# Some Considerations of Analyticity in the Many-Fermion Problem

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The analytic properties of Feynman amplitudes in the many-fermion problem are investigated by means of techniques recently developed for use in field theory. A prescription is given for calculation of discontinuities across branch cuts. This prescription leads to a reformulation of the theory in terms of quasi-particles; but considerations of unitarity are no longer so simple as they are in field theories dealing with stable particles. These analytic methods turn out to be very useful in electron transport problems.

## I. INTRODUCTION

A NUMBER of elegant techniques have been devised recently by Landau[1] and others[2] for the purpose of analyzing the singularities which occur in Feynman diagrams in field theory. In this paper we shall explore the possibility of applying some of these methods to the diagrams which arise in the many-fermion problem. In fact, the main purpose of this paper is to derive some mathematical results which will be used in a theory of residual resistivity in metals. Inasmuch as these mathematical techniques may be of interest outside the field of electrical transport theory, it seems worthwhile to present this material as a separate paper.

The system to be considered here is an interacting Fermi fluid at zero temperature, possibly in the presence of some static external field due to a lattice, impurities, or the like. The particle-particle interactions are instantaneous, describable by two-body potentials which are functions of the initial and final momenta only. We shall assume that the system is "normal," a term which these days is taken to mean that the analytic properties of various propagation functions are the same as those of each term in a perturbation expansion in powers of the particle-particle interaction strength.

Two important differences between ordinary field theories and the many-fermion problem severely restrict our application of arguments based on analyticity. The first is the absence in our problem of any symmetry principle which might play the role of Lorentz invariance. Even Galilean invariance must be ruled out if we are to consider an electron fluid bound within a fixed solid. The resulting separation between momentum and energy variables already is apparent in our description of the two-body forces. Because we do not want to assume special analytic properties for the two-body potentials, we shall have to content ourselves with analytic continuations only in the energy variables.

The second major difficulty is that there exists no true particle description of the elementary excitations of a Fermi fluid. The best we can do is to talk about quasi-particles, i.e., "clothed" electrons.[3] But a quasi-particle, even when very far away from any other quasi-particles, decays spontaneously, creating new quasi-particle-hole pairs. As a result, the exact eigenstates of the Fermi fluid are quite complicated, and cannot be labeled, even in an asymptotic sense, according to the number of elementary excitations which they contain. This situation will not seriously hinder our application of analytic techniques, but will reduce their utility for us. In field theory, discontinuities across branch cuts usually turn out to be reaction rates for physically observable processes. The analogous terms which arise in the many-body problem will not be interpreted quite so easily.

The saving feature of the normal Fermi fluid is the fact that quasi-particles are very nearly stable when they lie close to the Fermi surface. In particular, the decay rate vanishes as the square of the difference between the energy of the quasi-particle and the chemical potential.[4] It follows that the quasi-particle picture will be quite accurate in situations where the excitation energies are small, that is, when the perturbing fields are nearly static. We shall see that this particle picture plays an important role in this paper and in the work on dc conductivity which follows.

## II. LOCATION OF SINGULARITIES

We consider many-body diagrams containing at least two external lines, and we inquire about the analytic properties of the corresponding Feynman amplitudes as functions of the energy variables associated with these lines. Rules for the evaluation of such diagrams have been given by many authors.[5] We shall find it most economical to adopt Hugenholtz's graphical notation in which no internal interaction lines are drawn, each two-body interaction being indicated by a four-line vertex. Energy must be conserved at these vertices; but we shall make no special assumptions about momentum conservation.

One question which must be dispensed with immediately has to do with the presence of self-energy parts of the internal electron lines. As has been discussed elsewhere,[6] these self-energy parts are related to

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[1] L. D. Landau, Nuclear Phys. **13**, 181 (1959).

[2] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[3] L. D. Landau, Soviet Phys.-JETP, **3**, 920 (1957).

[4] J. M. Luttinger, Phys. Rev. **121**, 942 (1961).

[5] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957); J. Hubbard, *ibid.* **A240**, 539 (1957); N. M. Hugenholtz, Physica **23**, 481 (1957).

[6] J. S. Langer, Phys. Rev. **120**, 714 (1960). In particular, see Sec. V.

the shift of the chemical potential due to the electron-electron interactions. To any finite order in perturbation theory, this shift must be treated by a procedure similar to mass renormalization in field theory. This procedure can best be included in our formalism by the device of calculating with skeleton diagrams. That is, we eliminate all internal self-energy parts from each graph and substitute for each remaining line the complete single-particle propagator $S$. The function $S$ may be said to describe the propagation of a quasi-particle; thus it seems that we are forced to talk about quasi-particles at the very beginning of our analysis.

Of course we do not have an exact expression for the function $S$, but most of its essential features are well known. In particular, we know that it has the spectral representation[7]

$$S_{\mathbf{k},\mathbf{k}'}(\omega)=\lim_{\eta\to 0}\frac{1}{2\pi i}\left\{\int_{\mu}^{\infty}\frac{A_{\mathbf{k},\mathbf{k}'}(x)}{x-\omega-i\eta}dx+\int_{-\infty}^{\mu}\frac{B_{\mathbf{k},\mathbf{k}'}(x)}{x-\omega+i\eta}dx\right\}, \quad (2.1)$$

where $A$ and $B$ are Hermitian matrices in the momentum indices $\mathbf{k}$ and $\mathbf{k}'$, and $\mu$ is the chemical potential. For simplicity of notation, we shift $\mu$ to the origin and write

$$S_{\mathbf{k},\mathbf{k}'}(\omega)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathfrak{a}_{\mathbf{k},\mathbf{k}'}(\xi)}{\xi-\omega}d\xi, \quad (2.2)$$

with the understanding that the physical values of $S$ are to be found along a contour in the $\omega$ plane which starts at $-\infty$ in the third quadrant, passes through the origin, and goes to $+\infty$ in the first quadrant. The function $\mathfrak{a}(\xi)$ is peaked near some value of $\xi$ depending on the momentum subscripts. The width of this peak is a measure of the decay rate of the quasi-particle.

To start our analysis of any Feynman graph $G$, let us choose pure imaginary values of all the external energy variables. Then we can write

$$F_G=\int_{-\infty}^{\infty}d\xi_1\cdots\int_{-\infty}^{\infty}d\xi_L\, f(\xi_1,\cdots\xi_L)\times\int_{-i\infty}^{i\infty}d\omega_1\cdots\int_{-i\infty}^{i\infty}d\omega_l\frac{1}{(\xi_1-\Omega_1)\cdots(\xi_L-\Omega_L)}, \quad (2.3)$$

where the function $f$ contains all the factors $\mathfrak{a}$, the interactions, and the momentum sums. $L$ is the number of internal lines in $G$. The $\Omega_i$ are linear combinations of the external energies $\nu_i$ and the $l$ different $\omega$'s associated with independent closed loops. The coefficients of the $\nu$'s and $\omega$'s occurring in the $\Omega$'s are all $\pm 1$ or 0. Thus, for pure imaginary $\nu$'s, we may distort the contours for the $\omega$ integrations to run along the imaginary axis as indicated in Eq. (2.3).

If we hold the $\xi$'s fixed and perform the $\omega$ integrations in (2.3), we find the usual product of energy denominators which occurs in time-independent perturbation theory. The $\xi$'s play the role of the single-particle energies, positive values corresponding to particles and negative values to holes. In some region in the space of imaginary $\nu$'s, these denominators all will be non-vanishing; but they will give rise to poles when the $\nu$'s are continued away from the imaginary axis. These poles then turn into branch cuts after integration over the $\xi$'s.

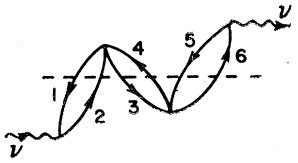


FIG. 1. A vacuum-polarization graph which gives rise to overlapping branch cuts.

It is well known that not all the energy denominators associated with a particular time-ordered graph will be retained as poles in $F_G$. For example, the graph drawn in Fig. 1 would contain a denominator $(\xi_2-\xi_1+\xi_4-\xi_3+\xi_6-\xi_5-\nu)$ arising from the intermediate state indicated by the dashed line. But if we sum over all time orders that can be achieved without changing the particle or hole character of any line, i.e., retain the signs of all the $\xi$'s, then we find only the denominators $(\xi_2-\xi_1-\nu)$, $(\xi_4-\xi_3+\nu)$, and $(\xi_6-\xi_5-\nu)$. In (2.3), if we choose the signs of all the $\xi$'s and then perform the $\omega$ integrations, we automatically perform this limited sum over time orders. In general, it is not difficult to pick out those energy denominators in a graph which will remain as singularities in $F$. Such an energy denominator must be associated with an intermediate state which has the property that, if all the lines occurring in the state were broken, then the entire graph would separate into just two pieces. Each of these pieces must be connected and must contain at least one of the external vertices. Although this prescription may seem obvious, it may be well to outline a proof of it via Landau's "reduced graphs."[1] In any case, we shall want to use the reduced graph notation in later applications.

We consider now only the $\omega$ integration in (2.3) for some fixed set of $\xi$'s. If we multiply and divide the integrand by the product of factors $(\xi_i+\Omega_i)$, we find in the denominator a product of positive-definite quantities. Remember that we have chosen pure imaginary $\nu$'s; and we can, for the moment, say that none of the $\xi$'s are zero. We next use Feynman's trick for combining these denominators, arriving at an expression of the form

$$(L-1)!\int_{-i\infty}^{i\infty}d\omega_1\cdots\int_{-i\infty}^{i\infty}d\omega_l\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_L\times\frac{\prod_{i=1}^{L}(\xi_i+\Omega_i)\delta(\sum_{i=1}^{L}\alpha_i-1)}{[\sum_{i=1}^{L}\alpha_i(\xi_i^2-\Omega_i^2)]^L}. \quad (2.4)$$

[7] V. M. Galitskii and A. B. Migdal, Zhur. Eksptl' i Teoret. Fiz. 34, 189 (1958) [translation: Soviet Phys.-JETP 34(7), 96 (1958)].

The denominator in (2.4) is a quadratic form in the $\omega$'s and can be diagonalized by a suitable translation and rotation in $\omega$ space. That is, $\omega_i \rightarrow \tilde{\omega}_i$ so that

$$D \equiv \sum_{i=1}^{L} \alpha_i(\xi_i^2-\Omega_i^2) \rightarrow -\sum_{i=1}^{l} A_i\tilde{\omega}_i^2 + \varphi(\alpha_1,\cdots,\nu_1,\cdots,\xi_1,\cdots), \quad (2.5)$$

where

$$\varphi = \min_{(\omega)} D. \quad (2.6)$$

The integrations over the $\tilde{\omega}_i$ can be performed without difficulty, leaving some power of $\varphi$ (at most $L-\frac{1}{2}l$) in the denominator. The integral can have no singularity unless $\varphi$ vanishes for some values of the $\alpha$'s. It follows that, when we allow the $\nu$'s to move away from the imaginary axis, the first singularity will develop when

$$\min_{(\alpha)} \varphi = 0. \quad (2.7)$$

Equation (2.7) is satisfied[1] when either

$$\xi_i-\Omega_i=0 \quad \text{or} \quad \alpha_i=0 \quad (2.8)$$

for each $i$, where the $\Omega_i$ are to be evaluated for that choice of the $\omega_i$ which minimizes $D$. This last condition leads to Landau's second set of equations:

$$\sum_i{}^{(l)} \alpha_i(\pm)_i\Omega_i = \sum_i{}^{(l)} \alpha_i(\pm)_i\xi_i = 0 \quad (2.9)$$

for each closed loop in the diagram. The signs $(\pm)_i$ are determined by whether the sense of the loop is the same as or is opposed to the direction of the $i$th line. Cutkosky[2] has pointed out that Eqs. (2.8) and (2.9) are both necessary and sufficient for singularities in $F$. His argument is applicable here, but only before integration over the $\xi$'s.

By definition, the reduced diagram is obtained by deleting from the original graph all those lines for which $\alpha_i=0$ and fusing the vertices connected by these lines. In this process we also may delete all lines which close upon themselves, and we need not consider reduced diagrams in which all the external vertices coalesce at the same point. The simplest reduced diagram associated with any graph consists of only two external vertices simply connected by a number of electron lines. All more complicated reduced graphs may be further reduced to a number of the simple graphs. The Landau equations (2.8) in such cases may be interpreted as telling us that several simple singularities may occur simultaneously. The electron lines remaining in a simple reduced graph comprise an intermediate state whose energy denominator gives rise to a singularity in $F$. This completes the proof of the rule for selection of such intermediate states.

To illustrate these techniques, let us consider the vertex diagram drawn in Fig. 2 and the associated reduced graphs shown in Fig. 3. On applying Eq. (2.8),

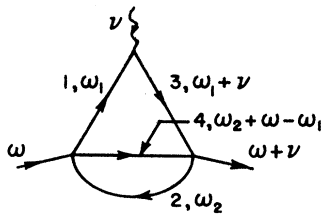


FIG. 2. A typical vertex diagram.

we find

$$\begin{aligned} &[\text{Fig. 3(a)}], && \nu=\xi_3-\xi_1; \\ &[\text{Fig. 3(b)}], && \omega+\nu=\xi_3+\xi_4-\xi_2; \\ &[\text{Fig. 3(c)}], && \omega=\xi_1+\xi_4-\xi_2. \end{aligned} \quad (2.10)$$

For Fig. 3(b), for example, Eqs. (2.9) are

$$\begin{aligned} [\text{Fig. 3(b)}], \quad & \alpha_3\xi_3-\alpha_4\xi_4=0; \\ & \alpha_4\xi_4+\alpha_2\xi_2=0. \end{aligned} \quad (2.11)$$

Since the $\alpha$'s are positive, these equations imply that $\xi_3$ and $\xi_4$ must be of the same sign and that $\xi_2$ must have the opposite sign.

## III. DISCONTINUITIES ACROSS BRANCH CUTS

The singularities discussed above are simple poles which generate branch cuts when the integrations over the $\xi$'s are carried out. We turn now to the problem of calculating the discontinuity across any one of these branch cuts.

The first step is to isolate each pole and to compute its residue. Let us confine our attention to a single simple reduced graph and look for the associated singularity in the complex plane of one of the external energies, say $\nu_1$, all other $\nu$'s being fixed. The reduced graph consists solely of a certain number, say $m$, of independent two-line loops. We may choose the $m$ $\omega$'s associated with these loops in such a way that $\Omega_1=\omega_1$, $\Omega_2$ is a linear combination of only $\omega_1$ and $\omega_2$, $\Omega_3$ a combination of $\omega_2$ and $\omega_3$, etc. That is, each $\omega$ occurs in only two lines of the reduced graph. Finally, we may arrange that $\nu_1$ occurs only in $\Omega_m$. Thus we consider a multiple integral of the form

$$I = \int_{-i\infty}^{i\infty} d\omega_1 \cdots \int_{-i\infty}^{i\infty} d\omega_m \frac{I_m(\omega_1,\cdots,\omega_m,\nu_1,\cdots)}{(\xi_1-\Omega_1)\cdots(\xi_m-\Omega_m)}, \quad (3.1)$$

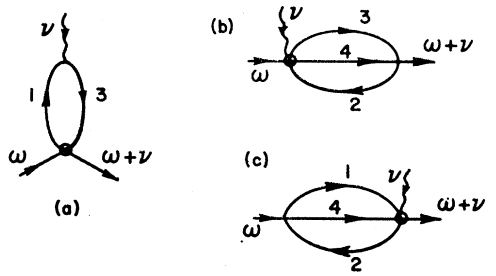


FIG. 3. The simple reduced graphs associated with the vertex diagram shown in Fig. 2.

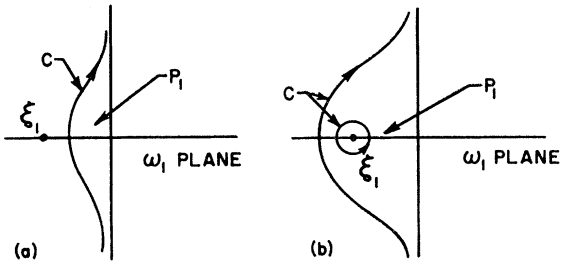


FIG. 4. Illustration of the pinch singularity occurring in Eq. (3.2).

where $I_m$ contains the remaining $\omega$ integrations. For the moment, we choose all the $\xi$'s so that no two simple singularities of the complete graph coincide in the $\nu_1$ plane.

Now suppose that all the $\omega$ integrations have been performed except that over $\omega_1$, i.e.,

$$I=\int_{-i\infty}^{i\infty} d\omega_1 \frac{1}{\xi_1-\omega_1} I_1(\omega_1, \nu_1, \cdots). \tag{3.2}$$

The relevant singularity arises when the $\omega_1$ contour is pinched between the pole at $\omega_1=\xi_1$ and that pole $P_1$ in $I_1$ associated with the singularity at $\Omega_2(\omega_1)=\xi_2$. This situation is illustrated in Fig. 4. $P_1$ moves toward $\xi_1$ as we continue $\nu_1$ away from the imaginary axis. The singular part of $I$ is obtained by distorting the contour $C$ as shown in Fig. 4(b) and retaining only the small circle about $\xi_1$. In a similar fashion, the pole $P_1$ is associated with a pinch in the $\omega_2$ plane, and we need retain only that part of the $\omega_2$ contour which consists of a small circle about the pole at $\Omega_2=\xi_2$. After this process has been repeated $m$ times, we find for the singular part of $I$

$$I_s=(\pm)(2\pi i)^m \frac{I_m(\omega_1', \cdots, \omega_m', \nu_1, \cdots)}{\xi_m-\Omega_m(\omega_m', \nu_1, \cdots)}. \tag{3.3}$$

Here the $\omega_i'$ are those values of the $\omega$'s which satisfy the Landau equations (2.8). The sign $(\pm)$ is determined by the sense in which we have integrated around each of the poles in carrying out the above procedure. Equations (2.9) must be satisfied in order that all these pinches may occur, i.e., that each pair of poles straddles the imaginary axis before analytic continuations in the $\nu$'s are performed. Because $\nu_1$ occurs linearly in $\Omega_m$ with coefficient $\pm 1$, Eq. (3.3) implies that $I$ contains a simple pole at the location determined by Eq. (2.8) and with a residue $\pm(2\pi i)^m I_m$.

We now must consider the possibility that one or more of the singularities of $I_m$ (associated with other reduced graphs) coincides with the pole found in Eq. (3.). In general we may avoid this possibility entirely by assigning to each $\xi_i$ a small imaginary part $\eta_i$ which will be allowed to vanish at the very end of the calculation. There is usually no difficulty in choosing the $\eta_i$ in such a way that the poles associated with different reduced graphs are displaced away from each other in the complex plane. We shall see in an example to follow how this displacement may be performed in a particularly convenient manner.

Once the above displacement has been performed, we may be assured that the explicit pole in (3.3) generates a well defined branch cut when we integrate over the $\xi$'s. According to the above derivation, the discontinuity across this cut may be computed by substituting for each of the lines in the reduced graph a factor of either

$$\int_0^\infty d\xi_i\, \mathfrak{A}(\xi_i)\delta(\xi_i-i\eta_i-\Omega_i)$$

or (3.4)

$$\int_{-\infty}^0 d\xi_i\, \mathfrak{A}(\xi_i)\delta(\xi_i-i\eta_i-\Omega_i),$$

depending upon whether Eqs. (2.9) require that $\xi_i$ be positive or negative, i.e., particle-like or hole-like. The rest of the diagram then is computed in the usual way, any singularities which arise being defined by the displacements $\eta_i$.

The technique of separating overlapping branch cuts by means of the displacements $\eta_i$ deserves examination in greater detail. Let us consider as an example the set of vacuum-polarization graphs of the form drawn in Fig. 1. The only simple reduced graph consists of just a single loop, and all more complicated reduced graphs merely indicate that the poles associated with various simple graphs may occur at the same place in the $\nu$ plane. Any one of these loops contributes a factor

$$F^{(1)}(\nu)=\int_0^\infty \rho(x)\left\{\frac{1}{x-\nu}+\frac{1}{x+\nu}\right\}dx, \tag{3.5}$$

where $x$ has the form $\xi_2-\xi_1$ in our previous notation, and a single $\omega$ integration has been performed. We omit momentum sums and potentials, and write, for a ladder diagram containing $L$ of these loops,

$$F^{(L)}(\nu)=\lim_{\epsilon_1, \cdots, \epsilon_L \to 0} \prod_{i=1}^{L} F_i^{(1)}(\nu), \tag{3.6}$$

where

$$F_i^{(1)}(\nu)=\int_0^\infty \rho(x)\left\{\frac{1}{x-i\epsilon_i-\nu}+\frac{1}{x-i\epsilon_i+\nu}\right\}dx. \tag{3.7}$$

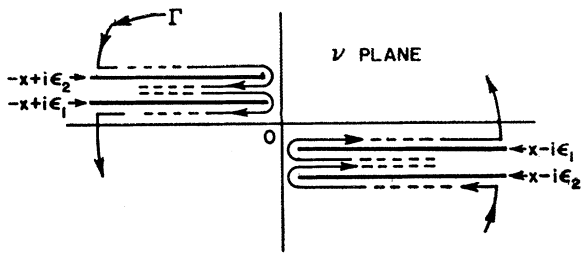


Fig. 5. The contour $\Gamma$ used in Eq. (3.10).

Here each $\epsilon$ is the difference between two $\eta$'s. The signs of the $\eta$'s have been chosen according to the convention [see Eq. (2.1)]

$$\xi_i \to \xi_i - i\eta_i; \quad \eta_i > 0 \quad \text{if} \quad \xi_i > 0; \quad \eta_i < 0 \quad \text{if} \quad \xi_i < 0. \tag{3.8}$$

Let the subscript $i$ label the loops in the order in which they occur in the diagram; and choose

$$\epsilon_1 < \epsilon_2 < \cdots < \epsilon_L. \tag{3.9}$$

Now $F^{(L)}$ contains $2L$ separate branch cuts as shown in Fig. 5. If we apply Cauchy's integral formula using the contour $\Gamma$, we find

$$F^{(L)}(\nu) = \frac{1}{2\pi i}\oint_\Gamma \frac{F^{(L)}(\nu')}{\nu'-\nu}d\nu'$$

$$= \sum_{j=1}^{L}\int_0^\infty \rho(x)dx\left\{\frac{1}{x-i\epsilon_j-\nu}\prod_{\substack{i=1\\ i\neq j}}^{L} F_i^{(1)}(x-i\epsilon_j)\right.$$

$$\left. + \frac{1}{x-i\epsilon_j+\nu}\prod_{\substack{i=1\\ i\neq j}}^{L} F_i^{(1)}(-x+i\epsilon_j)\right\}. \tag{3.10}$$

The discontinuity across the $j$th branch cut (for $y=\mathrm{Re}\nu>0$, say) is

$$[F^{(L)}(\nu=y-i\epsilon_j)]_d = 2\pi i\rho(y)\prod_{\substack{i=1\\ i\neq j}}^{L} F_i^{(1)}(y-i\epsilon_j)$$

$$\underset{\epsilon\to 0}{\longrightarrow} 2\pi i\rho(y)\{F^{(1)}(y+i\epsilon)\}^{j-1}\{F^{(1)}(y-i\epsilon)\}^{L-j}. \tag{3.11}$$

In order to calculate the complete discontinuity across the cut in $F^{(L)}$ when all the $\epsilon$'s go to zero, we may sum Eq. (3.11) over $j$. Finally, we should note that, if we sum all ladder diagrams, the discontinuity may be written in the form

$$[F(y)]_d = [\sum_L F^{(L)}(y)]_d$$

$$= 2\pi i \lim_{\epsilon\to 0} F(y+i\epsilon)\rho(y)F(y-i\epsilon)$$

$$= 2\pi i\rho(y)\,|F(y+i\epsilon)|^2, \tag{3.12}$$

which is the usual form of a unitarity sum.

The generalization of this technique to arbitrary diagrams is not difficult, the only tricky point being the choice of a convenient order for the $\eta_i$. On inspecting any diagram, we always can pick out the sets of singularities which might occur simultaneously. For example, in Fig. 6, we have marked the four possible singularities by dotted lines which indicate the associated intermediate states. We may consider the sequences (1, 2, 4) and (1, 3, 4) separately. Singularities 2 and 3 never can occur together because they correspond to different solutions of Eq. (2.9), i.e., different choices of particles and holes. The important point is that, once we have chosen a sequence of compatible singularities, there is no ambiguity in ordering the displacements of these singularities into the complex plane. Each sequence may be ordered separately. Because all simple reduced graphs contain only two external vertices, it is apparent that these remarks apply to graphs with an arbitrary number of external lines.

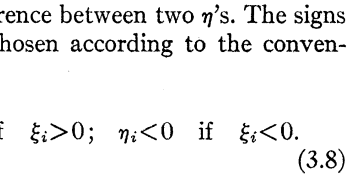


FIG. 6. A vacuum-polarization graph. The intermediate states associated with singularities are indicated by dashed lines.

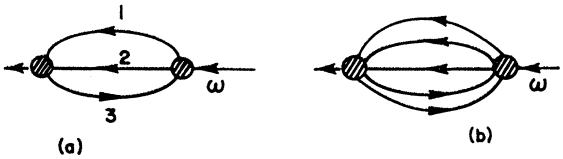


FIG. 7. Simple reduced graphs associated with the proper self-energy function.

## IV. SOME SIMPLE APPLICATIONS

As a conclusion to this paper we discuss two instances where the techniques discussed here provide some useful information.

First we consider the electron self-energy graph. As mentioned in the Introduction, it has been shown by Luttinger[4] that the imaginary part of the self energy goes to zero as $\omega^2$ near the Fermi surface. We may obtain this result via our techniques by noting that the self energy has much the same analytic properties as the propagator itself; i.e., it has a cut along the real axis and its imaginary part is just the discontinuity across this cut. All of the simple reduced graphs are of the form shown in Fig. 7. The contribution to the discontinuity from reduced graphs like Fig. 7(a) is (omitting momentum sums)

$$\mathrm{Im}G(\omega) \sim \int_0^\infty d\xi_1\int_0^\infty d\xi_2\int_{-\infty}^0 d\xi_3\, M_1 \mathfrak{A}(\xi_1)\mathfrak{A}(\xi_2)\mathfrak{A}(\xi_3)$$

$$\times M_2^*\delta(\xi_1+\xi_2-\xi_3-\omega)$$

$$\sim \omega^2 \text{ for small } \omega. \tag{4.1}$$

In (4.1), $M_1$ and $M_2^*$ are the vertex functions indicated by the small shaded circles in the figure. According to the discussion at the end of the last section, $M_1$ and $M_2^*$ are to be evaluated for $\omega$ just above and just below the real axis, respectively. These vertex functions may be complex but must be finite. Actually one must be rather

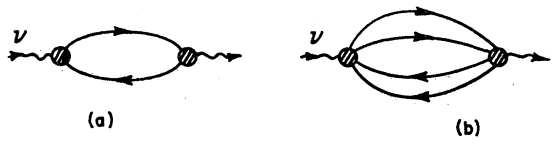


FIG. 8. Simple reduced graphs associated with the vacuum-polarization function.

careful before taking it for granted that all the vertex functions appearing in Fig. 7 are well behaved. In some cases there occur spurious divergences which resemble the infrared catastrophe in electrodynamics; but these divergences disappear after a suitable amount of diagram summation.

The final expression in Eq. (4.1) obviously is in accord with Luttinger's conclusion. All higher order reduced graphs contribute only higher powers of $\omega$ to Im$G$. To see this, note that the contribution of a reduced graph containing $n$ holes and $n+1$ particles may be written in the form

$$\begin{aligned}\mathrm{Im}G^{(n)}\sim&\int d\omega_1\cdots\int d\omega_{n+1}\,\delta(\omega_1+\cdots+\omega_{n+1}-\omega)\\&\times\int_0^\infty d\xi_1\int_{-\infty}^0 d\xi_1'\;\mathfrak{A}(\xi_1)\mathfrak{A}(\xi_1')\\&\times\delta(\xi_1-\xi_1'-\omega_1)\cdots\int_0^\infty d\xi_n\int_{-\infty}^0 d\xi_n'\\&\times\mathfrak{A}(\xi_n)\mathfrak{A}(\xi_n')\delta(\xi_n-\xi_n'-\omega_n)\int_0^\infty d\xi_{n+1}\\&\times\mathfrak{A}(\xi_{n+1})\delta(\xi_{n+1}-\omega_{n+1})M_1{}^{(n)}M_2{}^{(n)*}\\\sim&\int_0^\infty d\omega_1\cdots\int_0^\infty d\omega_{n+1}\,\omega_1\omega_2\cdots\omega_n\\&\times\delta(\omega_1+\cdots+\omega_{n+1}-\omega)\\\sim&\omega^{2n}\text{ for small }\omega.\end{aligned}\tag{4.2}$$

This is in exact agreement with Luttinger's result.

As a second example we consider vacuum-polarization graphs. The analysis is exactly the same as for the self-energy function, and we repeat it only because the result is useful in transport theory. All simple reduced graphs are of the form of Fig. 8. The polarization function has a cut along the real axis in the $\nu$ plane; and we are interested in the discontinuity across this cut. For a reduced graph with $n$ intermediate particle-hole pairs, this discontinuity has the form

$$\begin{aligned}[F(\nu)]_d{}^{(n)}\sim&\int d\nu_1\cdots\int d\nu_n\delta(\nu_1+\cdots+\nu_n-\nu)\\&\times\int_0^\infty d\xi_1\int_{-\infty}^0 d\xi_1'\;\mathfrak{A}(\xi_1)\mathfrak{A}(\xi_1')\\&\times\delta(\xi_1-\xi_1'-\nu_1)\cdots\int_0^\infty d\xi_n\int_{-\infty}^0 d\xi_n'\\&\times\mathfrak{A}(\xi_n)\mathfrak{A}(\xi_n')\delta(\xi_n-\xi_n'-\nu_n)\\&\times F_1{}^{(n)}(\nu)F_2{}^{(n)*}(\nu).\end{aligned}\tag{4.3}$$

If $\nu$ is small, we have

$$\begin{aligned}[F(\nu)]_d{}^{(n)}\sim&\int_0^\infty d\nu_1\cdots\int_0^\infty d\nu_n\,\nu_1\cdots\nu_n\\&\times\delta(\nu_1+\cdots+\nu_n-\nu)\sim\nu^{2n-1}.\end{aligned}\tag{4.4}$$

It must be emphasized that, in applying the methods outlined in this paper, we first must choose a particular skeleton diagram and then may evaluate the discontinuity across a branch cut by summing the contributions from the associated simple reduced graphs. To a certain extent this procedure may be reversed to yield expressions similar to unitarity relations. For example, the sum of all possible reduced graphs of the form of Fig. 8(a) may be obtained by inserting for each of the small shaded circles the sum of all proper vertex functions. This contribution to the discontinuity in $F$ can be expressed as

$$\begin{aligned}[F_{\mathbf{q}}(\nu)]_d{}^{(1)}\sim&\sum_{\mathbf{p}}\int_0^\infty d\xi_1\int_{-\infty}^0 d\xi_1'|\Lambda_{\mathbf{p},\mathbf{q}}(\nu)|^2\\&\times\mathfrak{A}_{\mathbf{p}+\mathbf{q}}(\xi_1)\mathfrak{A}_{\mathbf{p}}(\xi_1')\delta(\xi_1-\xi_1'-\nu)\\=&\sum_{\mathbf{p}}\int_{-\nu}^0 d\xi_1'|\Lambda_{\mathbf{p},\mathbf{q}}|^2\,\mathfrak{A}_{\mathbf{p}+\mathbf{q}}(\xi_1'+\nu)\mathfrak{A}_{\mathbf{p}}(\xi_1')\\\cong&\nu\sum_{\mathbf{p}}|\Lambda_{\mathbf{p},\mathbf{q}}|^2\,\mathfrak{A}_{\mathbf{p}+\mathbf{q}}(0)\mathfrak{A}_{\mathbf{p}}(0),\end{aligned}\tag{4.5}$$

where $\Lambda$ is the sum of vertex functions just mentioned. Equation (4.5) has the form of a unitarity sum over all intermediate states containing a single pair, except that the delta functions expressing energy conservation have been replaced by the spectral densities $\mathfrak{A}$. If the quasiparticles were stable, then the $\mathfrak{A}$'s would in fact be delta functions, and strict energy conservation would be retained.

On the other hand, we cannot sum all reduced graphs of the form Fig. 8(b) by inserting for the shaded circles the sum of all amplitudes for double-pair production. Such a procedure would lead to diagrams with internal

self-energy parts; i.e., diagrams which are not skeleton-like. This difficulty might be eliminated by reformulating the theory in terms of free-particle propagators and introducing some graph-by-graph renormalization scheme. However, the present formulation in terms of true propagators and skeleton graphs turns out to be particularly convenient for use in situations where there is present a static crystal or impurity field. In any case, the fact that quasi-particles are not stable implies that we cannot hope to find such simple relationships here as those which occur in true particle-field theories.

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# Theory of Impurity Resistance in Metals. II

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The many-body theory of impurity resistance which was developed in a previous paper is extended to include all corrections resulting from electron-electron interactions. The model used is a normal Fermi fluid in the presence of a small but finite concentration of randomly scattered, fixed impurities. The resulting expression for the conductivity may be interpreted in terms of independent single-electron-like excitations, or "quasi-particles." The combined effect of the impurities and the many-body interactions causes these quasi-particles to carry current at their group velocity; but there is no effective charge correction.

## I. INTRODUCTION

IN a recent paper[1] the author presented a theory of impurity-resistance in normal metals using as a model an interacting Fermi fluid and randomly placed scattering centers. The physical picture which emerged was that of single-electron-like excitations at the Fermi surface scattering from screened impurities. Interactions among these excitations, or "quasi-particles,"[2] were assumed to be negligible. It is the purpose of the present paper to complete the work of I by taking into account all of the many-body effects. The resulting expression for the resistivity will be exact to first order in the density of impurities and to all orders in the electron-electron interactions.

Although electron-electron collisions cannot contribute directly to the resistivity, there are several places where they play an important role in the theory. Perhaps the most important many-body effect is the screening of the impurities. In principle, however, this effect is included exactly when one calculates the amplitude for scattering of a single quasi-particle at an impurity according to the rules prescribed in I. This amplitude, which includes even the exchange interactions between the incident electron and the screening cloud, is merely a basic ingredient of the independent quasi-particle model. In this paper we shall be in search of more subtle effects.

One possible many-body effect has been emphasized recently in an article by Heine and Falicov.[3] These authors point out that it may be incorrect to neglect the interactions among the quasi-particles when calculating their acceleration in an external electric field. For example, the acceleration of a perfectly free electron gas in an external field is independent of the strength of the interactions between the electrons. If one uses an independent quasi-particle model to describe this gas, one destroys the Galilean invariance in a way which is quite inappropriate to this particular situation and arrives at an incorrect value for the acceleration. The introduction of fixed impurities or a lattice removes the Galilean invariance; but it is apparent that a purely kinetic formulation of transport theory in terms of quasi-particles requires careful examination from a fundamental point of view.

An even more subtle many-body effect, which apparently has not been pointed out before now, involves the current carried by a quasi-particle. In the usual Landau picture, a quasi-particle of wave vector $\mathbf{k}$ carries a current $e\mathbf{k}/m$ simply because momentum and charge are conserved in electron-electron collisions. When we make a dc measurement on an impure metal, however, we measure the current over a time much longer than any other time which appears in the problem. In particular, the period of the applied field must

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