucleon Eq. (A3), and the analogous Born-approximation cross

[24] For comparison of our formulas with those of Foldy,[23] we mention that in Eq. (6) of Foldy's paper a factor of $\frac{1}{2}$ is missing in front of each μ_n. The equations of Foldy and Wouthuysen,[22] contain a large number of misprints.

[25] Kinematic relations useful in obtaining the result in this form are: $q^2=2M\omega$; $\mathbf{q}\cdot(\hat{k}_1+\hat{k}_2)=\omega(1+\cos\theta)$; and $(\mathbf{q}\cdot\hat{k}_1)(\mathbf{q}\cdot\hat{k}_2)=q^2\cos\theta-q_\mu^2\cos^2(\theta/2)$. Also, since $\omega=q^2/2M$, we have set $q_\mu^2/M^2=q^2/M^2$ in the Darwin-Foldy term.

section computed, one obtains the generalized Rosenbluth expression[26]

$$d\sigma/d\Omega=\sigma'\{F_1^2+(q_\mu^2/4M^2)\times[2(F_1+KF_2)^2\tan^2(\theta/2)+K^2F_2^2]\}. \quad \text{(A11)}$$

To obtain the approximation appropriate to a nonrelativistic nucleon, we note that, for a given θ, the nucleon's recoil velocity $v=q/M$, appears explicitly only in q_μ^2/M^2, which to second order in v is q^2/M^2, since $\omega=q^2/2M$. Hence (A11) and (A9) agree through second order, as they should. In contrast, we note that, e.g., the diagonal terms of Eq. (19) of reference 5, which should in the same way agree through second order with Eq. (A11), fail to do so because the Darwin-Foldy term was neglected in the interaction.[27]

[26] M. N. Rosenbluth, Phys. Rev. **79**, 619 (1950).

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[27] The neglect of the Darwin-Foldy term in other calculations by the Stanford group has been pointed out by L. Durand, III, Phys. Rev. **115**, 1020 (1959).

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Non-Abelian Gauge Fields. Commutation Relations*

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The question is raised for non-Abelian vector gauge fields whether gauge invariance necessarily implies a massless physical particle. As a preliminary to studying this problem, the action principle is used to discover the independent dynamical variables of such gauge fields and construct their commutation relations.

INTRODUCTION

It is well known that gauge invariance intimately ties the electromagnetic field $A_\mu(x)$, $F_{\mu\nu}(x)$ to the set of all fields $\chi(x)$ that bear electrical charge. This internal property is described by a finite imaginary Hermitian matrix q with integer eigenvalues. A gauge transformation involves an arbitrary numerical function $\lambda(x)$. It is a linear homogeneous transformation for the charged fields $\chi(x)$, but an inhomogeneous one for the gauge field $A_\mu(x)$,

$$\chi(x)\rightarrow e^{iq\lambda(x)}\chi(x), \quad A_\mu(x)\rightarrow A_\mu(x)+\partial_\mu\lambda(x),$$
$$F_{\mu\nu}(x)\rightarrow F_{\mu\nu}(x).$$

Such transformations form an Abelian group, in which the gauge function,

$$\lambda(x)=\lambda^{(1)}(x)+\lambda^{(2)}(x),$$

describes the superposition of two individual transformations. The integer spectrum of charge is related to the compact structure of this group, which has the topology of the circle. Gauge invariance implies that local conservation of charge is not just a consequence of the equations of motion of the charge bearing fields but appears as an identity characteristic of the gauge field differential equations.

In this familiar situation the gauge field does not carry the internal property to which it is coupled. A different example is furnished by the gravitational field, for this couples with energy and momentum, to which all physical systems must contribute. In other respects, however, the requirement of general coordinate invariance is quite analogous to that of gauge invariance. There is an intermediate possibility in which the gauge field is coupled to, and also carries, internal rather than space-time properties. Then the gauge field retains the space-time transformation properties of the electromagnetic field. This is indicated by the tensor notation $\phi_{\mu a}$, $G_{\mu\nu a}$, where the index $a=1\cdots n$ refers to the internal space. For the gravitational field the latter is also a coordinate index, which requires fields of more complicated space-time transformation properties.

The gauge transformations of a field $\chi(x)$ that supports a number of internal properties, as represented by finite linearly independent matrices T_a, $a=1\cdots n$, can generally be stated explicitly only for infinitesimal transformations,

$$\chi(x)\rightarrow[1+i\sum_{a=1}^{n}T_a\delta\lambda_a(x)]\chi(x).$$

If these are to generate a transformation group, two

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