Form Factors and Vector Mesons*

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The 2π and 3π resonances are re-examined from the point of view that they are vector mesons coupled to conserved currents. The theory of unstable mesons is discussed and formulas are then derived for the emission and propagation of these mesons. The connection with electromagnetic form factors is then given, particularly for the simple case of infinite bare mass. The results are very similar to those of the dispersion method. Experimental manifestations of universality (connected with the conserved vector current) are discussed. Applications are then made to the decay of π^0 and a group of related phenomena, including several "pole" experiments. Also, the contribution of the 2π resonance to π-N scattering is discussed briefly from the vector meson point of view. Finally, we compare the vector meson approach to the alternative method using dispersion relations applied to presumably dynamical resonances. We conclude that the dynamical picture is an interesting special case of the vector meson theory with infinite bare mass, a case in which the mass and coupling constant are determined and the behavior at high energies is less singular. The methods we develop are applicable to the dynamical case.

I. INTRODUCTION

In recent years a considerable amount of effort has been expended on predicting the effects of a proposed P-wave resonance in pion-pion scattering on various elementary particle processes. This resonance was first suggested[1] in order to explain some features of the isotopic vector electromagnetic form factors of the nucleon. It has also been suggested[2] that there may be a resonance in the three-pion system (with $J=1$, $I=0$); that would facilitate an understanding of the isotopic scalar form factors. Some attempts[3] have been made to demonstrate the existence of the 2π resonance on a dynamical basis. Calculations involving the exchange of such a resonant state tend to be rather cumbersome, however, if the dynamical approach is used. That is especially true of the 3π resonance, since processes like $3\pi \rightarrow N+\bar{N}$ are genuinely complicated.

A different picture is employed by Sakurai,[4] who treats the $\pi\pi$ resonance as an unstable vector meson coupled to the isotopic spin current. He has also suggested the existence of an $I=0$ vector meson (coupled to the hypercharge current), which corresponds to the 3π resonance. In a generalization of Sakurai's work, Gell-Mann[5] predicts in addition four strange vector mesons in two doublets each with $I=\frac{1}{2}$. A ninth vector meson coupled to the baryon current may exist as well.[4,5]

The vector meson approach simplifies greatly the approximate theoretical discussion of the resonances, as we shall show in a number of cases. We concentrate primarily on the description of the $\pi\pi$ resonance as a particle coupled universally to the isotopic spin. The treatment is particularly simple if the bare mass is infinite.

We then take up the relation of such a description to the dynamical model. We conclude that essentially all the results can be carried over to the dynamical theory, which can be regarded as a special case of the vector meson theory with infinite bare mass. The special case is characterized by a mass and coupling constant that are determined and by less singular high energy behavior, which may be necessary for consistency.

The vector mesons are all unstable and some, at least, decay very rapidly. Let us adopt the notation of reference 5. The dominant decay mode for the ρ meson (the $I=1$ particle) is into two pions with a lifetime of the order of 10^{-23} sec. The $I=0$ meson, called ω^0, can decay into $\pi^0+\gamma$ with a lifetime of about 10^{-18} sec, or, if it is sufficiently massive, into three pions with a much shorter lifetime. A prerequisite of any discussion of the effects of the vector mesons, then, is an understanding of the properties of unstable particles and how to compute with them. We shall therefore begin with an illustration, in terms of a simple model field theory, of the behavior of an unstable particle. The concepts so developed will then be extended to the vector mesons in question, and to calculations of their effects.

II. MODEL FIELD THEORY AS A GUIDE

In order to remove all nonessential complications, we shall ignore spin. We are then interested in describing the properties of an unstable scalar meson (called σ), which couples to pairs of another meson (called π). To make everything explicit and obvious, we shall assume the interaction of σ and π is described by a specific relativistic model field theory.[6]

First, if the σ meson did not exist, the $\pi\pi$ (S-wave) scattering amplitude predicted by the model field

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[1] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

[2] Y. Nambu, Phys. Rev. **106**, 1366 (1957); G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); M. Baker and F. Zachariasen, *ibid.* **118**, 1659 (1960).

[4] J. J. Sakurai, Ann. Phys. **11**, 1 (1960).

[5] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961 (unpublished).

[6] F. Zachariasen, Phys. Rev. **121**, 1851 (1961); see also B. Lee and M. Vaughn, Phys. Rev. Letters **4**, 578 (1960).

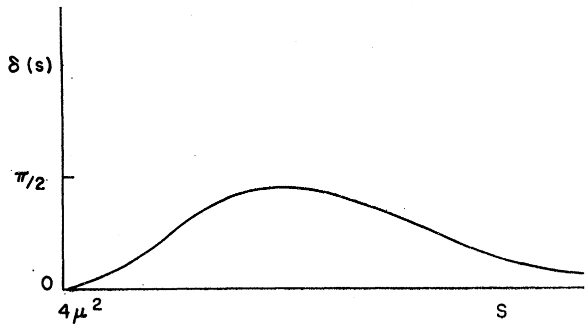

FIG. 1. The dynamical case: the phase shift in the absence of a CDD pole.

theory would be

$$T(s)=\frac{\lambda}{\left\{1+\dfrac{\bar{\lambda}}{16\pi^2}(s-s_0)\displaystyle\int_{4\mu^2}^{\infty}\left[\frac{s'-4\mu^2}{s'}\right]^{\frac{1}{2}}\frac{ds'}{(s'-s_0)(s'-s)}\right\}}=\bar{\lambda}/\bar{D}_0(s),\qquad(2.1)$$

where s is the square of the center-of-mass energy and where $\bar{\lambda}$ is a constant defined by $T(s_0)=\bar{\lambda}$. s_0 is chosen less than $4\mu^2$ so that $\bar{\lambda}$ will be real. The s-wave $\pi\pi$ phase shift is related to $T(s)$ by

$$T(s)=-16\pi\left(\frac{s}{s-4\mu^2}\right)^{\frac{1}{2}}\sin\delta(s)e^{i\delta(s)},\qquad(2.2)$$

for $s>4\mu^2$. If $\bar{\lambda}>0$ or if $\bar{\lambda}$ is sufficiently negative, there is a bound state (or a ghost). We are not interested in such a possibility, so let us restrict ourselves to $\bar{\lambda}<0$ and fairly small. The phase shift for this case is sketched in Fig. 1. There is no resonance, since for s-wave scattering there is no centrifugal barrier, and a purely attractive potential such as the simple model provides cannot give rise to a resonance. In the physical case, however, where σ has spin one, the interesting state is the p-wave and a resonance can occur. If one does occur under these conditions, it is a "dynamical" resonance.

Now let us assume again the existence of the σ meson and see what effect it produces. If the bare mass of the σ meson is $m_{\sigma0}^2$, the existence of the meson is equivalent to adding a Castillejo-Dalitz-Dyson[7] pole at $m_{\sigma0}^2$ to the denominator function $\bar{D}_0(s)$ in Eq. (2.1). Thus we now have $\bar{D}_0(s)\rightarrow\bar{D}(s)$ in (2.1) where

$$\bar{D}(s)=1+\frac{\bar{\lambda}}{16\pi^2}(s-s_0)\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-s_0)(s'-s)}+\frac{R}{s-m_{\sigma0}^2}\left(\frac{s_0-s}{s_0-m_{\sigma0}^2}\right).\qquad(2.3)$$

[7] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

It is necessary to require that $R<0$.[7] Now the existence of the CDD pole forces $\mathrm{Re}\bar{D}(s)$ to have a zero at one point at least. Let this point be m_σ^2, which we define as the physical mass of the σ meson. If $m_\sigma^2<4\mu^2$, then $\mathrm{Im}\bar{D}(m_\sigma^2)=0$ and $\bar{D}(m_\sigma^2)=0$, in which case $T(s)$ has a pole at m_σ^2. In this case the σ meson is stable. The renormalized coupling constant g for the $\sigma\pi\pi$ coupling is defined as the square root of the residue of $T(s)$ at the pole; that is,

$$\frac{1}{g^2}=\frac{d}{ds}\frac{1}{T(s)}\bigg|_{s=m_\sigma^2}.$$

If $m_\sigma^2>4\mu^2$, on the other hand, then

$$\mathrm{Im}\bar{D}(m_\sigma^2)=\frac{\bar{\lambda}}{16\pi}\left(\frac{m_\sigma^2-4\mu^2}{m_\sigma^2}\right)^{\frac{1}{2}}\neq0,\qquad(2.4)$$

and we have from Eq. (2.2)

$$\sin\delta(m_\sigma^2)\exp[i\delta(m_\sigma^2)]=i.$$

There is a resonance at m_σ^2. We may now define the renormalized coupling constant g of the unstable particle, in analogy to the stable case by

$$\frac{1}{g^2}=\frac{d}{ds}\mathrm{Re}\left(\frac{1}{T(s)}\right)\bigg|_{s=m_\sigma^2},\qquad(2.5)$$

and hence explicitly

$$\frac{1}{g^2}=\frac{\bar{\lambda}}{16\pi^2}\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-m_\sigma^2)^2}-\frac{R}{(m_\sigma^2-m_{\sigma0}^2)^2}.\qquad(2.6)$$

If we define

$$\lambda=g^2/(m_\sigma^2-m_{\sigma0}^2),\qquad(2.7)$$

and

$$D(s)=\left(\frac{s-m_{\sigma0}^2}{s-m_\sigma^2}\right)\frac{\lambda}{\bar{\lambda}}\bar{D}(s),\qquad(2.8)$$

then we may write the scattering amplitude in the form

$$T(s)=\frac{\lambda+g^2/(s-m_\sigma^2)}{D(s)},\qquad(2.9)$$

where

$$D(s)=1+\frac{s-m_\sigma^2}{16\pi}\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\left(\lambda+\frac{g^2}{s'-m_\sigma^2}\right)\frac{ds'}{(s'-m_\sigma^2)(s'-s)}.\qquad(2.10)$$

The analogy with the stable case is thereby made obvious.[6] The form of the solution is identical; the only difference is that in the unstable case $m_\sigma^2>4\mu^2$ and therefore $\mathrm{Im}D(s)$ has a pole at m_σ^2 which cancels the pole in the numerator of (2.9) so that $T(s)$ goes through a resonance at m_σ^2.

An alternative description of an unstable particle which has been suggested several times[8,9] is that it manifests itself through a pole of the scattering amplitude on its second Riemann sheet. The real part of the position of the pole is defined as the mass, the imaginary part is the width of the particle. We can demonstrate, within the framework of the explicit solutions offered by the model field theory, that this property holds and those definitions of the mass and width agree with ours for small width.

The existence of a pole on the second sheet of T is the same as the existence of a zero on the second sheet of $\bar{D}$. $\bar{D}$, on its second sheet, is given by

$$\bar{D}^{(2)}(z)=\bar{D}(z)+2i\,\mathrm{Im}\bar{D}(z),$$

for $y=\mathrm{Im}z<0$. The function $\mathrm{Re}\bar{D}$ vanishes on the real axis at m_σ^2; this was our definition of the physical mass of the unstable particle. Therefore, we can write

$$\bar{D}^{(2)}(z)=\frac{\bar{\lambda}}{16\pi^2}(z-m_\sigma^2)\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-m_\sigma^2)(s'-z)} -\left(\frac{z-m_\sigma^2}{z-m_{\sigma0}^2}\right)\frac{R}{(m_\sigma^2-m_{\sigma0}^2)}+\frac{2i\bar{\lambda}}{16\pi}\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}.$$

If $\bar{D}^{(2)}(z)=0$ for $z=x+iy$ where y is small, then we must have approximately

$$\frac{\bar{\lambda}}{16\pi^2}(x-m_\sigma^2)\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-m_\sigma^2)(s'-x)} -\left(\frac{x-m_\sigma^2}{x-m_{\sigma0}^2}\right)\frac{R}{(m_\sigma^2-m_{\sigma0}^2)}=0,$$

and

$$\frac{\bar{\lambda}}{16\pi^2}iy\int_{4\mu^2}^{\infty}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-m_\sigma^2)(s'-x)} -\frac{iy}{x-m_{\sigma0}^2}\frac{R}{m_\sigma^2-m_{\sigma0}^2}+\frac{iy}{x-m_{\sigma0}^2}\frac{x-m_\sigma^2}{x-m_{\sigma0}^2}\frac{R}{m_\sigma^2-m_{\sigma0}^2} +\frac{i\bar{\lambda}}{16\pi}\left(\frac{x-4\mu^2}{x}\right)^{\frac{1}{2}}=0.$$

The first of these equations is clearly satisfied with $x=m_\sigma^2$; the second may then also be satisfied with

$$y=-\frac{g^2}{16\pi}\left(\frac{m_\sigma^2-4\mu^2}{m_\sigma^2}\right)^{\frac{1}{2}},$$

where g^2 is defined by Eq. (2.6).

Thus for small widths, that is, small g^2, the pole on the second sheet occurs on $m_\sigma^2-im_\sigma\Gamma$, where m_σ is the mass, defined as the position of the zero of $\mathrm{Re}\bar{D}(s)$, and

$$\Gamma=\frac{g^2}{16\pi}\left(\frac{m_\sigma^2-4\mu^2}{m_\sigma^4}\right)^{\frac{1}{2}}$$

is the width expressed in terms of the renormalized coupling constant defined by Eq. (2.5). For larger values of g^2, of course, the mass and width defined in our way will not be precisely the same as the mass and width defined in terms of the pole on the second sheet. Let us now abandon the second sheet and return to develop further properties of unstable particles.

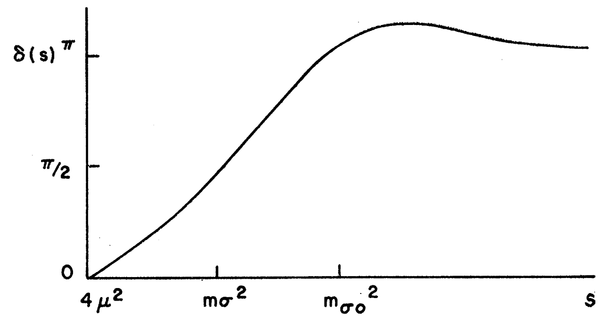

FIG. 2. The case of an unstable particle: the phase shift in the presence of one CDD pole.

We may observe at this point that $m_{\sigma0}^2=\infty$ is obtained by setting $\lambda=0$ and allowing g^2 to be arbitrary instead of given by Eq. (2.6). The dynamical case, where the σ particle does not exist, recurs if g^2 has the value given in (2.6) with $m_{\sigma0}^2=\infty$, or $R=0$. This is easily verified by comparing with Eq. (2.1).

As in the purely dynamical case, there is a bound state (or ghost) if $\lambda>0$ and we are not interested in this. However, if $\lambda<0$ we find the behavior of the phase shift shown in Fig. 2. There is no bound state so the phase shift starts at zero, rises through $\pi/2$ at m_σ^2, through π at $m_{\sigma0}^2$, and drops asymptotically back down to π from above as $s\rightarrow\infty$. Thus, we see that $\delta(\infty)-\delta(4\mu^2)=\pi$ for the case of a single unstable particle.[10] Only if $m_{\sigma0}^2-m_\sigma^2$ is finite and small does the scattering amplitude pass through zero near the resonance. If $m_{\sigma0}^2=\infty$ the phase shift just rises asymptotically to π from below as $s\rightarrow\infty$, without ever crossing π. We recall that in the dynamical case the phase shift rises through $\pi/2$ at the resonance, then comes back down through $\pi/2$, and reaches zero at infinity.

Next we may observe that the residue at the pole in $\mathrm{Im}D(s)$ is

$$(s-m_\sigma^2)\,\mathrm{Im}D(s)\,|_{(s=m_\sigma^2)} =\frac{g^2}{16\pi}\left(\frac{m_0^2-4\mu^2}{m_0^2}\right)^{\frac{1}{2}}=m_\sigma\Gamma, \quad (2.11)$$

[8] R. Oehme (to be published); M. Lévy, Nuovo cimento 13, 115 (1959); R. Jacob and R. G. Sachs, Phys. Rev. 121, 350 (1961).
[9] R. Peierls, *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press, New York, 1954), p. 296.
[10] M. Ruderman and C. Sommerfield, Bull. Am. Phys. Soc. 4, 375 (1959).

where Γ is the decay rate of the σ particle. This justifies the definition of the renormalized $\sigma\pi\pi$ coupling constant g by Eq. (2.5).

The form factor for the dissociation of the σ particle into two pions is given[6] by

$$F_{\sigma\pi\pi}(s)=1/D(s). \tag{2.12}$$

In contrast to the stable situation, where $F_{\sigma\pi\pi}(m_\sigma^2)=1$, we have in the unstable case the result that $F_{\sigma\pi\pi}(m_\sigma^2)=0$. The coefficient of the zero is

$$F_{\sigma\pi\pi}'(m_\sigma^2)=-i/m_\sigma\Gamma. \tag{2.13}$$

Finally, we can introduce a weakly coupled particle (scalar photon) of zero bare and physical masses, coupled to the pion. The form factor $F_\pi(s)$ for this new particle can be computed in the model field theory by dispersion theory. The only possible intermediate states are the $\pi\pi$ state and the σ meson state. From these states we find

$$\mathrm{Im}F_\pi(s)=F_\pi^*(s)\sin\delta(s)e^{i\delta(s)} -\pi\delta(s-m_\sigma^2)F_{\sigma\pi\pi}(s)A, \tag{2.14}$$

where A is an amplitude describing the transition of a virtual scalar photon into a σ meson. Since $F_{\sigma\pi\pi}(m_\sigma^2)=0$, the second term in (2.14), from the σ-meson intermediate state, in fact does not contribute. This is a generally true statement: namely, the single virtual unstable particle intermediate state never contributes to the absorptive part of any process.

We can relate $F_\pi(s)$ to $F_{\sigma\pi\pi}(s)$ by observing that

$$\mathrm{Im}F_{\sigma\pi\pi}(s)=F^*_{\sigma\pi\pi}(s)\sin\delta(s)e^{i\delta(s)}, \tag{2.15}$$

which, when compared with (2.14) yields the result

$$F_\pi(s)=\frac{m_\sigma^2}{m_{\sigma0}^2}\frac{s-m_{\sigma0}^2}{s-m_\sigma^2}\frac{F_{\sigma\pi\pi}(s)}{F_{\sigma\pi\pi}(0)}. \tag{2.16}$$

In deriving (2.16), use is made of the relations $F_{\sigma\pi\pi}(m_\sigma^2)=0$, $F_\pi(m_{\sigma0}^2)=0$,[11] and $F_\pi(0)=1$. If $m_{\sigma0}^2=\infty$, an especially simple relation obtains:

$$F_\pi(s)=\frac{-m_\sigma^2}{s-m_\sigma^2}\frac{F_{\sigma\pi\pi}(s)}{F_{\sigma\pi\pi}(0)}. \tag{2.17}$$

For weak coupling of the σ to π mesons, we see that in the region $s<0$, for example, [where $\mathrm{Im}D(s)=0$ so $F_{\sigma\pi\pi}(s)=1/\mathrm{Re}D(s)$] Eq. (2.17) gives the approximate result

$$F_\pi(s)\approx-m_\sigma^2/(s-m_\sigma^2). \tag{2.18}$$

Thus for weak coupling the result of calculating the photon form factor is the same as it would have been if one had (incorrectly) included the one σ meson intermediate state (which actually gives zero) in the dispersion calculation and dropped the two pion state.

While the results of this section are all obtained within the framework of the model field theory, many of the statements are in fact general. We shall devote the next section to repeating, for the actual case of unstable vector mesons and a complete field theory, the development of the basic formulas given here.

[11] See Sec. IV.

III. INTERACTION OF AN UNSTABLE VECTOR MESON

We shall now use the ρ meson and π mesons as examples, with their correct spins and charges. Let $T_{\pi\pi}^{11}(s)$ denote the P-wave pion-pion scattering amplitude. The phase shift is expressed in terms of $T_{\pi\pi}^{11}(s)$ by[12]

$$T_{\pi\pi}^{11}(s)=-12\pi\left[\frac{s}{(s-4\mu^2)^3}\right]^{\frac{1}{2}}\sin\delta(s)e^{i\delta(s)}. \tag{3.1}$$

Now, associated with the existence of the vector ρ meson of isotopic spin one, there is a resonance in the $\pi\pi$ P-wave scattering. We define the physical mass m_ρ^2 of the ρ meson as the position of this resonance. Furthermore, we define the renormalized $\rho\pi\pi$ coupling constant $\gamma_{\rho\pi\pi}$ by

$$\frac{1}{\gamma_{\rho\pi\pi}^2}=\frac{d}{ds}\,\mathrm{Re}\frac{1}{T_{\pi\pi}^{11}(s)}\bigg|_{s=m_\rho^2}. \tag{3.2}$$

We can write[13] in general

$$T_{\pi\pi}^{11}(s)=\bar{N}(s)/\bar{D}(s), \tag{3.3}$$

where $\bar{N}$ is an analytic function with only a left-hand cut and $\bar{D}$ is analytic with only a right-hand cut. Thus, for $s>4\mu^2$,

$$\mathrm{Im}\bar{D}(s)=\bar{N}(s)\,\mathrm{Im}\frac{1}{T_{\pi\pi}^{11}(s)}. \tag{3.4}$$

Note that in the elastic region, $4\mu^2<s<16\mu^2$,

$$\mathrm{Im}\bar{D}(s)=\frac{1}{12\pi}\left[\frac{(s-4\mu^2)^3}{s}\right]^{\frac{1}{2}}\bar{N}(s). \tag{3.5}$$

If there is a ρ meson of bare mass $m_{\rho0}^2$ and physical mass m_ρ^2, then $\mathrm{Re}\bar{D}(m_\rho^2)=0$ and $\bar{D}(s)$ has a pole at $m_{\rho0}^2$, by definition. In the physical case, the situation is complicated by the fact that apparently $m_\rho^2>16\mu^2$, so that m_ρ^2 lies in the inelastic region and the decay $\rho\rightarrow4\pi$ can occur. In the remainder of this section, we shall ignore this complication[14] and assume $m_\rho^2<16\mu^2$. Since m_ρ^2 is then in the elastic region, we have from Eq. (3.5) that

$$\mathrm{Im}\bar{D}(m_\rho^2)=\frac{1}{12\pi}\left[\frac{(m_\rho^2-4\mu^2)^3}{m_\rho^2}\right]^{\frac{1}{2}}\bar{N}(m_\rho^2). \tag{3.6}$$

[12] We use the notation: T matrix $=(1/s)\sum_{l,I}P_l(\cos\vartheta)Q_Iq^lT_{\pi\pi}^{lI}(s)$, where S matrix $=1-(2\pi)^4i\delta^4(p_f-p_i)[T$ matrix$]$ and where Q_I is the isotopic projection operator.

[13] M. Baker, Ann. Phys. 4, 271 (1958). See also reference 3.

[14] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

Furthermore,

$$\frac{1}{\gamma_{\rho\pi\pi}^2}=\frac{d}{ds}\frac{\mathrm{Re}\bar{D}(s)}{\bar{N}(s)}\bigg|_{(s=m_\rho^2)}=\frac{1}{\bar{N}(m_\rho^2)}\frac{d}{ds}\mathrm{Re}\bar{D}(s)\bigg|_{m_\rho^2}. \quad (3.7)$$

Let us define a new function $D(s)$, which also has only a right-hand cut, by

$$D(s)=\frac{\gamma_{\rho\pi\pi}^2}{\bar{N}(m_\rho^2)}\left(\frac{s-m_{\rho 0}^2}{s-m_\rho^2}\right)\left(\frac{1}{m_\rho^2-m_{\rho 0}^2}\right)\bar{D}(s). \quad (3.8)$$

$D(s)$ thus has neither a pole at $m_{\rho 0}^2$ nor a zero at m_ρ^2. In fact, it is easy to see from (3.7) that

$$\mathrm{Re}D(m_\rho^2)=1, \quad (3.9)$$

and from (3.6) that $\mathrm{Im}D(s)$ has a pole at m_ρ^2 with residue

$$(s-m_\rho^2)\,\mathrm{Im}D(s)\,|_{(s=m_\rho^2)}=\frac{\gamma_{\rho\pi\pi}^2}{12\pi}\left[\frac{(m_\rho^2-4\mu^2)^3}{m_\rho^2}\right]^{\frac{1}{2}}=m_\rho\Gamma, \quad (3.10)$$

where Γ is the decay rate of the ρ meson. In terms of D, we have

$$T_{\pi\pi}^{11}(s)=\left(\frac{\bar{N}(s)}{\bar{N}(m_\rho^2)}\right)\frac{\lambda+\gamma_{\rho\pi\pi}^2/(s-m_\rho^2)}{D(s)}, \quad (3.11)$$

where the constant λ is defined by

$$\gamma_{\rho\pi\pi}^2/\lambda=m_\rho^2-m_{\rho 0}^2. \quad (3.12)$$

An infinite bare mass for the ρ meson is then equivalent to $\lambda=0$.

The entire development has been carried out in parallel with the model field theory described in Sec. II, and the important results obtained there, such as the methods of defining the renormalized mass and coupling constant of the unstable particle, have therefore been extended to real field theory. We may pursue the analogy further by observing that the form factor of the $\rho\pi\pi$ vertex, which we shall call $F_{\rho\pi\pi}(s)$, is just

$$F_{\rho\pi\pi}(s)=1/D(s), \quad (3.13)$$

in the elastic region near m_ρ^2.

Therefore, just as in the model theory, we have

$$F_{\rho\pi\pi}(m_\rho^2)=0. \quad (3.14)$$

That the coefficient of the zero is again given by

$$F_{\rho\pi\pi}'(m_\rho^2)=-i/m_\rho\Gamma, \quad (3.15)$$

may be seen from Eq. (3.10).

It may be of value to elaborate a bit further on Eq. (3.14). Let us define $V_{\rho\pi\pi}(s)$ as the sum of all proper vertex graphs—that is, the usual renormalized vertex function—for the $\rho\pi\pi$ vertex. We choose $V_{\rho\pi\pi}(m_\rho^2)=1$. The form factor is then expressed as

$$F_{\rho\pi\pi}(s)=[\Delta_{F1}(s)/\Delta_F(s)]V_{\rho\pi\pi}(s), \quad (3.16)$$

where $\Delta_{F1}(s)$ is the renormalized ρ-meson propagator, and $\Delta_F(s)=(s-m_\rho^2)^{-1}$. Now if the ρ particle is unstable, only $\mathrm{Re}\Delta_{F1}^{-1}(m_\rho^2)=0$, so that $\Delta_{F1}(s)$ does not have a pole at m_ρ^2. Therefore, the ratio $\Delta_{F1}(m_\rho^2)/\Delta_F(m_\rho^2)=0$. From Eq. (3.15), we see that

$$\mathrm{Im}\Delta_{F1}^{-1}(m_\rho^2)=+m_\rho\Gamma, \quad (3.17)$$

as is, of course, to be expected.

The difference between the form factors for stable and unstable particles is thus easily seen: if the ρ particle were stable, then near $s=m_\rho^2$ we would have

$$F_{\rho\pi\pi}(s)\approx\frac{s-m_\rho^2}{s-m_\rho^2+O(s-m_\rho^2)^2};$$

then $F_{\rho\pi\pi}(m_\rho^2)=1$. If, on the other hand, ρ is unstable, we have instead

$$F_{\rho\pi\pi}(s)\approx\frac{s-m_\rho^2}{s-m_\rho^2+O(s-m_\rho^2)^2+im_\rho\Gamma+iO(s-m_\rho^2)}$$

in the vicinity of $s=m_\rho^2$, and hence

$$F_{\rho\pi\pi}(m_\rho^2)=0.$$

Since the ratio of Δ_{F1} to Δ_F vanishes at m_ρ^2, not only $F_{\rho\pi\pi}$ but the form factor describing the vertex where a ρ meson interacts with any other particle—e.g., a nucleon—also vanishes at m_ρ^2. This means that, in the dispersion theory, the single ρ particle intermediate state never contributes to any given absorptive part. The ρ meson makes itself felt only through the resonance it produces in the $\pi\pi$ system, which, of course, is allowed as one of the intermediate states in the same absorptive part.

IV. ELECTROMAGNETIC FORM FACTORS AND UNIVERSALITY

Let us now proceed to introduce electromagnetic form factors (of pions and nucleons, for example) and see what effects the unstable vector mesons may be expected to produce. We first discuss the pion electromagnetic form factor, which we shall call $F_\pi(s)$, and normalize so that $F_\pi(0)=1$.

The photon is coupled to the electromagnetic current, which is conserved and has two parts, an isotopic vector and an isotopic scalar part. The ρ meson, we shall assume (in accord with references 4 and 5), is also coupled to the conserved isotopic spin current. Thus the ρ^0 meson and the isovector part of the photon are coupled to the same unrenormalized current, and we can write the unrenormalized field equations

$$\begin{aligned}-\Box^2\hat{A}_\mu{}^V(x)&=\tfrac{1}{2}e_0\hat{j}_\mu,\\(-\Box^2+m_{\rho 0}^2)\hat{\rho}_\mu{}^0(x)&=\gamma_0\hat{j}_\mu.\end{aligned} \quad (4.1)$$

Here, the carets denote unrenormalized Heisenberg field operators, and the index V distinguishes the isovector part of the photon field from the isoscalar part.

The symbol $\rho_\mu{}^0$, of course, denotes the neutral ρ-meson field operator.

The corresponding renormalized field equations are

$$-\Box^2 A_\mu{}^V(x)=\tfrac{1}{2}e_0 Z_{3\gamma}{}^{-\frac{1}{2}}\hat{j}_\mu = j_\mu{}^\gamma, \qquad (4.2)$$
$$(-\Box^2+m_\rho{}^2)\rho_\mu{}^0(x)=\gamma_0 Z_{3\rho}{}^{-\frac{1}{2}}\hat{j}_\mu+\delta m_\rho{}^2\rho_\mu{}^0 = j_\mu{}^\rho.$$

Now the pion electromagnetic form factor receives contributions only from the isovector part of the photon field, so we may write the definitions of the form factors as follows:

$$\frac{-ie}{(4\omega_1\omega_2)^{\frac{1}{2}}}F_\pi(s)(q_1-q_2)_\mu(\delta_{\sigma_1 1}\delta_{\sigma_2 2}-\delta_{\sigma_1 2}\delta_{\sigma_2 1}) = \langle q_1\sigma_1, q_2\sigma_2{}^{(-)}|j_\mu{}^\gamma(0)|0\rangle, \qquad (4.3)$$

and

$$\frac{-i\gamma_{\rho\pi\pi}}{(4\omega_1\omega_2)^{\frac{1}{2}}}F_{\rho\pi\pi}(s)(q_1-q_2)_\mu(\delta_{\sigma_1 1}\delta_{\sigma_2 2}-\delta_{\sigma_1 2}\delta_{\sigma_2 1}) = \langle q_1\sigma_1, q_2\sigma_2{}^{(-)}|j_\mu{}^\rho(0)|0\rangle. \qquad (4.4)$$

Here, q_1, σ_1 and q_2 σ_2 are the 4-momenta and charges of two pions.

Note that

$$\langle n|j_\mu{}^\rho(0)|0\rangle=(-P_n{}^2+m_\rho{}^2)\langle n|\rho_\mu(0)|0\rangle, \qquad (4.5)$$

for any state n. Using this fact, it is easy to extract from Eqs. (4.3) and (4.4) a relation between F_π and $F_{\rho\pi\pi}$; namely,

$$eF_\pi(s)=\left(\frac{e_0}{\gamma_0}\right)\left(\frac{Z_{3\rho}}{Z_{3\gamma}}\right)^{\frac{1}{2}}\left(\frac{s-m_{\rho0}{}^2}{s-m_\rho{}^2}\right)\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(s). \qquad (4.6)$$

If we use the fact that $F_\pi(0)=1$, we may rewrite (4.6) as

$$F_\pi(s)=\left(\frac{m_\rho{}^2}{m_{\rho0}{}^2}\right)\left(\frac{s-m_{\rho0}{}^2}{s-m_\rho{}^2}\right)\left(\frac{F_{\rho\pi\pi}(s)}{F_{\rho\pi\pi}(0)}\right). \qquad (4.7)$$

This may be further simplified if we assume that $m_{\rho0}{}^2=\infty$ to

$$F_\pi(s)=\frac{-m_\rho{}^2}{s-m_\rho{}^2}\left(\frac{F_{\rho\pi\pi}(s)}{F_{\rho\pi\pi}(0)}\right). \qquad (4.8)$$

Equations (4.7) and (4.8) are identical with the results [Eqs. (2.16) and (2.17)] obtained on the basis of the model field theory. Because of the restricted number of intermediate states in the model, it was not necessary to assume explicitly a relation between the bare currents of the photon and the ρ meson, as we did here. In Sec. VII we discuss the extent to which the dynamical resonance approach (using dispersion relations) really differs from the vector meson approach in which the photon and meson have bare currents that are proportional.

The denominator $s-m_\rho{}^2$ in (4.7) and (4.8) shows that the result obtained in Sec. III that $F_{\rho\pi\pi}(m_\rho{}^2)=0$ is in fact necessary, for otherwise the electromagnetic form factor would have a singularity at the ρ-meson mass, which is a manifest absurdity. What happens instead is that $F_\pi(s)$ goes through a maximum at $m_\rho{}^2$. This may be seen explicitly in the model theory; more generally, from Eq. (3.15) we have

$$F_\pi(m_\rho{}^2)=im_\rho\Gamma^{-1}F_{\rho\pi\pi}(0), \qquad (4.9)$$

which is a rather large number. Furthermore, F_π is pure imaginary at $m_\rho{}^2$, reflecting the fact that in the elastic region the phase of F_π is the phase of the P-wave $\pi\pi$ scattering, which goes through 90° at $m_\rho{}^2$.

Exactly analogous arguments may evidently be carried over from the nucleon form factors. By the same means as were used in obtaining (4.7) and (4.8), we get

$$F_{1,2}{}^V(s)=\left(\frac{m_\rho{}^2}{m_{\rho0}{}^2}\right)\left(\frac{s-m_{\rho0}{}^2}{s-m_\rho{}^2}\right)\frac{F_{1,2}{}^\rho(s)}{F_{1,2}{}^\rho(0)}, \qquad (4.10)$$

where $F_{1,2}{}^V$ are the standard nucleon isovector charge and moment form factors, normalized to one at $s=0$, and $F_{1,2}{}^\rho$ are the "charge" and "moment" form factors for the $\rho N\bar{N}$ vertex, defined by

$$\frac{1}{(4E'E)^{\frac{1}{2}}}(\bar{u}_{p'}|\gamma_{\rho NN}F_1{}^\rho(s)\gamma_\mu+\mu_{\rho NN}F_2{}^\rho(s)\sigma_{\mu\nu}(p'+p)_\nu|v_{\bar{p}}) = \langle p'\bar{p}^{(-)}|j_\mu{}^\rho(0)|0\rangle, \qquad (4.11)$$

where $\gamma_{\rho NN}$ is the renormalized ρNN coupling constant and $\mu_{\rho NN}$ is the "anomalous moment" of the nucleon.

According to references 4 and 5, the ω^0 meson, with $I=0$, is coupled to the conserved isoscalar part of the electromagnetic current. If this is so, we may write

$$F_{1,2}{}^S(s)=\left(\frac{m_\omega{}^2}{m_{\omega0}{}^2}\right)\left(\frac{s-m_{\omega0}{}^2}{s-m_\omega{}^2}\right)\frac{F_{1,2}{}^\omega(s)}{F_{1,2}{}^\omega(0)}, \qquad (4.12)$$

where $F_{1,2}{}^S$ are the nucleon isoscalar charge and moment form factors, and the form factors for the $\omega^0 NN$ vertex $F_{1,2}{}^\omega$ are defined in analogy to (4.11).

Now in what we have done so far, the renormalized ρ-meson coupling constants to different particles—e.g., pions, nucleons, etc., are all different. Yet the ρ meson is coupled to a conserved current, and we know that if it had zero mass all these constants would be identical as a result of the Ward identity. We should next like to exploit this observation to correlate $\gamma_{\rho\pi\pi}$ and $\gamma_{\rho NN}$, for example.

If $m_\rho{}^2$ were zero, the renormalized coupling constant of ρ to anything would be universal. Consequently, the constant defined by

$$\gamma_\rho=\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(0), \qquad (4.13)$$

should be universal; that is, we expect relations like

$$\gamma_\rho=\gamma_{\rho NN}F_1{}^\rho(0). \qquad (4.14)$$

This equality can be tested experimentally. From Eq.

(4.9) we see that

$$\gamma_\rho = i\gamma_{\rho\pi\pi} m_\rho \Gamma^{-1} F_\pi^{-1}(m_\rho^2). \tag{4.15}$$

Hence, if $F_\pi(m_\rho^2)$ is measured (as it may be in the reaction $e+\bar{e}\to\pi+\pi$, for example), γ_ρ can be found. Similarly, from Eq. (4.14) we expect

$$\gamma_\rho = i\gamma_{\rho NN}(m_\rho/\Gamma)(1/F_1^V(m_\rho^2)), \tag{4.16}$$

so if the nucleon form factors are also measured at m_ρ^2 [a somewhat harder experimental job than measuring $F_\pi(m_\rho^2)$] the two values of γ_ρ can be compared and the universality observed, if it exists. We shall see below that by extrapolation we can, at least approximately, avoid the necessity of measuring $F_1^V(m_\rho^2)$ directly.

There is one further interesting point to be made in connection with the topics under discussion in this section. From (4.6) we find

$$F_{\rho\pi\pi}(0) = \left(\frac{e}{e_0}\right)\left(\frac{\gamma_0}{\gamma_{\rho\pi\pi}}\right)\left(\frac{Z_{3\gamma}}{Z_{3\rho}}\right)^{\frac{1}{2}}\left(\frac{m_\rho^2}{m_{\rho 0}^2}\right). \tag{4.17}$$

To lowest order in e, we can replace $e/e_0 Z_{3\gamma}^{\frac{1}{2}}$ by unity. Next note that $F_{\rho\pi\pi}(0)=d_\rho(0)V_{\rho\pi\pi}(0)$ while $\gamma_{\rho\pi\pi} =\gamma_0 V_{\rho\pi\pi}^{-1}(0) Z_{1\rho}^{-1}(0) Z_{3\rho}^{\frac{1}{2}} Z_2 = (\gamma_0) Z_{3\rho}^{\frac{1}{2}} V_{\rho\pi\pi}^{-1}(0)$ by the Ward identity. Hence, we get

$$d_\rho(0) = \left(\frac{m_\rho^2}{m_{\rho 0}^2}\right)\left(\frac{1}{Z_{3\rho}}\right). \tag{4.18}$$

Thus we see that the bare mass $m_{\rho 0}^2$ of the vector meson is no more divergent than $Z_{3\rho}$.

We may now rewrite Eqs. (4.10) and (4.12), assuming $m_{\rho 0}^2$ and $m_{\omega 0}^2$ are infinite, as

$$F_1^V(s) = \frac{-m_\rho^2}{s-m_\rho^2}\left(\frac{\gamma_{\rho NN}}{\gamma_\rho}\right) F_1^\rho(s), \tag{4.19}$$

and

$$F_1^S(s) = \frac{-m_\omega^2}{s-m_\omega^2}\left(\frac{\gamma_{\omega NN}}{\gamma_\omega}\right) F_1^\omega(s). \tag{4.20}$$

In the region measured by electron-proton scattering experiments, $s<0$ and the F's are purely real. We know that $\mathrm{Re}(1/F_1^\rho)=1$ at $s=m_\rho^2$ and $\mathrm{Re}(1/F_1^\omega)=1$ at $s=m_\omega^2$. So we might try to approximate

$$F_1^{\rho,\omega}(s) \approx 1+(s-m_{\rho,\omega}^2)\left(\frac{1}{m_{\rho,\omega}^2}\right)\left(1-\frac{\gamma_{\rho,\omega}}{\gamma_{\rho,\omega NN}}\right),$$

in the $s<0$ region. Thus (4.19) and (4.20) become

$$F_1^V(s) \approx \left(\frac{\gamma_{\rho NN}}{\gamma_\rho}\right)\left(\frac{-m_\rho^2}{s-m_\rho^2}\right)+\left(1-\frac{\gamma_{\rho NN}}{\gamma_\rho}\right), \tag{4.21}$$

$$F_1^S(s) \approx \left(\frac{\gamma_{\omega NN}}{\gamma_\omega}\right)\left(\frac{-m_\omega^2}{s-m_\omega^2}\right)+\left(1-\frac{\gamma_{\omega NN}}{\gamma_\omega}\right). \tag{4.22}$$

Precisely this analysis of the experimental data has been made and yields[15]

$$\begin{aligned} m_\rho^2 &\approx 20\mu^2, \\ m_\omega^2 &\approx 9\mu^2, \\ \gamma_{\rho NN}/\gamma_\rho &\approx 1.2, \\ \gamma_{\omega NN}/\gamma_\omega &= 0.56. \end{aligned} \tag{4.23}$$

It is worth emphasizing again that Eqs. (4.21) and (4.22) look basically just like the "pole" contributions from a single vector meson intermediate state, even though such a "pole" does not actually exist. The "pole" type approximation used in (4.21) and (4.22) is only valid far away from the position of the supposed "pole."

V. VERTICES CONNECTED WITH π^0 DECAY

The close relationship between the photon and the ρ^0 and ω^0 mesons may be explored more generally. We have already seen that the ρ^0 and the isovector part of the photon are coupled to the same unrenormalized conserved current; the same is true of ω^0 and the isoscalar part of the photon. As a result, one may expect to be able to correlate other vertices involving ρ^0 or ω^0 and photons in the same way that the ρ^0 and ω^0 form factors were connected with electromagnetic form factors in Sec. IV. The most interesting such vertex is that involving one neutral pion and two vector particles. This vertex can be identified in such processes as the decays $\pi^0\to\gamma+\gamma$ and $\omega^0\to\pi^0+\gamma$, in poles in ρ^0 and ω^0 photoproduction, and in poles in reactions like $\pi+N\to\rho+N$, or $\pi+N\to\omega+N$.

Associated with each of the possible $\pi^0\to$ two-vector-particle vertices we can define an effective renormalized coupling constant, which is just the amplitude for the decay described by the vertex. Thus, we have constants $f_{\pi\rho\omega}$, $f_{\pi\rho\gamma}$, $f_{\pi\omega\gamma}$, $f_{\pi\gamma\gamma}$ associated with the four possible vertices and the four possible decays $\rho^0\to\omega^0+\pi$, $\rho^0\to\pi^0+\gamma$, $\omega^0\to\pi^0+\gamma$, and $\pi^0\to\gamma+\gamma$.

The basic assumption we shall use is again that the unrenormalized current operators for the ρ^0 and ω^0 are the same as those for the isovector and isoscalar parts of the photon, respectively. Thus,

$$\sqrt{3}\gamma_0 \mathfrak{J}_\mu^{\omega 0} = (e_0/2)\mathfrak{J}_\mu^{s},$$

and

$$\gamma_0 \mathfrak{J}_\mu^{\rho 0} = (e_0/2)\mathfrak{J}_\mu^{V}. \tag{5.1}$$

The ratio $\sqrt{3}$ between the ρ^0 and ω^0 currents is a consequence of unitary symmetry.[5] When we express our results in terms of renormalized constants, they are independent of unitary symmetry.

From Eq. (5.1), by the method used in Sec. IV, it is straightforward to relate matrix elements of the corre-

[15] R. Hofstadter and R. Herman, Phys. Rev. Letters **6**, 293 (1961).

sponding renormalized currents: we find

$$\langle n|j_\mu{}^S|m\rangle=\left(\frac{e_0}{2\sqrt{3}\gamma_0}\right)\left(\frac{Z_{3\omega^0}}{Z_{3\gamma}}\right)^{\frac{1}{2}}\left(\frac{s-m_{\omega0}{}^2}{s-m_\omega{}^2}\right)\langle n|j_\mu{}^{\omega^0}|m\rangle,$$

and (5.2)

$$\langle n|j_\mu{}^V|m\rangle=\left(\frac{e_0}{2\gamma_0}\right)\left(\frac{Z_{3\rho^0}}{Z_{3\gamma}}\right)^{\frac{1}{2}}\left(\frac{s-m_{\rho0}{}^2}{s-m_\rho{}^2}\right)\langle n|j_\mu{}^{\rho^0}|m\rangle.$$

Here, n, m are arbitrary states, and $s=-(P_n-P_m)^2$, where $P_{n,m}$ are the total 4-momenta of the states.

Now the photon will be treated to first order, so $e_0=e$ and $Z_{3\gamma}=1$. Furthermore, we have $\gamma_\rho=\gamma_0 Z_{3\rho}{}^{\frac{1}{2}}d_\rho(0)$ and $\gamma_\omega=\gamma_0 Z_{3\omega}{}^{\frac{1}{2}}d_\omega(0)$ by the Ward identity. [γ_ρ and γ_ω, remember, are defined, for example, by $\gamma_\rho=\gamma_{\rho NN}F_1{}^\rho(0)$ and may be thought of as the coupling constants that ρ and ω would have if their masses were zero.] Finally, we note (4.20), which tells us that $d_\rho(0)m_{\rho0}{}^2Z_{3\rho}=m_\rho{}^2$ and $d_\omega(0)m_{\omega0}{}^2Z_{3\omega}=m_\omega{}^2$. These observations, when applied to (5.2) result in

$$\langle n|j_\mu{}^S|m\rangle=\frac{e}{2\sqrt{3}\gamma_\omega}\left(\frac{-m_\omega{}^2}{s-m_\omega{}^2}\right)\langle n|j_\mu{}^{\omega^0}|m\rangle, \quad (5.3a)$$

and

$$\langle n|j_\mu{}^V|m\rangle=\frac{e}{2\gamma_\rho}\left(\frac{-m_\rho{}^2}{s-m_\rho{}^2}\right)\langle n|j_\mu{}^{\rho^0}|m\rangle, \quad (5.3b)$$

if we assume the bare masses $m_{\rho0}{}^2$ and $m_{\omega0}{}^2$ are infinite.

If n is taken to be the nucleon-antinucleon state, and m the vacuum, (5.3) reproduces (4.10) and (4.12).

Equations (5.3) may be applied, for example, to the $\pi^0\gamma\gamma$ vertex if we choose n to be a $\pi^0\gamma$ state and m to be the vacuum. The form factor for this vertex, as a function of the momentum of one of the photons, is $F_{\pi\gamma\gamma}(s)$ where

$$\frac{f_{\pi\gamma\gamma}F_{\pi\gamma\gamma}(s)}{(4k_2\omega)^{\frac{1}{2}}}\epsilon_{\alpha\beta\mu\gamma}(k_1)_\alpha(k_2)_\beta(e_2)_\gamma=\langle k_2e_2,q^{(-)}|j_\mu(0)|0\rangle, \quad (5.4)$$

and $s=-k_1{}^2$. From Eq. (5.3) we have

$$f_{\pi\gamma\gamma}F_{\pi\gamma\gamma}(s)=\frac{e}{2\gamma_\rho}\frac{-m_\rho{}^2}{s-m_\rho{}^2}f_{\rho\pi\gamma}F_{\rho\pi\gamma}(s)+\frac{e}{2\sqrt{3}\gamma_\omega}\frac{-m_\omega{}^2}{s-m_\omega{}^2}f_{\omega\pi\gamma}F_{\omega\pi\gamma}(s), \quad (5.5)$$

where $F_{\rho\pi\gamma}$ and $F_{\omega\pi\gamma}$ are the corresponding form factors for $\rho\rightarrow\pi+\gamma$ and $\omega\rightarrow\pi+\gamma$. In the same "pole" approximation used at the end of Sec. IV for the electromagnetic form factors, which we hope may be valid for values of s well below $m_\omega{}^2$, we may replace $F_{\rho\pi\gamma}(s)$ and $F_{\omega\pi\gamma}(s)$ by unity and obtain

$$f_{\pi\gamma\gamma}F_{\pi\gamma\gamma}(s)\approx\frac{e}{2\gamma_\rho}\left(\frac{-m_\rho{}^2}{s-m_\rho{}^2}\right)f_{\rho\pi\gamma}+\frac{e}{2\sqrt{3}\gamma_\omega}\left(\frac{-m_\omega{}^2}{s-m_\omega{}^2}\right)f_{\omega\pi\gamma}. \quad (5.6)$$

From (5.6) we can obtain a relationship between the π^0 decay rate and either the rate for $\omega^0\rightarrow\pi^0+\gamma$ or $\rho^0\rightarrow\pi^0+\gamma$. The first of these is presumably the dominant decay mode of the ω^0 if $m_\omega{}^2<9\mu^2$; otherwise the ω^0 decays primarily into π^+, π^- and π^0. First set $s=0$ in (5.6). This gives

$$f_{\pi\gamma\gamma}\approx\frac{e}{2}\left(\frac{f_{\rho\pi\gamma}}{\gamma_\rho}\right)+\frac{e}{2}\left(\frac{f_{\omega\pi\gamma}}{\sqrt{3}\gamma_\omega}\right). \quad (5.7)$$

Now note that in the vertex $\rho\pi\gamma$ the photon must couple through the isoscalar current, while in the vertex $\omega\pi\gamma$ it must couple through the isovector current. Therefore, exactly the same argument and approximations which led to (5.7) yield

$$f_{\rho\pi\gamma}\approx\frac{e}{2\sqrt{3}}\left(\frac{f_{\rho\pi\omega}}{\gamma_\omega}\right), \quad (5.8)$$

and

$$f_{\omega\pi\gamma}\approx\frac{e}{2}\left(\frac{f_{\rho\pi\omega}}{\gamma_\rho}\right). \quad (5.9)$$

Comparing these, we see that

$$\frac{f_{\rho\pi\gamma}}{\gamma_\rho}\approx\frac{f_{\omega\pi\gamma}}{\sqrt{3}\gamma_\omega}, \quad (5.10)$$

which, coupled with (5.7), gives

$$f_{\pi\gamma\gamma}\approx\frac{e}{\sqrt{3}\gamma_\omega}f_{\omega\pi\gamma}. \quad (5.11)$$

From this we find the ratio of the rate for $\omega^0\rightarrow\pi^0+\gamma$ to the π^0 decay rate to be

$$\frac{\Gamma(\omega^0\rightarrow\pi^0+\gamma)}{\Gamma(\pi^0\rightarrow\gamma+\gamma)}\approx\frac{1}{2}\left(\frac{m_\omega{}^2-\mu^2}{m_\omega\mu}\right)^3\left(\frac{\gamma_\omega}{e}\right)^2. \quad (5.12)$$

Returning to (5.6) and substituting (5.8) and (5.9), we find an expression for the amplitude for $\pi^0\rightarrow\gamma+\gamma$ where one of the γ's is virtual:

$$f_{\pi\gamma\gamma}F_{\pi\gamma\gamma}(s)\approx\frac{e^2}{4\sqrt{3}\gamma_\rho\gamma_\omega}f_{\rho\pi\omega}\left\{\frac{-m_\rho{}^2}{s-m_\rho{}^2}+\frac{-m_\omega{}^2}{s-m_\omega{}^2}\right\}. \quad (5.13)$$

By examining the mass-distribution of Dalitz pairs, as suggested by Berman and Geffen,[16] one can measure

[16] S. Berman and D. Geffen, Nuovo cimento 28, 1192 (1960).

the form factor in (5.13). Our crude approximation suggests a dependence on s of the form $(1+as)$ with a *positive* and of the order of $\frac{1}{2}(m_\rho^{-2}+m_\omega^{-2})$. We cannot see, within the framework of a single resonance approximation, how a could turn out negative, as suggested.[17] Preliminary experiments, however, do indicate a negative value[18]; if they continue to do so, then we must conclude that the two resonances do not dominate the form factor.

The $\omega\pi\gamma$ vertex also appears in the process $\gamma+p\rightarrow\omega^0+p$ through a pole term due to a single π^0 meson. The contribution of this pole term to the differential cross section for the ω^0 photoproduction is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}=6\left(\frac{m_N}{m_N^2-\mu^2}\right)^3\frac{g_{\pi NN}^2}{4\pi}\Gamma_{\omega\rightarrow\pi+\gamma}\frac{kq^2\omega_q^2\mu^2}{EE'(E+k)}\times\frac{(1-\beta_q\cos\vartheta)^2}{[2m_N^2-2EE'(1-\beta\beta'\cos\vartheta)-\mu^2]^2},\quad(5.14)$$

where k and q are the photon and ω^0 momenta, respectively, E and E' the initial and final nucleon energies, and β, β', and β_q the initial and final nucleon velocities and the ω velocity.

In the same way as the constant $f_{\omega\pi^0\gamma}$ appears in ω^0 photoproduction; the analogous constant $f_{\rho^0\pi^0\gamma}$, which is the amplitude for the decay $\rho^0\rightarrow\pi^0+\gamma$, may be measured in a π^0 pole term in ρ^0 photoproduction. The pole contribution to the cross section is the same as Eq. (5.14) with $f_{\omega\pi\gamma}$ replaced by $f_{\rho\pi\gamma}$ and m_ω replaced by m_ρ.

Inverting the roles of π^0 and ω^0 in ω^0 photoproduction, and π^0 and ρ^0 in ρ^0 photoproduction, we see that there are "pole" terms produced by ω^0 and ρ^0 contributing to π^0 photoproduction. As we have seen in the foregoing, of course, these are not true poles, since ω^0 and ρ^0 are unstable. Stated more precisely, there is a contribution to π^0 photoproduction from two and three pion exchanges; the existence of ρ^0 and ω^0 produces resonances in these systems and the over-all effect, if the resonances are narrow, is to make the contribution look as if they were "poles" due to ρ^0 and ω^0 if one is not right at the places where the "poles" should be. If these "poles" could be isolated experimentally, the constants $f_{\omega\pi^0\gamma}$ and $f_{\rho\pi^0\gamma}$ could again be measured.

Finally, the constant $f_{\pi\rho\omega}$ can be measured through another "pole" term in the reaction $\pi+N\rightarrow\rho+N$ or in the reaction $\pi+N\rightarrow\omega+N$. In the first case, the "pole" is due to a ω^0 meson; in the second case, to a ρ meson.

VI. CONTRIBUTION OF THE ρ MESON TO $\pi-N$ SCATTERING

In this section we translate into the language of vector mesons the work on the contribution of the 2π resonance to πN scattering, as given for instance by Bowcock *et al.*[19]

In the BCL paper, the s-wave charge-exchange πN scattering amplitude is written as the sum of two parts. The first represents the contribution of the 2π resonance. The other comes from all other singularities, which are treated as if they occurred at infinite mass. (How such an approximation can be justified, we have no idea; we are merely transcribing.)

The approximate formula for the difference of $I=\frac{1}{2}$ and $I=\frac{3}{2}$ s-wave scattering amplitudes is then

$$f_{s^{\frac{1}{2}}}-f_{s^{\frac{3}{2}}}=A\frac{m_N}{W}\frac{\omega}{k^2}\ln\left(1+\frac{4k^2}{m_\rho^2}\right)+B\omega,\qquad(6.1)$$

where we neglect the "magnetic" coupling of the ρ meson to the nucleon since we will work at low energies. The contribution of higher singularities is lumped into the arbitrary constant B. Here ω is the pion energy, k the momentum, and W the total energy in the c.m. system.

Now the constant A is proportional to the product of the pion and nucleon coupling constants to the ρ meson or 2π resonance. In our language, we have

$$A=3\frac{\gamma_{\rho NN}\gamma_{\rho\pi\pi}}{4\pi}.\qquad(6.2)$$

We know that $\gamma_{\rho\pi\pi}$ is connected directly with the width of the 2π resonance by Eq. (3.10). From their point of view, BCL obtain the same relation. Thus, A is proportional to the width Γ_ρ multiplied by $\gamma_{\rho NN}/\gamma_{\rho\pi\pi}$. If we have the ratio $\gamma_{\rho NN}/\gamma_{\rho\pi\pi}$ and if we believe the approximation, then we may fit Eq. (6.1) to the data on πN scattering and so measure Γ_ρ.

In the vector meson theory, we know that $\gamma_{\rho NN}/\gamma_{\rho\pi\pi}$ is just $F_{\rho\pi\pi}(0)/F_1{}^\rho(0)$. [See (4.13) and (4.14).] It must be approximately one. If we want to know it better, we can use the rough determination of $F_1{}^\rho(0)=\gamma_\rho/\gamma_{\rho NN}$ from the nucleon form factors in Eq. (4.23) and obtain

$$1/F_1{}^\rho(0)\approx1.2.\qquad(6.3)$$

But the determination of $F_{\rho\pi\pi}(0)=\gamma_\rho/\gamma_{\rho\pi\pi}$ must await a measurement of the pion electromagnetic form factor; for the time being, we can only try the approximation

$$F_{\rho\pi\pi}(0)\approx1,\quad\gamma_{\rho NN}/\gamma_{\rho\pi\pi}\approx1.2,\qquad(6.4)$$

and hope for the best.

From the point of view of BCL, the problem of determining $\gamma_{\rho NN}/\gamma_{\rho\pi\pi}$ is a dynamical one, but they dismiss previous attempts at calculating it as too unreliable and evaluate it instead *from experiment.* They look at the nucleon electromagnetic form factor and use what amounts to the same method as in the previous paragraph. They take the number 1.2 (or rather an earlier

[17] How-Sen Wong, Phys. Rev. 121, 289 (1961).
[18] N. Samios, Phys. Rev. 121, 275 (1961).
[19] Bowcock, Cottingham, and Lurié, Phys. Rev. Letters 5, 386 (1960); referred to hereafter as BCL.

version of it) from experiment, use a dispersion theory approximation in which essentially $F_{\rho\pi\pi}(0)\approx 1$, and obtain just the estimate of Eq. (6.4). A determination of $\gamma_{\rho\pi\pi}^2/4\pi$ from πN scattering then yields a result of the order of $\frac{2}{3}$ with their parameters or $\frac{1}{2}$ with ours.

Note that without vector mesons and conserved currents, there is no principle of universality as such. Yet it is not unreasonable in the work of BCL, Chew, etc., that $\gamma_{\rho\pi\pi}/\gamma_{\rho NN}$ should come out of the order unity, since $\gamma_{\rho\pi\pi}/\gamma_\rho$ is, for them as for us, the coefficient of the 2π resonance term in the pion electric form factor and $\gamma_{\rho NN}/\gamma_\rho$ is the corresponding coefficient in the nucleon isovector electric form factor. For the dispersion theorists, it is to be expected that the 2π resonance will dominate these form factors and therefore that the numbers should both be of the order unity.

VII. SUMMARY OF RESULTS

We have derived some detailed conclusions from Sakurai's theory of vector mesons coupled to conserved currents. Many of them are known from the dispersion-theoretic treatment of presumably "dynamical" resonances. Let us list the most important ones, using the ρ meson as an example. We shall take its bare mass infinite and we shall neglect the decay mode $\rho\rightarrow 4\pi$.

First of all, ρ appears as a resonance at $s=m_\rho^2$ in $\pi\pi$ scattering with $I=1$, $J=1$. We have

$$T_{\pi\pi}^{11}(s)=-12\pi s^{\frac{1}{2}}(s-4\mu^2)^{-\frac{3}{2}}\sin\delta(s)\exp i\delta(s)=N(s)/D(s), \quad (7.1)$$

where $N(s)$ and $D(s)$ have the following properties near $s=m_\rho^2$:

$$N(s)\approx\gamma_{\rho\pi\pi}^2/(s-m_\rho^2),\quad \mathrm{Re}D(s)\approx 1,\quad \mathrm{Im}D(s)\approx im_\rho\Gamma_\rho/(s-m_\rho^2), \quad (7.2)$$

with

$$\Gamma_\rho=\tfrac{1}{3}(\gamma_{\rho\pi\pi}^2/4\pi)(m_\rho^2-4\mu^2)^{\frac{3}{2}}m_\rho^{-2}. \quad (7.3)$$

We see that the position and width of the resonance determine m_ρ^2 and $\gamma_{\rho\pi\pi}^2/4\pi$. Away from $s=m_\rho^2$, N and D are analytic functions of s with branch cuts on the negative and positive s axes, respectively.

The "form factor" for the dissociation $\rho\rightarrow 2\pi$ is the product

$$F_{\rho\pi\pi}=V_{\rho\pi\pi}d_\rho, \quad (7.4)$$

where V is the sum of all proper vertex graphs and d is the propagator for ρ divided by the free propagator $(-s+m_\rho^2)^{-1}$. We have the relation

$$F_{\rho\pi\pi}=D^{-1}, \quad (7.5)$$

and so near $s=m_\rho^2$ the form factor has the behavior

$$F_{\rho\pi\pi}\approx\frac{s-m_\rho^2}{s-m_\rho^2+im_\rho\Gamma_\rho+\cdots}, \quad (7.6)$$

where the denominator is corrected by real terms $O[(s-m_\rho^2)^2]$ and imaginary terms $iO(s-m_\rho^2)$.

The ρ meson is coupled to the conserved isotopic spin current. Thus, at zero momentum transfer, it has a universal interaction with the isotopic spin **I**. As an example of a particle carrying isotopic spin, let us take the nucleon. We define the renormalized coupling constant $\gamma_{\rho NN}$ at momentum transfer m_ρ^2, and we consider the form factor F_1^ρ for the "electrical" coupling of ρ to the nucleon. Then the universality at zero momentum transfer is expressed by the rule

$$\gamma_\rho=\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(0)=\gamma_{\rho NN}F_1^\rho(0),\text{ etc.} \quad (7.7)$$

Just as all form factors would be normalized to unity for a stable meson, so they all have the form (7.6) near m_ρ^2 for the actual case of instability:

$$F_1^\rho\approx\frac{s-m_\rho^2}{s-m_\rho^2+im_\rho\Gamma_\rho+\cdots}=\left(1+\frac{im_\rho\Gamma_\rho+\cdots}{s-m_\rho^2}\right)^{-1}\quad\text{near } s=m_\rho^2. \quad (7.8)$$

Of course, ρ also has a "magnetic" interaction with the nucleon with a "strong magnetic" moment $\mu_{\rho NN}$ and a form factor F_2^ρ normalized exactly as in Eq. (7.8).

In the region of negative s, the F's are all purely real. The imaginary part of $1/F$, which varies so rapidly near $s=m_\rho^2$, is now gone; only the real part, which equals unity at m_ρ^2 and varies rather slowly, is present. We may therefore try to approximate each F in the region of small negative s by an expansion of the real part in a power series in $s-m_\rho^2$. If we keep only the first two terms, we may use (7.7) to obtain

$$F_1^\rho\approx 1+\frac{s-m_\rho^2}{m_\rho^2}\left(1-\frac{\gamma_\rho}{\gamma_{\rho NN}}\right)+\cdots\quad\text{for small negative } s. \quad (7.9)$$

Now the actual measurement of $\gamma_{\rho NN}$ can be discussed in connection with πN scattering. In Sec. VI, we have presented a very rough treatment of the low-energy s wave based on the exchange of a single ρ. A much better approach is to consider high-energy πN scattering at small momentum transfers and extrapolate to the "pole" at $t=m_\rho^2$. Such a procedure would determine $\gamma_{\rho NN}^2\gamma_{\rho\pi\pi}^2$ unambiguously if ρ were stable. Since the breadth of the ρ state is in fact of the order of a hundred Mev, we must take into account the instability, which turns the "pole" into what is merely a large lump. Since there is no longer a true pole in the crossed p-wave channel, the other partial waves in the crossed channel are not completely negligible even at the extrapolated value $t=m_\rho^2$. If we can somehow neglect or correct for the small contribution of the other crossed partial waves, then the extrapolation gives us effectively the p-wave amplitude for the annihilation $N+\bar{N}\rightarrow\pi+\pi$ in the neighborhood of the unphysical energy for which $t=m_\rho^2$. If $Q(t)$ is the $I=1$, $J=1$ annihilation amplitude, we have explicitly that

$$\frac{1}{\gamma_{\rho NN}\gamma_{\rho\pi\pi}}=\frac{d}{dt}\mathrm{Re}\frac{1}{Q(t)}\bigg|_{t=m_\rho^2}. \tag{7.10}$$

in analogy to Eq. (3.2).

If we measure $\gamma_{\rho\pi\pi}^2$ from $\pi\pi$ scattering and then $\gamma_{\rho NN}^2$ from πN scattering, we must still apply the correction factors $F_{\rho\pi\pi}(0)$, $F_1^\rho(0)$ in order to check the exact universality relation (7.7). These form factors are, however, directly related to electromagnetic ones, since the source currents for ρ and the isovector part of the electromagnetic field are essentially the same.

In *all* problems each matrix element for a virtual isovector γ ray (to lowest order in e) can be expressed in terms of the corresponding matrix element for a virtual ρ meson by multiplying by the factor

$$\frac{e}{2\gamma_\rho}\frac{-m_\rho^2}{s-m_\rho^2}. \tag{7.11}$$

The simplest example is provided by the relations between form factors $F_{\rho\pi\pi}$, etc., for the meson and the electromagnetic form factors F_π, etc.:

$$eF_\pi(s)=\frac{e}{2\gamma_\rho}\frac{-m_\rho^2}{s-m_\rho^2}\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(s)$$

$$=\frac{-m_\rho^2}{s-m_\rho^2}\frac{F_{\rho\pi\pi}(s)}{F_{\rho\pi\pi}(0)}e, \tag{7.12}$$

$$F_1^V(s)=\frac{-m_\rho^2}{s-m_\rho^2}\frac{F_1^\rho(s)}{F_1^\rho(0)}, \tag{7.13}$$

and so forth.

Thus in $e^++e^-\rightarrow\pi^++\pi^-$, we can measure the pion electric form factor at the resonance:

$$F_\pi(m_\rho^2)=im_\rho\Gamma_\rho^{-1}F_{\rho\pi\pi}^{-1}(0)=im_\rho\Gamma_\rho^{-1}\gamma_{\rho\pi\pi}/\gamma_\rho. \tag{7.14}$$

where we have used Eqs. (7.12), (7.6), and (7.7). For the nucleon electric form factor, if we make use of the approximation (7.9) for F_1^ρ, we may put

$$F_1^V(s)\approx\frac{\gamma_{\rho NN}}{\gamma_\rho}\left(\frac{-m_\rho^2}{s-m_\rho^2}\right)+\left(1-\frac{\gamma_{\rho NN}}{\gamma_\rho}\right), \tag{7.15}$$

for small $s<0$. Applying the correction factors so determined to the constants $\gamma_{\rho NN}$ and $\gamma_{\rho\pi\pi}$, we can really check the universality.

We have discussed another application of the conversion factor (7.11), namely the comparison of such vertices as $\pi^0\gamma\gamma$ (where one photon is isovector and the other isoscalar) and $\pi\rho^0\gamma$. The latter contributes important "poles" in the photoproduction processes $\gamma+p\rightarrow\pi^0+p$ and $\gamma+p\rightarrow\rho^0+p$. If we assume the $\pi^0\rho^0\gamma$ vertex varies slowly with the virtual mass of ρ^0, then we can use Eq. (7.11) to connect its value with the π^0 lifetime.

So far in our summary we have used the ρ meson as an example. But parallel results have been found, of course, for the ω. All told, the conserved vector current theory provides a simple and coherent picture of all the processes which these two resonances dominate.

The question naturally arises how much of the picture is identical with that obtained by dispersion-theoretic methods for dynamical resonances.

Evidently the dispersion relations are common to the two points of view. The only matters at issue are, so to speak, the boundary conditions on the dispersion relations, viz.: the number of subtractions, the values of subtraction constants, and the number and character of CDD poles, if any.

From the vector meson point of view, the most important result is universality. The constants $\gamma_{\rho\pi\pi}$, $\gamma_{\rho NN}$, etc., can all be measured in "pole" experiments and by decay widths. They should all be roughly equal; much more important, when they are corrected by the factors $F_{\rho\pi\pi}(0)$, $F_1^\rho(0)$, etc., all of which can be determined from electromagnetic form factors, the resulting quantities γ_ρ must all be exactly equal.

The same universality statement, however, is also true in the dynamical theory, where the vector mesons are viewed as dynamical resonances, and where no statement is made about the "current" to which those dynamical states are coupled. The fact that the resonances, for example the ρ, occur in states which can be reached by a photon, together with the fact that the photon is universally coupled, is sufficient to guarantee the universality of the ρ coupling, when the correction factors $F(0)$ are applied.

As we have seen in Sec. IV, the content of the universality statement for the ρ meson is that

$$\gamma_{\rho\pi\pi}F_{\rho\pi\pi}(0)=\gamma_{\rho NN}F_1^\rho(0)=\text{etc.} \tag{7.16}$$

Equivalent to this is the statement that the associated electromagnetic form factors have "poles"[20] with residues in the ratios

$$\frac{(s-m_\rho^2)F_\pi(s)\,|_{s=m_\rho^2}}{(s-m_\rho^2)F_1^V(s)\,|_{s=m_\rho^2}}=\frac{\gamma_{\rho\pi\pi}}{\gamma_{\rho NN}},\quad\text{etc.} \tag{7.17}$$

Equation (7.17) is however always true. For (if we ignore the instability of the ρ meson) the poles in the electromagnetic form factors are due to an intermediate state of one ρ meson. The contributions of these pole terms are just

$$\frac{A\gamma_{\rho\pi\pi}}{s-m_\rho^2};\quad\frac{A\gamma_{\rho NN}}{s-m_\rho^2};\quad\text{etc.,}$$

where A is the amplitude of a photon to make a ρ meson on its mass shell. The ratio of residues is thus just $\gamma_{\rho\pi\pi}/\gamma_{\rho NN}$ as required by Eq. (7.17).

[20] Actually, the form factors will not have true poles, but only bumps, because of the instability of the ρ meson; this is however not an essential complication and we have discussed it above.

Now, if the universality statement of the vector meson point of view can be obtained equally well without regarding the vector mesons as elementary particles, what remains as the difference between this viewpoint and the dynamical one? Perhaps the simplest answer is given in terms of the model field theory of Sec. II. With the mechanical mass of the vector meson set equal to infinity, the dynamical theory appears as a special case of the vector meson theory in which a particular relation holds involving the coupling constant of the particle and its physical mass. For this special case, the change in the phase shift between $s=4\mu^2$ and $s=\infty$ is zero instead of π and consequently the function $\bar{D}(s)$, which is given by

$$\bar{D}(s)=\exp\left[-\frac{s-s_0}{\pi}\int\frac{\delta(s')}{(s'-s_0)(s'-s)}ds'\right],$$

goes like a constant, rather than linearly in s, at infinity. In other words, the coefficient of s at infinity is just equal to zero.

In the model theory, setting this coefficient equal to zero is all that is necessary in order to convert the elementary particle theory of the resonance (with infinite bare mass) into the dynamical theory. In a complete theory, the number of conditions that are needed to define the "dynamical case" is presumably greater, but we can probably carry through the analogy and treat the dynamical theory as a special case by improving some conditions at infinity on the elementary particle theory. These conditions may well be needed for consistency of the dispersion relations.

To sum up, then, it would appear that everything we have concluded on the basis of the vector meson approach can be applied to the dynamical theory; the only practical differences will be that the masses and coupling constant, which for the vector meson theory are arbitrary parameters, become in the dynamical framework predictable constants, and that the high energy behavior becomes less singular.

For the ρ meson, the more singular behavior may be intolerable, since it is related to the unrenormalizability of the field theory for an elementary ρ. For other particles, like the ω meson, for which renormalizable field theories can be constructed, both hypotheses may be logically tenable—that of an elementary ω^0 corresponding to a CDD pole and that of a dynamical ω^0. In such a case, differences in high energy behavior may lead to the possibility of experimental discrimination between the two situations.

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