$\rho=0.5$. If it is confirmed by further experimental work, the presence of electric quadrupole may affect the analysis of the data at lower energies and the limits of virtual π^0 lifetime as determined in reference 7. Moreover, it will help to understand the asymmetry in the angular distribution of π^+ photoproduction at high energy.

In a recent paper Minami,[21] by describing the proton Compton effect in terms of shadow scattering due to photoproduction of pions, predicts a strong and broad peak in the cross section corresponding to the second photopion resonance. Although the dispersion relations of (10) contain many theoretical uncertainties at these energies, we have applied them at 760 Mev. For $\rho=0$ and $\rho=0.25$ the differential cross-section in units of 10^{-32} cm^2/sterad is

$$\frac{d\sigma}{d\Omega_c}(90°)=12.75,\ 14.95,$$

correspondingly. This estimation agrees qualitatively with the predictions of reference 21 as well as with the value $(d\sigma/d\Omega_c)(90°)=(13.0\pm6.0)\times10^{-32}$ cm^2/sr reported from Frascati.[22] Since the resonant behavior seems to be clearly reflected on the proton Compton effect, further experimental work in that region will add very useful information regarding the character and the details of the second resonance.

In the present calculation the effect of the third photopion resonance has been entirely neglected; it is not difficult, however, to include it in the calculation of Sec. 2. This effect, as well as that of the $T=1$, $J=1$ pion-pion resonance (which enters through double pion photoproduction) is certainly negligible in the region of the first resonance, but could be very significant in that of the second. Even more significant might be the two-pion exchange contribution in a Low-type process. A practical form of this contribution is now under investigation.

ACKNOWLEDGMENTS

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Neutron Gas*

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We assume that the neutron-neutron potential is well-behaved and velocity-dependent. We can then apply perturbation theory to find the energy per particle of a neutron gas, in the range of Fermi wave numbers $0.5<k_f<2$ f^{-1}. The energy through first order is found in closed form, or by a single numerical integration. We use two different velocity-dependent potentials adjusted to fit observed nucleon-nucleon ^1S and ^1D phase shifts. In the range of densities $0.5<k_f<1$ f^{-1}, our two potentials give nearly the same energy/particle (within 0.5 Mev); our values tend to run an Mev below values found by Brueckner *et al.*, for the Gammel-Thaler potential. Wider divergences appear at higher densities. Our values, and Brueckner's are higher than those found by Salpeter by a semiempirical approach. A crude estimate of the second-order energy for our potentials indicates that perturbation theory converges rapidly in the density range considered. Our results suggest that at moderately low densities the energy/particle in a many-body system is insensitive to the shape or nonlocal character of the assumed two-body potential.

INTRODUCTION

THE neutron gas is a good proving ground for many-body calculations for two reasons. First, the neutron-neutron potentials are rather well known, since they must fit the accurately determined proton-proton phase shifts.[1] (We assume charge independence, and have not corrected for Coulomb effects.) Second, there is no experimental data on the neutron gas, to prejudice us for or against any special calculation.

We are not comparing with experiment, but we have two interesting comparisons to make. First, certain terms are neglected in any calculational method for a many-body problem. We need estimates to show that the neglected terms are small in comparison with the terms considered. Second, Bég[2] has discussed whether two potentials (one static and the other velocity-dependent) which give the same phase shifts in the

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[2] M. A. B. Bég, Ann. Phys. **13**, 110 (1961).

two-body problem will give identical results in the many-body problem. This identity should hold if the particles are well separated (i.e., separation distance greater than range of the two-body force). We shall use perturbation theory to calculate the energy of a neutron gas for three different well-behaved velocity-dependent potentials.[3] Brueckner, Gammel, and Kubis[4] have already given values of the energy calculated for the static Gammel-Thaler potential with repulsive core. Also, Sood and Moszkowski[5] have calculated the energy of a neutron gas at low densities. Comparisons among these different results should give a test of the equivalence of static and velocity-dependent potentials. (Of course, this comparison assumes that each calculation method is satisfactory. We cannot prove *both* the validity of the calculational methods and of the equivalence of different potentials by one comparison.)

We also wish to examine the question of the dependence of the energy of the neutron gas on the assumed shape of the well-behaved neutron-neutron potential. Recently, de Swart and Dullemond[6] and Levinger *et al.* have found independently that there is a shape-independent relation connecting the volume integral of a two-body potential with the two effective range parameters. Specifically, for a square-well potential of infinite scattering length, the volume integral $\int v(r)d^3r=(\pi^3/3)(\hbar^2/M)b$, where b is the intrinsic range. The coefficient $(\pi^3/3)$ varies by only several percent for the six different well-behaved central potentials considered: square, Gaussian, Yukawa, exponential, Hulthén, and Jost.[7] Since the first-order potential energy term is dominated by the ordinary integral, which is just proportional to the volume integral of the potential, we might suppose that the properties of the neutron gas would be insensitive to the potential shape. This supposition is supported by a comparison at moderate densities of the neutron gas energy for our two different shapes.

We would like to make it completely clear that we are *not* using a velocity-dependent potential as a device to calculate the properties of the neutron gas for a static potential with repulsive core. Instead, we take the attitude that there is no firm evidence as to whether the two-body potential is static, or velocity-dependent.[8] We choose, arbitrarily, the velocity-dependent case, and for this case calculate the energy of the neutron gas. *If* Bég's and de Swart's arguments apply we will find very nearly the same energy as others find assuming a static potential. Otherwise, we will have a different energy; but our assumption and result may have as much relation to physical reality as the calculations with a static repulsive core.

We shall follow Salpeter's approach[9] in using different methods for the neutron gas for three different density regions: $k_fb\ll1$; $k_fb\approx1$; and $k_fb\gg1$. Here k_f is the wave number at the Fermi surface, and b is the intrinsic range of the neutron-neutron force.

In the low density region, where $k_fb\ll1$, the phase-shift approximation should work well. That is, the properties of the many-body system are determined from the t matrix for an isolated pair. Salpeter has given the energy/particle for this region in analtyical form: there is no bound state.

At moderate densities ($k_fb\approx1$) the Pauli principle drastically changes the t matrix from its value for an isolated pair. Several different calculational methods have been used in this region. Brueckner *et al.*[4] calculate the t matrix in the neutron gas, and give numerical results for the energy/particle for $0.637\leq k_f\leq1.27$ f^{-1}. (They, and other authors, quote the radius parameter $r_0=1.912/k_f$; they treat $1.5\leq r_0\leq3.0$ f.) Sood and Moszkowski[5] treat the low-density edge of the moderate density region ($k_f\leq0.5$ f^{-1}) by correcting the free-pair t matrix for the effects of the Pauli principle. (To facilitate the calculation, they assume a separable potential.) They also include the collective pairing effect. Salpeter treats the moderate density region ($0.637\leq k_f\leq1.91$ f^{-1}) by extrapolating the Weizsäcker semiempirical mass formula for nuclear matter to the case of a pure neutron gas. His answers are sensitive to the assumed value of the symmetry energy coefficient, so he quotes four different results for different assumptions. He believes his case *d* (highest symmetry energy coefficient, and highest energy for the neutron gas) is the most probable, so we shall compare with this case.

In the high-density region ($k_fb\gg1$) two new effects are estimated by Salpeter. First, the high Fermi energy favors the formation of hyperons. Second, the potential energy is sensitive to the assumed behavior of the two-body potential at small distances. In particular, for Salpeter's assumption of a static repulsive core of radius 0.45 f, the potential energy becomes infinite for $r_0=0.25$ f, or $k_f=7.65$ f^{-1}. Salpeter uses the "cell method" to treat the region $3.2<k_f<7.65$ f^{-1}.

In this paper, we shall confine our calculations to the moderate density region $0.5\leq k_f\leq2.0$ f^{-1}. We note that in the low-density region where the phase shift approximation is valid, Bég's equivalence holds in a trivial manner. In the high-density region Bég's equivalence clearly fails: e.g., compare the infinite potential energy for a static repulsive core (for $k_f>7.65$ f^{-1}) with the finite very high value for our well-behaved velocity-dependent potential. We wish to find out how well Bég's equivalence holds in the moderate-density region.

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TABLE I. Singlet-even phase shifts.[a]

Energy	^1S $\hat{v}_\alpha$	^1S $\hat{v}_\beta$	^1S $\hat{v}_\gamma$	^1S Breit	^1D $\hat{v}_\alpha$	^1D $\hat{v}_\beta$	^1D $\hat{v}_\gamma$	^1D Breit
20	0.87	0.85	0.90	0.86	0.00	···	0.01	0.02
100	0.22	0.45	0.40	0.38	0.00	0.05	0.09	0.07
180	−0.08	0.17	0.14	0.14	0.01	0.13	0.11	0.12
260	−0.19	−0.03	−0.04	−0.03	0.01	0.18	0.10	0.16
340	−0.20	−0.19	−0.20	−0.20	−0.02	0.27	0.08	0.18

[a] Our potentials $\hat{v}_\alpha$, $\hat{v}_\beta$, and $\hat{v}_\gamma$ are given in Eqs. (1) through (6). The ^1S and ^1D phase shifts, in radians, calculated for these three potentials are taken from references 12 and 13; Coulomb effects are neglected. The phase shifts of Breit *et al.*, reference 1, are from their curves YLAM for proton-proton scattering. The laboratory energy is given in Mev.

In this paper, we shall use the perturbation expansion of the energy. We assume a well-behaved two-body potential so that we can use perturbation theory. It is assumed velocity-dependent for two reasons. First, for a well-behaved potential, velocity-dependence is needed to fit the ^1S phase shifts of Breit *et al.*[1] Second, a static potential $J(r)$, chosen so that $J(r)\leq 0$, could cause the same collapse in the neutron gas that is familiar in the nuclear matter problem. The saturation conditions are modified slightly, but in any case are not met by the approximately Serber character of the empirical two-body potential.

The purpose of this paper is to calculate the first-order term in the energy. It is known[10,11] that for nuclear forces the second-order term $E^{(2)}$ is significant at low densities (that is, it is comparable to the first-order term). On the other hand, due to the effects of the Pauli principle, $E^{(2)}$ is expected to be relatively small at moderate densities. We shall later make a crude estimate of $E^{(2)}$, which confirms these conclusions. We calculate the energy through first order for an arbitrary exchange mixture, and later specialize to the particular exchange mixture used by Brueckner *et al.* to facilitate comparison with their results. Numerical results are given for three different velocity-dependent potentials, but it is trivial to calculate the energy for any other assumptions concerning the shape or exchange character of the velocity-dependent potential.

THE TWO-BODY POTENTIAL

We choose first a velocity-dependent potential $\hat{v}_\alpha$ developed by Razavy *et al.*[12] with the form

$$\hat{v}_\alpha = -V_0 J_1(r) - (\lambda/M)\mathbf{p}\cdot J_2(r)\mathbf{p}. \tag{1}$$

Here $\mathbf{p}$ is the operator $-i\hbar$ grad.

$$\begin{aligned} J_1(r)=J_2(r)&=1, \quad r<b,\\ &=\tfrac{1}{2}, \quad r=b,\\ &=0, \quad r>b; \end{aligned} \tag{2}$$

$V_0=16.9$ Mev; $\lambda=-0.21$; $b=2.4$ f.

This potential fits the accepted low-energy parameters: effective range $=2.65$ f and scattering length $=-23.6$ f. Table I shows that $\hat{v}_\alpha$ gives ^1S phase shifts in rough agreement with Breit's values; but the calculated ^1D phase shifts are unsatisfactory, being much too small.

The potential $\hat{v}_\beta$ uses a combination[12] of square-well, delta-function, and Yukawa shapes:

$$\hat{v}_\beta = -V_0 J_1(r) - (\hbar^2\lambda/Mc)\delta(r-c) - (\lambda/M)\mathbf{p}\cdot J_2(r)\mathbf{p}, \tag{3}$$

$$\begin{aligned} V_0 J_1(r)&=51 \text{ Mev}, \quad r<b \qquad J_2(r)=1, \quad r<c,\\ &=-\text{OPEP}, \quad r>b; \qquad\quad =\tfrac{1}{2}, \quad r=c, \\ &\qquad\qquad\qquad\quad =0, \quad r>c; \end{aligned} \tag{4}$$

$$\text{OPEP}=-10.83\exp(-0.708\,r)/0.708\,r \text{ Mev},$$

$$b=1.6 \text{ f}, \quad c=0.5 \text{ f}, \quad \lambda=-1.64.$$

This potential has four adjustable parameters, and as shown in the table gives a satisfactory fit to the ^1S, and ^1D phase shifts from 20 to 340 Mev. [The one pion exchange potential (OPEP) used for $r>1.6$ f, does not contain adjustable parameters.] We have not determined the effective range parameters for this potential.

Other work fitting phase shifts with velocity-dependent two-body potentials has been done at Birmingham by Green (private communication) and at Louisiana State University by Rojo and Simmons.[13] Both groups used a form similar to Eq. (1):

$$\hat{v}_\gamma = -V_0 J_1(r) + (1/M)[p^2\omega(r) + \omega(r)p^2]. \tag{5}$$

We choose, with energies in Mev and lengths in f,

$$\begin{aligned} V_0 J_1(r) &= [1+2\omega(r)]\{112\exp(-1.4r)\\ &\qquad - (\hbar^2/M)[\omega'(r)]^2/[1+2\omega(r)]^2\}, \end{aligned} \tag{6}$$

$$\omega(r)=5\exp(-3.6\,r).$$

[It turns out that this complicated choice of $J_1(r)$ gives a simple form for an "effective potential" that can be used in a transformed Schrödinger equation.]

Table I shows that $\hat{v}_\gamma$ gives a good fit to Breit's ^1S phase shifts, and a fair fit to his ^1D phase shifts. This potential fits the low-energy parameters: effective range $=2.65$ f and scattering length $=-23.6$ f.

Each of these potentials is an example of a nonlocal potential, expanded in powers of the operator $\mathbf{p}$, with terms beyond p^2 being omitted. This expansion would be dubious at very high values of the relative momentum. However, for our present work with a neutron gas at moderate densities, we are interested only in moderate values of the relative momentum. We choose an expansion up to p^2 terms that fits the two-body data up to 340 Mev (lab system), and then use this expansion only for energies appreciably less than this value.

In this preliminary work, we make a rough guess as to the potential in (triplet) odd states: namely, we follow a simplified form of the potential used by Brueckner

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[13] O. Rojo, Ph.D. dissertation, Louisiana State University, 1961 (unpublished); L. M. Simmons, M. S. thesis, Louisiana State University, 1961 (unpublished).

et al. in which the attractive potential is weakly attractive in odd states (one-tenth that in even states, if the same range is chosen), and the repulsive core is the same in all states. For our potentials, this suggests taking the static term as weakly attractive in odd states, and taking the velocity-dependent terms as the same in all states. Of course, in a complete treatment one should include the tensor and spin-orbit forces in the odd states, and adjust the exchange character of the static and velocity-dependent terms in our central potentials to fit the observed odd phase shifts.

CALCULATION OF FIRST-ORDER ENERGY

We use perturbation theory [10,14] with the unperturbed Hamiltonian $\hat{H}_0=p^2/2M$, and the perturbation $\hat{v}$ given in the previous section. The energy through first order, denoted by E, is

$$E=\langle\Phi|\hat{H}_0|\Phi\rangle+\langle\Phi|\hat{v}|\Phi\rangle, \tag{7}$$

Φ is a determinant composed of plane-waves, so the first term is the usual $(\frac{3}{5})NT_f$, where N is the total member of particles, and $T_f=\hbar^2k_f^2/2M$ is the kinetic energy at the Fermi surface. [Note that for the neutron gas $k_f=1.912/r_0$, where the volume $\Omega=(\frac{4}{3})\pi r_0^3N=3\pi^2N/k_f$.]

The second term is conveniently calculated in four parts: the ordinary and the exchange integrals for the static and the velocity-dependent terms in the potential, respectively.

The simplest is $v_0(k_f)$, the ordinary integral for the static term. For each pair of neutrons, we have for potential $\hat{v}_\alpha$ of Eq. (1)

$$\langle\phi(\mathbf{r})|-V_0J_1(r)|\phi(\mathbf{r})\rangle=-(4\pi V_0/\Omega)\int J_1(r)r^2dr. \tag{8}$$

For a Serber force, there are $\frac{1}{8}N^2$ pairs interacting in even (spin-singlet) states. The ordinary static term per particle for a Serber force becomes

$$v_0(k_f)=-(V_0/6\pi)k_f^3\int J_1(r)r^2dr. \tag{9}$$

For a different exchange mixture, the static ordinary term contributes c_1v_0 where (Euler)

$$c_1=1+3V_-/V_+=1.3. \tag{10}$$

Here V_-/V_+ is the ratio of static forces (of the same shape and range) in odd and even states. Following Brueckner *et al.*, we give the numerical value in Eq. (10) for $V_-/V_+=0.1$.

The ordinary term $w_0(k_f)$ for the velocity-dependent term is also very easy to calculate. We write the wave function in terms of the relative coordinate $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and the relative wave number for a specified pair of neutrons

$$\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2). \tag{11}$$

The wave function ϕ is proportional to exp $(i\mathbf{k}\cdot\mathbf{r})$. Then, for $\hat{v}_\alpha$ and $\hat{v}_\beta$,

$$\langle\phi|(-\lambda/M)\mathbf{p}\cdot J_2(r)\mathbf{p}|\phi\rangle=-(4\pi\hbar^2\lambda/M\Omega)k^2\int J_2(r)r^2dr. \tag{12}$$

We have used $\mathbf{p}\phi=\hbar\mathbf{k}\phi$ for the $\mathbf{p}$ following $J_2(r)$. For the $\mathbf{p}$ preceding J_2, we have used the Hermitian property of the operator in performing the integration. In treating all pairs, we obtain the mean square average of the relative wave number which is

$$\langle k^2\rangle_{\rm av}=(\tfrac{3}{10})k_f^2. \tag{13}$$

Again considering $\frac{1}{8}N^2$ pairs, the ordinary term per particle for a Serber force becomes

$$w_0(k_f)=-(\lambda/20\pi)(\hbar^2/M)k_f^5\int J_2(r)r^2dr. \tag{14}$$

This equation holds also for a velocity-dependence of the form $\hat{v}_\gamma$ of Eq. (5) [replace $-\lambda J_2(r)$ by $2\omega(r)$].

For a different force mixture, we use $c_2w_0(k_f)$ where

$$c_2=1+3\lambda_-/\lambda_+=4. \tag{15}$$

The numerical value applies to a Wigner character for the velocity-dependent term ($\lambda_-=\lambda_+=\lambda$).

If $\lambda<0$ (as is needed to fit the ^1S phase shifts) and $c_1>0$, $c_2>0$, the ordinary terms and kinetic energy combine to give:

$$\tfrac{3}{10}\hbar^2k_f^2/M+c_1v_0(k_f)+c_2w_0k_f=a_2k_f^2-a_3k_f^3+a_5k_f^5 \tag{16}$$

with coefficients a_2, a_3, and a_5 each positive. Collapse of the neutron gas will not occur. Quantitative calculations, including exchange integrals are needed to find whether the system is bound.

The exchange term $v_e(k_f)$ for the static part of the potential is calculated for a given pair as follows,

$$\langle\phi(\mathbf{r})|-V_0J_1(r)|\phi(-\mathbf{r})\rangle=-(4\pi V_0/\Omega)\int J_1(r)(\sin2kr/2kr)r^2dr. \tag{17}$$

The usual method of calculation[15] involves performing the Fourier transform given by Eq. (17), and then averaging over the distribution function for k. Because of the complicated forms we shall use for $J_2(r)$ we follow the different route of first integrating over the distribution function for k, and then performing the integral over r. Using the same factors as in Eq. (9), we have,

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[15] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. of Phys. **3**, 241 (1958).

TABLE II. Some useful functions.[a]

y	$f_1(y)$	$f_2(y)$	$f_3(y)$	$f_4(y)$
0	0	0	0	0
1	0.0396	0.015	0.011	0.003
2	0.272	0.097	0.297	0.027
3	0.706	0.27	1.49	0.095
4	1.14	0.57	3.07	0.14
5	1.30	0.83	1.78	0.16
6	1.08	1.06	−3.65	0.15
7	0.60	1.19	−19.9	0.11
8	0.16	1.26	−31.1	0.047

[a] The functions $f_1(y)$, $f_2(y)$, $f_3(y)$, and $f_4(y)$ are given in closed form in Eqs. (19), (21), (24), and (26), respectively.

for a Serber force,

$$v_e(k_f)=-(V_0/2\pi)\int J_1(r)f_1(2k_fr)dr/r, \quad (18)$$

where the function $f_1(y)$ is defined by

$$f_1(y)=-(12/y^2)\sin y+(3/y-12/y^3)\cos y +3/y+12/y^3. \quad (19)$$

For square-well or delta-function shapes, the integral of Eq. (18) is easily given in closed form. For a square well of range b, we have

$$v_e(k_f)=-(V_0/2\pi)f_2(2k_fb), \quad (20)$$

$$f_2(y)=4\cos y/y^3+4\sin y/y^2+\cos y/y +\mathrm{Si}(y)-3/y-4/y^3. \quad (21)$$

For a different exchange mixture, v_e has a coefficient c_1' given by

$$c_1'=1-3V_-/V_+=0.7, \quad (22)$$

for $V_-/V_+=0.1$.

The exchange term $w_e(k_f)$ for the velocity-dependent part of $\hat{v}$ can also be calculated by two equivalent methods. Following the method used above for $v_e(k_f)$ we have, for a Serber force of form $\hat{v}_\alpha$ or $\hat{v}_\beta$,

$$w_e(k_f)=(\hbar^2\lambda/8M\pi)\int J_2(r)f_3(2k_fr)dr/r^3, \quad (23)$$

where the function $f_3(y)$ is defined by

$$f_3(y)=(-30+360/y^2)\sin y +(3y-144/y+360/y^3)\cos y -(36/y+360/y^3). \quad (24)$$

For $\hat{v}_\alpha$ or $\hat{v}_\beta$ with a square-well form for $J_2(r)$ the integral of Eq. (23) gives

$$w_e(k_f)=(\hbar^2\lambda k_f^2/2\pi M)f_4(2k_fc), \quad (25)$$

$$f_4(y)=72/y^5+12/y^3-72\cos y/y^5-72\sin y/y^4 +24\cos y/y^3+3\sin y/y^2. \quad (26)$$

The four functions given above are tabulated in Table II.

For the exponential form $\omega(r)=ve^{-\beta r}$ of $\hat{v}_\gamma$, the exchange integral

$$w_e(k_f)=(3\beta^2v\hbar^2/4\pi M)f_5(2k_f/\beta), \quad (27)$$

$$f_5(y)=\tfrac{1}{4}y-\tan^{-1}y+(1/y)[(1+\tfrac{1}{2}y^{-2})\ln(1+y^2)-\tfrac{1}{2}]. \quad (28)$$

Note from Tables III and IV that $w_e(k_f)$ of Eq. (23) has a sign opposite to the ordinary term $w_0(k_f)$ of Eq. (14). This sign difference occurs because in the operation $\langle\phi(\mathbf{r})|\mathbf{p}\cdot J_2(r)\mathbf{p}|\phi(\pm\mathbf{r})\rangle$, the gradient operator following J_2 gives a different sign when operating on $\phi(-\mathbf{r})$ for the exchange term than when operating on $\phi(\mathbf{r})$. However, for the form $\hat{v}_\gamma$, $w_0(k_f)$ and $w_e(k_f)$ have the same sign, since $p^2\phi(-\mathbf{r})=p^2\phi(\mathbf{r})$.

For a different force mixture, $w_e(k_f)$ has a coefficient,

$$c_2'=1-3\lambda_-/\lambda_+=-2 \quad (29)$$

for $\lambda_-/\lambda_+=1$.

Combining, the energy per particle, through first order, is given by

$$E/N=\tfrac{3}{5}(\hbar^2k_f^2/2M)+c_1v_0(k_f)+c_1'v_e(k_f) +c_2w_0(k_f)+c_2'w_e(k_f) \quad (30)$$

$$c_1=1.3,\quad c_1'=0.7,\quad c_2=4,\quad c_2'=-2. \quad (31)$$

The function $c_1v_0(k_f)$ is proportional to k_f^3 and is negative; $c_2w_0(k_f)$ is proportional to k_f^5 and is positive. The exchange integrals are equal in magnitude to the corresponding ordinary integrals at low density $[v_e(k_f)\approx v_0(k_f)$ and $|w_e(k_f)|\approx|w_0(k_f)|]$, but become much smaller at high density:

$$[|v_e|\ll|v_0| \quad \text{and} \quad |w_e|\ll|w_0|].$$

NUMERICAL RESULTS

The kinetic energy of the neutron gas has the simple expression

$$\tfrac{3}{5}(\hbar^2k_f^2/2M)=12.5k_f^2. \quad (32)$$

(Throughout this paper, the wave number k_f at the Fermi surface is given in f^{-1}, and the energy/particle is given in Mev.)

The ordinary integral for the static term $[v_0(k_f)$ of Eq. (9)] evaluated for the potentials $\hat{v}_\alpha$, $\hat{v}_\beta$, and $\hat{v}_\gamma$ of Eqs. (1), (3), and (5) respectively, is given in Table III. The ordinary integral for the velocity-dependent term [w_0 of Eq. (14)] for these three potentials is also given in Table III. Note the large spread in the values of $w_0(k_f)$.

The exchange integrals [v_e of Eqs. (18) and (20), and w_e of Eqs. (23) and (25)] were evaluated in closed form for square-well and delta-function shapes, and by

TABLE III. Ordinary integrals for potential energy.[a]

	Potential $\hat{v}_\alpha$	Potential $\hat{v}_\beta$	Potential $\hat{v}_\gamma$
Static $v_0(k_f)$	$-4.12\ k_f^3$	$-3.02\ k_f^3$	$-4.02\ k_f^3$
Vel.-dep. $w_0(k_f)$	$0.64\ k_f^5$	$0.045\ k_f^5$	$0.28\ k_f^5$

[a] The ordinary integral $v_0(k_f)$ due to the static term is given in Eq. (9); the ordinary integral $w_0(k_f)$ due to the velocity-dependent term is given in Eq. (14). The two-body potentials $\hat{v}_\alpha$, $\hat{v}_\beta$, and $\hat{v}_\gamma$ are given in Eqs. (1) through (6).

numerical integration on a desk computer for the other shapes. The results are given in Table IV.

We substitute the numerical results of Eq. (32), and Tables III and IV, in Eqs. (30) and (31) to find the energy/particle of a neutron gas vs density. The results of E/N for different values of k_f are given in Table V; we also include values taken from graphs of Brueckner *et al.*[4] and of Salpeter.[9]

At the lowest density considered, $k_f=0.5$ f^{-1}, we can also compare with Sood and Moszkowski's[5] calculations. Using the Gammel-Thaler potential, with an approximate correction for the effects of the Pauli principle, they obtain 1.6 Mev/particle. They also give two different estimates of the gap due to the collective pairing energy: this lowers E/N to either 1.0 or 1.3 Mev/particle. At this density our three potentials each give 2.3 Mev/particle (within 0.1 Mev) which is about a Mev higher than the Sood-Moszkowski value. All these results are considerably higher than Salpeter's semi-empirical value of 0.3 Mev. The difference between ours and the Sood-Moszkowski value is in part due to the second-order correction to our result, estimated in the next section.

At k_f of 0.637 or 0.75 f^{-1}, our results for three different potentials continue to agree reasonably well with each other, supporting the arguments for shape-independence of the binding energy at moderate density ($k_f b<2$, where b is the intrinsic range). The Salpeter value seems definitely low, as compared to the others.

At k_f around 1 f^{-1}, we see a definite divergence among the results of our three different potentials. This divergence is not unexpected, since, for example, the ^1D phase shift is becoming appreciable for $k_f b>2$, and our potential $\hat{v}_\alpha$ seriously underestimates the ^1D phase shift in the two-nucleon problem. We should disregard the results for $\hat{v}_\alpha$, and take some sort of weighted mean of the results for $\hat{v}_\beta$ and $\hat{v}_\gamma$. This weighted mean seems to be below the values of Brueckner *et al.*

At the highest density considered, $k_f=2$f^{-1}, we see a divergence among our values for $\hat{v}_\beta$ and $\hat{v}_\gamma$: Shape-independence is no longer valid. As discussed in the Introduction, we would expect to see significant differences also between calculations for a velocity-dependent potential and calculations for a static repulsive core.

TABLE IV. Exchange integrals.[a]

Wave number	Potential $\hat{v}_\alpha$		Potential $\hat{v}_\beta$		Potential $\hat{v}_\gamma$	
k_f	$v_e(k_f)$	$w_e(k_f)$	$v_e(k_f)$	$w_e(k_f)$	$v_e(k_f)$	$w_e(k_f)$
0.5	−0.4	−0.02	−0.3	−0.001	−0.4	+0.008
0.75	−1.2	−0.12	−0.6	−0.006	−1.0	+0.06
1.0	−2.1	−0.21	−1.2	−0.04	−1.7	+0.29
1.5	−3.2	−0.31	−1.5	−0.29	−2.8	+1.3
2.0	−3.5	−0.06	+2.7	−1.2	−3.7	+4.1

[a] k_f is the wave-number at the Fermi surface, in f^{-1}. The exchange integral $v_e(k_f)$ for the static term is given by Eqs. (18) through (21); the exchange integral $w_e(k_f)$ for the velocity-dependent term is given by Eqs. (23) through (28). The two-body potentials $\hat{v}_\alpha$, $\hat{v}_\beta$, and $\hat{v}_\gamma$ are given in Eqs. (1) through (6).

TABLE V. Energy/particle for a neutron gas.[a]

k_f	Our results for potential $\hat{v}_\alpha$	$\hat{v}_\beta$	$\hat{v}_\gamma$	Brueckner *et al.*	Salpeter
0.5	2.3	2.4	2.2	···	0.3
0.637	3.5	3.8	3.4	4	0.5
0.75	4.8	5.0	4.2	5	0.8
1.0	8.7	8.1	6.8	8	2
1.27	16.3	12.0	10.1	12.5	4
1.5	27.8	15.8	14.4	···	7
2.0	86.8	28.8	33.2	···	···

[a] The wave number at the Fermi surface, k_f, is given in f^{-1}. All energies are given in Mev/particle. Our potentials $\hat{v}_\alpha$, $\hat{v}_\beta$, and $\hat{v}_\gamma$ are given in Eqs. (1) through (6); the energies are through first order in perturbation theory. The values of Brueckner *et al.* are from their graph, reference 4; the values of Salpeter are from his curve *d* in reference 9.

It is very easy to study other force mixtures. For instance, at $k_f=1$ f^{-1}, if we use an exact Serber mixture for the static term in $\hat{v}_\beta$ and keep a Wigner mixture for the velocity-dependent term, we have $c_1=c_1'=1$, $c_2=4$, $c_2'=-2$; then $E/N=8.6$ Mev (cf. 8.1 Mev in Table V). The reason for the 0.5 Mev/particle increase is simple: we are now considering zero static force in a relative p-state rather than the weak attraction used in Table V ($c_1=1.3$; $c_1'=0.7$).

Of course, E/N is independent of the exchange mixture at very low k_f. At $k_f\approx 2$ f^{-1}, the dependence on the exchange mixtures is greater than that shown in the above paragraph.

DISCUSSION

We shall make a *crude* estimate of the second-order energy $E^{(2)}$ for our potential $\hat{v}_\alpha$ in order to consider the convergence of the Rayleigh-Schrödinger perturbation series as applied to the neutron gas problem. This calculation of $E^{(2)}$ is crude in three respects. First, only the central force, is considered. Noncentral $\mathbf{L}\cdot\mathbf{S}$ and tensor forces give zero contribution to the first-order energy but may be important for $E^{(2)}$. Second, we calculate only the ordinary term for $E^{(2)}$. In this calculation, we further neglect the velocity-dependent term and replace the square-well by an equivalent Gaussian potential [see reference 14, Table VIII]

$$-V_0 J_1(r)\approx -V_0\exp(-r^2/\beta^2);\quad \beta=1.74\text{ f};\quad V_0=35\text{ Mev}. \tag{33}$$

With these approximations, we merely need to recalculate factors for a neutron gas, to use Euler's calculation of $E^{(2)}$ for nuclear matter. For a neutron gas, we find

$$E^{(2)}/N=-(c_3/2^5\times 5\pi)(M\beta^2/\hbar^2)g(x). \tag{34}$$

The coefficient c_3 is given for a Serber force by $c_3=(V_0)^2$. Here $x=k_f\beta$ and the function $g(x)$ is tabulated by Euler[10] and Levinger *et al.*[14]

Substituting V_0 and β from Eq. (33),

$$E^{(2)}/N=-0.17g(1.74k_f). \tag{35}$$

TABLE VI. Estimate of second-order energy.[a]

k_f	$E^{(2)}/v_0(k_f)$
$k_f b \ll 1$	$3.8\ k_f^3/4.2\ k_f^3 = 90\%$
0.5	$0.17/0.5 = 35\%$
1.0	$0.41/4.2 = 10\%$
2.0	$0.5/34 = 2\%$

[a] k_f, the wave number at the Fermi surface, is given in f^{-1}; b is the intrinsic range. The second-order energy $E^{(2)}$ is estimated crudely in Eq. (35). $v_0(k_f)$ is the ordinary integral for the potential energy, due to the static term: see values in Table III. The absolute values of the energies are given in Mev/particle.

We follow Swiatecki[11] in using $E^{(2)}/\langle v \rangle$ as an estimate of the convergence of the perturbation series. We take the expectation value of the potential $\langle v \rangle \approx v_0(k_f)$ from Table III. $E^{(2)}/N$ is given by Eq. (35). Table VI confirms Swiatecki's result for the nuclear matter problem: perturbation theory converges poorly at low density, and well at high density.

The value $E^{(2)} \approx -0.3$ Mev/particle gives a *crude* estimate of the error of our first order calculation of E/N given in Table V. Since the kinetic energy is large, $(\langle T \rangle \gg |\langle v \rangle|)$ the percentage error in E/N is small. Note that this situation for the neutron gas is much more favorable for our perturbation-theory calculation than in the nuclear matter problem. In the latter case $\langle T \rangle \approx |\langle v \rangle|$ so that E/A is the result of near cancellation of two large terms.

We further note that our use of a velocity-dependent potential gives a more rapidly converging perturbation theory expansion than does use of the Gammel-Thaler potential. Levinger *et al.*[14] found that $E^{(2)}/A$ consisted of the sum of three different singlet terms, having value at $k_f = 1.4$ f^{-1} of $+10$, -7, and $+2$ Mev/particle respectively. These numbers (even when reduced by the factor $\frac{2}{3}$ appropriate to the neutron gas problem) are much larger than $E^{(2)}/N$ given in Table VI.

Table V shows that our two velocity-dependent potentials $\hat{v}_\beta$ and $\hat{v}_\gamma$ give fair agreement for the value of the energy of the neutron gas for the range $0.5 < k_f < 1$ f^{-1}. This agreement at moderate densities supports the belief that the properties of the many-body system are insensitive to the shape[6] of the two-body potential, so long as it reproduces the moderate energy phase shifts. The results for $\hat{v}_\beta$ and $\hat{v}_\gamma$ seem below those of Brueckner *et al.*, particularly when we consider that including the second-order energy would lower our values further. This difference may be due to the dependence of the energy of the many-body system on the local vs nonlocal character[2] of the two-body force. On the other hand, the difference may be due to the different calculational methods used.

From the variation principle we know that *if* some specified well-behaved potential is the "true" two-body potential, then the energy calculated through first order represents an upper limit on the true energy of the many-body system. (Of course, we are neglecting the possibility of many-body forces). It is generally believed that the energy calculated by the phase-shift approximation represents a lower limit for the energy. We might then be able to bracket the energy between these two limits. The supposition as to the phase shift approximation seems very reasonable on physical grounds, but we do not know if it has been proved.

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