Mandelstam Representation for Dirac Potential Scattering

K. W. EDWARDS*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The scattering amplitude for Dirac potential scattering is shown to have a Mandelstam representation if the potential is a superposition of Yukawa potentials. This representation together with the unitarity condition is used to obtain an integral equation for the weight functions. The subtraction in energy makes it necessary to know the scattering amplitude's dependence on momentum transfer at zero energy in order to solve for the weight functions. Dispersion relations for the partial wave amplitudes are obtained.

I. INTRODUCTION

THE purpose of this paper is to point out that the work of Blankenbecler *et al.*[1] on the Mandelstam representation for Schrödinger potential scattering can readily be extended to the Dirac case. The amplitude for the scattering of a Dirac particle by a scalar potential that is a superposition of Yukawa potentials is shown to be an analytic function of complex momentum transfer outside of a cut along the real axis. The proof consists of demonstrating that every successive Born term has a cut which moves farther out in the momentum transfer plane while the exact remainder is analytic in a Lehmann ellipse of arbitrarily large size. It is shown that a knowledge of the potential along with the Mandelstam representation and unitarity is not sufficient to determine the weight functions of the representation, as the presence of a subtraction in the energy requires in addition a knowledge of the scattering amplitude as a function of momentum transfer at zero energy.

II. PROOF OF ANALYTICITY

The Dirac equation is $(E-H)\psi=V\psi$, where ψ is a four-component spinor wave function and

$$H=-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m$$

is the free-particle Hamiltonian. $\boldsymbol{\alpha}$ and β are the usual Dirac matrices. The solutions of the free-particle equation $(E-H)\phi=0$ are taken to be plane waves so that for a particle of momentum $\mathbf{k}$, $\phi_k=u(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$, where $u(\mathbf{k})$ is a four-component spinor normalized to $u^\dagger u=1$. The matrix element which describes the scattering of a particle from an initial state i to a final state f is given by $M_{fi}=(\phi_f,V\psi_i)$. If we define a T matrix by $M_{fi}=u_f^\dagger T(fi)u_i$, then the Dirac equation can be used to reduce T to the form $T=a+\beta b$, where a and b are scalar functions of energy and momentum transfer and

$$\begin{aligned}T(fi)&=\tilde{V}(\mathbf{k}_f-\mathbf{k}_i)+\int\frac{d\mathbf{q}}{(2\pi)^3}\tilde{V}(\mathbf{k}_f-\mathbf{q})\frac{1}{E-\boldsymbol{\alpha}\cdot\mathbf{q}-\beta m+i\epsilon}T(qi)\\&=\tilde{V}(\mathbf{k}_f-\mathbf{k}_i)-\int\frac{d\mathbf{q}}{(2\pi)^3}\tilde{V}(\mathbf{k}_f-\mathbf{q})\frac{E+\boldsymbol{\alpha}\cdot\mathbf{q}+\beta m}{q^2-k^2-i\epsilon}T(qi),\end{aligned}\tag{2.1}$$

where $\tilde{V}(\mathbf{q})$ is the Fourier transform of the potential. The formal solution to this equation is given by

$$T(fi)=\tilde{V}(\mathbf{k}_f-\mathbf{k}_i)+\int d\mathbf{q}_1d\mathbf{q}_2\,\tilde{V}(\mathbf{k}_f-\mathbf{q}_1)G(\mathbf{q}_1,\mathbf{q}_2;E)\tilde{V}(\mathbf{q}_2-\mathbf{k}_i),\tag{2.2}$$

where G is the Fourier transform of the exact Green's function:

$$G(\mathbf{q}_1,\mathbf{q}_2;E)=\int\frac{d\mathbf{r}}{(2\pi)^6}e^{-i\mathbf{q}_1\cdot\mathbf{r}}\frac{1}{E-H-V(r)+i\epsilon}e^{i\mathbf{q}_2\cdot\mathbf{r}}.$$

As in BGKT we shall assume that

$$V(r)=\int_\mu^\infty dx\,\sigma(x)\frac{e^{-xr}}{r}.$$

This potential has the advantage that its Fourier transform has a particularly useful form:

$$\tilde{V}(\mathbf{k}_f-\mathbf{k}_i)=4\pi\int_\mu^\infty dx\frac{\sigma(x)}{x^2+t},$$

where $t=(\mathbf{k}_f-\mathbf{k}_i)^2$. We see that $\tilde{V}(\mathbf{k}_f-\mathbf{k}_i)$ is analytic in the t-plane cut from $-\infty$ to $-\mu^2$. What remains to be proven is that $a-\tilde{V}$ and b are analytic in the t-plane cut along the real axis. Equation (2.2) for T has the same form as the scattering amplitude for nonrelativistic potential scattering. The fact that G now has the form

$$G=G_1+\beta mG_2+\boldsymbol{\alpha}\cdot\mathbf{q}_1G_3+\boldsymbol{\alpha}\cdot\mathbf{q}_2G_4\tag{2.3}$$

produces no essential complications. The Lehmann coordinates are introduced and one of the azimuthal angles is integrated over while the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$ is kept constant. The result is that T is analytic in a Lehmann ellipse in the complex $\cos\theta$ plane of semimajor axis $1+2\mu^2/k^2$. The absolute minimum for t, $4\mu^2$, is just the maximum value of t for which the

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* Now at Physics Dept., State University of Iowa, Iowa City, Iowa.

[1] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960); hereafter referred to as BGKT.

proof of ordinary dispersion relations given earlier by Khuri and Treiman[2] holds. If we iterate Eq. (2.1) $n-2$ times and then use Eq. (2.2), we have

$$T=\tilde{V}+T_2+\cdots+T_n+R_n, \tag{2.4}$$

where

$$T_n=(-1)^{n-1}\int\frac{d\mathbf{q}_1}{(2\pi)^3}\cdots\frac{d\mathbf{q}_{n-1}}{(2\pi)^3}\frac{4\pi\sigma(x_1)dx_1}{x_1^2+(\mathbf{k}_f-\mathbf{q}_1)^2}\frac{E+\beta m+\boldsymbol{\alpha}\cdot\mathbf{q}_1}{q_1^2-k^2-i\epsilon}\cdots\frac{E+\beta m+\boldsymbol{\alpha}\cdot\mathbf{q}_{n-1}}{q_{n-1}^2-k^2-i\epsilon}\frac{4\pi\sigma(x_n)dx_n}{x_n^2+(\mathbf{q}_{n-1}-\mathbf{k}_i)^2},$$

and R_n is the exact remainder,

$$R_n=\int d\mathbf{q}_1d\mathbf{q}_2\,T_n(E,\mathbf{k}_f\cdot\mathbf{q}_1)G(\mathbf{q}_1,\mathbf{q}_2;E)\,\tilde{V}(\mathbf{q}_2-\mathbf{k}_i).$$

Using an inductive proof we can show that T_n is analytic in the complex $\cos\theta$ plane outside of a cut running from $\cos\theta=1+n^2\mu^2/2k^2$ to ∞. Then the Lehmann technique can be used again to show that R_n is an analytic function of $\cos\theta$ inside an ellipse centered at the origin with semimajor axis $y_0=1+(n+1)^2\mu^2/2k^2$. Since this region is indefinitely enlarged as we iterate farther, we conclude that $a-\tilde{V}$ and b are analytic in the $\cos\theta$ plane cut from $1+2\mu^2/k^2$ to ∞, except for a possible essential singularity at infinity which we however suppose not to occur. The cut in the $t=2k^2(1-\cos\theta)$ plane then extends from $t=-4\mu^2$ to $-\infty$.

III. UNITARITY CONDITION

Having proved the analytic properties of $T(E,t)$ as a function of momentum transfer t, we now wish to write dispersion relations which display this analyticity. For these purposes we write T in the form

$$T=A(E,t)+\boldsymbol{\alpha}\cdot\mathbf{k}_f\boldsymbol{\alpha}\cdot\mathbf{k}_iB(E,t). \tag{3.1}$$

Similarly the T matrix for antiparticle scattering can be written

$$T^c=A^c+\boldsymbol{\alpha}\cdot\mathbf{k}_f\boldsymbol{\alpha}\cdot\mathbf{k}_iB^c.$$

Then from KT[2] we see that if we include one bound s state, then

$$\begin{aligned}
A(E,t)&=A(0,t)+\frac{E\Gamma_A}{E_B(E-E_B)}+\frac{E}{\pi}\int_m^\infty\frac{dE'}{E'}\left\{\frac{\mathrm{Im}A(E',t)}{E'-E-i\epsilon}-\frac{\mathrm{Im}A^c(E',t)}{E'+E}\right\},\\
B(E,t)&=\frac{\Gamma_B}{E-E_B}+\frac{1}{\pi}\int_m^\infty dE'\left\{\frac{\mathrm{Im}B(E',t)}{E'-E-i\epsilon}+\frac{\mathrm{Im}B^c(E',t)}{E'+E}\right\},\\
A^c(E,t)&=A(0,t)+\frac{E\Gamma_A}{E_B(E+E_B)}+\frac{E}{\pi}\int_m^\infty\frac{dE'}{E'}\left\{\frac{\mathrm{Im}A^c(E',t)}{E'-E-i\epsilon}-\frac{\mathrm{Im}A(E',t)}{E'+E}\right\},\\
B^c(E,t)&=-\frac{\Gamma_B}{E+E_B}+\frac{1}{\pi}\int_m^\infty dE'\left\{\frac{\mathrm{Im}B^c(E',t)}{E'-E-i\epsilon}+\frac{\mathrm{Im}B(E',t)}{E'+E}\right\},
\end{aligned}\tag{3.2}$$

where E_B is the energy of the bound state and Γ_A, Γ_B are the residues of the amplitudes at this bound state. The simplest form of $\mathrm{Im}A$, etc., that will display our proven analyticity properties and satisfy the condition that $S_{fi}=\delta_{fi}-2\pi i\delta(E_f-E_i)M_{fi}$ be unitary is

$$\begin{aligned}
\mathrm{Im}A(E,t)&=\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\rho_1(E,t')}{t'+t}+g_1(E),\\
\mathrm{Im}B(E,t)&=\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\rho_2(E,t')}{t'+t}+g_2(E),\\
\mathrm{Im}A^c(E,t)&=\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\rho_3(E,t')}{t'+t}+g_3(E),\\
\mathrm{Im}B^c(E,t)&=\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\rho_4(E,t')}{t'+t}+g_4(E).
\end{aligned}\tag{3.3}$$

Then T may be written in the form

$$\begin{aligned}
T(E,t)&=L_1(E)+\tilde{V}(t)+\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\mathcal{A}(E,t')+\phi(t')}{t+t'}+\boldsymbol{\alpha}\cdot\mathbf{k}_f\boldsymbol{\alpha}\cdot\mathbf{k}_i\left[L_2(E)+\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\mathcal{B}(E,t')}{t'+t}\right],\\
L_1&=\frac{E\Gamma_A}{E_B(E-E_B)}+\frac{E}{\pi}\int_m^\infty\frac{dE'}{E'}\times\left(\frac{g_1(E')}{E'-E-i\epsilon}-\frac{g_3(E')}{E'+E}\right),\\
A(0,t)&=\tilde{V}(t)+\int_{4\mu^2}^\infty\frac{dt'}{\pi}\frac{\phi(t')}{t'+t}\\
L_2&=\frac{\Gamma_B}{E-E_B}+\int_m^\infty\frac{dE'}{\pi}\left(\frac{g_2(E')}{E'-E-i\epsilon}+\frac{g_4(E')}{E'+E}\right),\\
\mathcal{A}(E,t)&=\frac{E}{\pi}\int_m^\infty\frac{dE'}{E'}\left(\frac{\rho_1(E',t)}{E'-E-i\epsilon}-\frac{\rho_3(E',t)}{E'+E}\right),\\
\mathcal{B}(E,t)&=\int_m^\infty\frac{dE'}{\pi}\left(\frac{\rho_2(E',t)}{E'-E-i\epsilon}+\frac{\rho_4(E',t)}{E'+E}\right).
\end{aligned}\tag{3.4}$$

[2] N. N. Khuri and S. B. Treiman, Phys. Rev. **109**, 198 (1958); hereafter referred to as KT.

The unitary requirement for S becomes the following condition for T:

$$i[T(fi)-T^{\dagger}(if)]=\frac{k}{8\pi}\int d\Omega_q\, T^{\dagger}(qf)(E+\beta m+\boldsymbol{\alpha}\cdot\mathbf{q})T(qi). \tag{3.5}$$

If we now substitute (3.4) in (3.5) and take the discontinuity across the cut in the t plane, we obtain equations for the weight functions of the form

$$\begin{aligned}\rho_i(E,t)=&\int_{\mu^2}^{\infty}\frac{dt_1}{\pi}\int_{\mu^2}^{\infty}\frac{dt_2}{\pi}K(k^2,t;t_1,t_2)\,\tilde{V}(t_1)\,\tilde{V}(t_2)X_i(t,t_1,t_2,E)\\&+\int_{\mu^2}^{\infty}\frac{dt_1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt_2}{\pi}K(k^2,t;t_1,t_2)\,\tilde{V}(t_1)\left\{\int_m^{\infty}dE'\sum_j A_{ij}(E,E',t,t_1,t_2)\rho_j(E',t_2)+\phi(t_2)\right\}Y_i(t,t_1,t_2,E)\\&+\int_{4\mu^2}^{\infty}\frac{dt_1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt_2}{\pi}K(k^2,t;t_1,t_2)\left\{\int_m^{\infty}dE_1\sum_j B_{ij}(E,E_1,t,t_1,t_2)\rho_j(E_1,t_1)+\phi(t_1)\right\}\\&\times\left\{\int_m^{\infty}dE_2\sum_j C_{ij}(E,E_2,t,t_1,t_2)\rho_j(E_2,t_2)+\phi(t_2)\right\}Z_i(t,t_1,t_2,E),\end{aligned} \tag{3.6}$$

where K is the same as in the Schrödinger case:

$$K(k^2,t;t_1,t_2)=\frac{\pi}{2}\frac{\Theta\{t-t_1-t_2-(t_1t_2/2k^2)-(t_1t_2)^{\frac{1}{2}}[16k^4+4k^2(t_1+t_2)+t_1t_2]^{\frac{1}{2}}/2k^2\}}{\{k^2[t-(t_1^{\frac{1}{2}}+t_2^{\frac{1}{2}})^2][t-(t_1^{\frac{1}{2}}-t_2^{\frac{1}{2}})^2]-tt_1t_2\}^{\frac{1}{2}}}.$$

The Θ function in K [$\Theta(x)=0$ for $x<0$ and $\theta(x)=1$ for $x>0$] determines the actual regions of integration for each term. In order for Θ to be nonvanishing for small k^2, $t>t_1t_2/k^2$; for large k^2, $t>(t_1^{\frac{1}{2}}+t_2^{\frac{1}{2}})^2$. Thus the first term is nonvanishing in a region of the k^2, t plane bounded by $t=\mu^4/k^2$ for small k^2 and $t=4\mu^2$ for large k^2; the second term, $4\mu^4/k^2$ and $9\mu^2$; the third term, $16\mu^4/k^2$ and $16\mu^2$. This is best illustrated by Fig. 1. As shown in the diagram there is a region of the k^2, t plane in which the ρ's are given exactly by the first term. This first term is determined entirely by the potential. If we now substitute this exact value for the ρ's into the second term we can generate a larger region over which the ρ's are known exactly if we know $\phi(t)$. By taking this new expression for the ρ's and substituting it back into the equation, we generate a still larger region over which ρ_i is known exactly. In this manner the entire k^2, t plane can be obtained through simultaneous iteration of the four weight functions. If the subtraction in energy was not present, a knowledge of the potential would be sufficient to determine the weight functions ρ_i exactly as in the Schrödinger case.

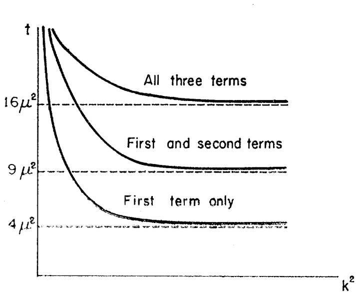

FIG. 1. Regions in which the K function is nonvanishing.

IV. DISPERSION RELATIONS FOR PARTIAL WAVE AMPLITUDES

The partial wave decomposition of the Dirac particle scattering amplitude is essentially that of pion-nucleon scattering. We follow the usual notation and introduce the conventional scattering amplitude $f_{fi}(\theta)$, a two-by-two matrix in Pauli spin space. The differential cross section is given by

$$d\sigma_{fi}/d\Omega=|f_{fi}(\theta)|^2, \tag{4.1}$$

f_{fi} can be related to M_{fi} by the fact that $d\sigma_{fi}/d\Omega=[E^2/(2\pi)^2]|M_{fi}|^2$, so that, if

$$f_{fi}=f_1+\boldsymbol{\sigma}\cdot\hat{k}_f\boldsymbol{\sigma}\cdot\hat{k}_if_2, \tag{4.2}$$

then

$$f_1=\frac{E+m}{4\pi}[A+(E-m)^2B],$$

$$f_2=\frac{E-m}{4\pi}[A+(E+m)^2B].$$

The partial wave expansions for f_1 and f_2 are given by

$$\begin{aligned}f_1&=\textstyle\sum_l(f_{l+}P_{l+1}'-f_{l-}P_{l-1}'),\\f_2&=\textstyle\sum_l(f_{l-}-f_{l+})P_l',\end{aligned} \tag{4.3}$$

where P_l are the Legendre polynomials, $P_l'=dP_l(z)/dz$, and $f_{l\pm}=k^{-1}\exp(i\delta_{l\pm})\sin\delta_{l\pm}$, $\delta_{l\pm}$ being the phase shift for a state with total angular momentum $j=l\pm\frac{1}{2}$ and parity $(-1)^l$. Inverting the equations for $f_{l\pm}$ we find that

$$f_{l\pm}=\int_{-1}^{1}\frac{dz}{2}[f_1P_l(z)+f_2P_{l\pm1}(z)]. \tag{4.4}$$

When we substitute our representation of A and B into f_1 and f_2 and substitute these into $f_{l\pm}$, the resulting integral is

$$J_l=\frac{1}{2}\int_{-1}^{1}dz\frac{P_l(z)}{t'+2k^2(1-z)}. \tag{4.5}$$

Thus the term involving the Fourier transform of the potential is cut in the k^2 plane from $-\frac{1}{4}\mu^2$ to $-\infty$, while the double integral term is cut from $-\mu^2$ to $-\infty$. Application of Cauchy's theorem to $f_{l\pm}$ considered as a function of the variable $s=k^2$ produces the dispersion relation

$$f_{l\pm}(s)=\sum_i\frac{c_i}{s-s_i}+\frac{1}{\pi}\int_0^\infty ds'\frac{\operatorname{Im}f_{l\pm}(s')}{s'-s-i\epsilon}+\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4}ds'\frac{\operatorname{Im}f_{l\pm}(s')}{s'+s}, \tag{4.6}$$

if it can be assumed that $\operatorname{Im}f_{l\pm}(s)\to0$ as $s\to\infty$.

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