Charged Boson Gas*

LESLIE L. FOLDY
Case Institute of Technology, Cleveland, Ohio
(Received June 28, 1961)

The ground state energy and elementary excitations of a charged gas of bosons at high densities are examined by use of the method developed by Bogoliubov for boson gases. It is conjectured, but not herein established, that this method yields exact results in the high-density limit analogous to those obtained by Gell-Mann and Brueckner, and by Sawada, in the corresponding case of a charged fermion gas. The ground state energy is essentially correlation energy, and is therefore negative, and its magnitude varies as the one-fourth power of the density at high densities. The elementary excitations have for low momenta the energy appropriate to plasma waves, and for high momenta the energy appropriate to single-particle excitation. There is therefore an energy gap, suggesting that the gas is both a superfluid and a superconductor at low temperatures. At low densities the behavior of a charged gas is independent of statistics; hence, such a gap must disappear as the system is expanded.

I

RECENTLY much attention has been devoted to the study of the low-lying states of systems composed of many identical particles in view of applications to such physical systems as superfluids, superconductors, normal metals, and to the nuclear many-body problem.[1] While many of these investigations have been directed towards models which may be considered representative of real physical situations such as those described above, there are also virtues in studying certain systems for which no analogous physical system is known in view of the insight such investigations yield into the behavior of many-body systems in general. It may be remarked that in the case where the forces between the particles are of a Coulomb character, one is dealing with a particularly simple situation in that the ground state of the system is characterized by a single parameter, namely the density of the system, or more precisely by the ratio of the mean particle separation to the Bohr radius, since the potential in this case has no intrinsic range. The electron gas, in particular, has been studied[2] because of its application to electrons in metals, but the charged boson gas seems to have been largely neglected since no known system corresponds to it. Schafroth,[3] however, made an exploratory survey of the properties of such a system on the basis that it may be a suitable model for a superconductor. His investigation is based on the assumption of an ideal gas (which corresponds to a self-consistent field approximation), and primary attention was directed to the phenomena associated with the Bose-Einstein condensation. In particular, he found that for the ground state the system has zero energy in this approximation since there is neither a Fermi energy nor an exchange energy (at least at high densities) for a charged Bose gas.

In studying Bogoliubov's method[4] as applied to a low-density system of bosons interacting with short-range forces, the author noted that the basic validity condition for the approximation could also be satisfied when the system was at sufficiently high densities that there were many particles in a sphere whose radius is the range of the force. It was, therefore, natural to examine the case of the long-range Coulomb force. One finds here that the validity condition is satisfied when the density is so large that there are many particles within a sphere whose radius is the Bohr radius. In this situation there is a qualitative change from the behavior of a free gas since the Coulomb force leads to an irregular perturbation on the free system. Application of the Bogoliubov method leads to the result that the ground state of the system has a negative correlation energy whose magnitude increases with the one-fourth power of the density at high densities. Perhaps more interesting is the fact that the elementary excitations of the system have, for small momenta, energies characteristic of plasma oscillations (or waves) which pass over smoothly for large momenta to the energies characteristic of single particle excitation. Since one would intuitively expect such a character for the excitations of a dense charged gas, one is led to conjecture that the Bogoliubov method leads to the exact solutions in the high-density limit. A similar conjecture would follow from the similarity of the "diagrams" which are taken into account in the Bogoliubov method for the Bose gas and those taken into account by Gell-Mann and Brueckner, and Sawada[2] in their exact calculation of the correlation energy of a charged fermion gas in the high-density limit. Further investigation is planned to determine if this conjecture is in fact true, but the present results are published without this verification having yet been performed.

If in fact the Bogoliubov method does yield a correct picture of the high-density charged boson gas, then

* Work supported in part by the U. S. Atomic Energy Commission.

[1] For general reference, see *The Many Body Problem* edited by C. DeWitt (John Wiley & Sons, New York, 1959).

[2] M. Gell-Mann and K. Brueckner, Phys. Rev. **106**, 364 (1957); K. Sawada, Phys. Rev. **106**, 372 (1957). A more comprehensive review may be found in reference 1.

[3] M. R. Schafroth, Phys. Rev. **100**, 463 (1955), also J. M. Blatt and S. T. Butler, *ibid.* **100**, 476 (1955).

[4] N. N. Bogoliubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).

since the excitation spectrum has a finite energy gap given by the plasma frequency, one would anticipate that such a gas behaves both as a superconductor and as a superfluid at sufficiently low temperatures. In this case it would provide one of the simplest models on which these peculiar phenomena may be quantitatively studied. Without further ado, we present a summary of the Bogoliubov method applied to such a charged boson system.

II

We consider a gas of spinless bosons each with a charge e contained in a volume Ω (periodic boundary conditions) with a uniform rigid background canceling charge distribution. The Hamiltonian in second-quantized representation for the system is then

$$H=\sum_{\mathbf{k}} t_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+\frac{1}{2\Omega}\sum_{\mathbf{k}''}\sum_{\mathbf{k}'}\sum_{\mathbf{k}}{}' g_k a_{\mathbf{k}''-\mathbf{k}}^\dagger a_{\mathbf{k}'+\mathbf{k}}^\dagger a_{\mathbf{k}''} a_{\mathbf{k}'}, \quad (1)$$

where

$$t_k=\hbar^2k^2/2m, \quad g_k=4\pi e^2/k^2,$$

and $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ are creation and destruction operators for particles of momentum $\hbar\mathbf{k}$, satisfying the usual commutation relations for bosons. The prime on the summation indicates that the term with $k=0$ is to be omitted; the absence of this term is the consequence of the assumed background charge. The approximations of Bogoliubov are then: (1) the dropping of those terms in the second sum in the above Hamiltonian which contain fewer than two creation or destruction operators for particles of momentum zero; (2) the replacement of a_0 and $a_0^\dagger$ in the remaining terms by the c numbers $N_0^{\frac{1}{2}}$, where N_0 is the mean occupation number of the zero-momentum state and is assumed to be very large.

Bogoliubov has discussed these approximations and given arguments to support their validity in the case where the system contains a large number of particles, and the major fraction of the particles are contained in the state of zero momentum. We shall find *a posteriori* that the latter condition is satisfied in our case in the *high-density limit.* The second approximation above forfeits the constancy in number of particles so that the total number of particles N is to be identified with

$$N=N_0+\langle\textstyle\sum_{\mathbf{k}}{}'\, a_k^\dagger a_k\rangle, \quad (2)$$

where the second term represents the expectation value of the number of particles in states other than the zero-momentum state.

With these approximations the Hamiltonian takes the form

$$H=\textstyle\sum_{\mathbf{k}}{}'[(t_k+n_0g_k)a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+\frac{1}{2}n_0g_k(a_{\mathbf{k}}a_{-\mathbf{k}}+a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)], \quad (3)$$

where

$$n_0=N_0/\Omega. \quad (4)$$

This Hamiltonian is diagonalized by the canonical transformation,

$$a_{\mathbf{k}}=c_k\alpha_{\mathbf{k}}-s_k\alpha_{-\mathbf{k}}^\dagger, \quad (5)$$

where

$$s_k=[(t_k+n_0g_k-\epsilon_k)/2\epsilon_k]^{\frac{1}{2}}, \quad (6)$$

$$c_k=[(t_k+n_0g_k+\epsilon_k)/2\epsilon_k]^{\frac{1}{2}}, \quad (7)$$

$$\epsilon_k=[2n_0g_kt_k+t_k^2]^{\frac{1}{2}}=\hbar[\omega_p^2+\hbar^2k^4/4m^2]^{\frac{1}{2}}, \quad (8)$$

with

$$\omega_p=(4\pi n_0e^2/m)^{\frac{1}{2}}, \quad (9)$$

the plasma frequency of the system, at least insofar as the actual density,

$$n=N/\Omega, \quad (10)$$

can be approximated by n_0.

Under this transformation, the Hamiltonian then takes the form

$$H=U_0+\textstyle\sum_{\mathbf{k}}{}'\ \epsilon_k\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}, \quad (11)$$

where the ground-state energy of the system U_0 is given by

$$U_0=\sum_{\mathbf{k}}{}'(\epsilon_k-t_k-n_0g_k)=\frac{\Omega}{2\pi^2}\int_0^\infty(\epsilon_k-t_k-n_0g_k)k^2dk; \quad (12)$$

here the last step follows on allowing the volume Ω to become infinitely large while maintaining n fixed.

Before discussing the transformed Hamiltonian, we derive Bogoliubov's validity criterion by calculating the expectation value of the number of particles in states of nonzero momentum:

$$(N-N_0)/N_0=(n-n_0)/n_0=\frac{1}{2\pi^2n_0}\int_0^\infty s_k^2k^2dk. \quad (13)$$

Transforming the variable of integration to

$$\xi=(\hbar^2/4\pi mn_0e^2)^{\frac{1}{4}}k,$$

one obtains

$$(n-n_0)/n_0=Qr_{s0}^{\frac{3}{4}}, \quad (14)$$

where

$$r_{s0}=(3/4\pi)^{\frac{1}{3}}me^2/(\hbar^2n_0^{\frac{1}{3}}), \quad (15)$$

and[5]

$$Q=\frac{1}{\pi}\left(\frac{1}{3}\right)^{\frac{1}{4}}\int_0^\infty\left[\frac{\xi^4+2}{(\xi^4+4)^{\frac{1}{2}}}-\xi^2\right]d\xi=0.2114. \quad (16)$$

Thus we see that most of the particles are in the zero-momentum state at high densities (small r_{s0}) and that the fraction in this state approaches unity in the high density limit. The rms fluctuation in the number

[5] The precise values of the constants Q and S are

$$Q=\frac{1}{3\pi}(\tfrac{4}{3})^{\frac{1}{4}}F(\pi/2,1/\sqrt{2}),$$

$$S=\frac{32}{5\pi}(\tfrac{3}{4})^{\frac{1}{4}}[F(\pi/2,1/\sqrt{2})-2K(\pi/2,1/\sqrt{2})],$$

where F and K are complete elliptic integrals of the first and second kinds, respectively.

of particles in states of nonzero momentum is easily determined to be of the same order as the mean occupation number of these states. This indicates that the validity criterion is not upset by fluctuations and at the same time shows why occupation numbers of states of low momentum, which can have a large average value for a large system, nevertheless cannot also be treated classically as c numbers.

Turning now to the ground state energy, one finds on making the same transformation of the variable of integration in (12) that the energy per particle in rydbergs is given by

$$u_0 = U_0/NR_H = S(n_0/n)r_{s0}^{-\frac{3}{4}}, \tag{17}$$

where[5]

$$S = \frac{2}{\pi}(3)^{\frac{1}{4}} \int_0^\infty [\xi^2(4+\xi^4)^{\frac{1}{2}} - \xi^4 - 2]d\xi = -1.606. \tag{18}$$

We may rewrite this result in terms of the more customary variable

$$r_s = (3/4\pi)^{\frac{1}{3}} me^2/\hbar^2 n^{\frac{1}{3}}, \tag{19}$$

since from (14) we have

$$n/n_0 = 1 + Qr_{s0}^{\frac{3}{4}},$$

and

$$r_s = (n_0/n)^{\frac{1}{3}} r_{s0} = r_{s0}[1 + Qr_{s0}^{\frac{3}{4}}]^{-\frac{1}{3}} \tag{20}$$

which determines r_{s0} implicitly in terms of r_s. For sufficiently small r_{s0}, we have

$$r_{s0} = r_s + \tfrac{1}{3}Qr_s^{7/4}, \tag{21}$$

and

$$u_0 = Sr_s^{-\frac{3}{4}} - (5/4)SQ = -1.606 r_s^{-\frac{3}{4}} + 0.424. \tag{22}$$

It is not clear from our analysis that the constant term in (22) is correct or even that u_0 possesses an expansion in powers of $r_s^{\frac{3}{4}}$ as this equation would suggest, since the order in r_s in which the present calculation is valid, in view of the first Bogoliubov approximation, is not obvious. A more intensive analysis of the problem is necessary for clarification of this point. On the other hand, there is no particular reason to suspect that the first term is not correctly given by the analysis here presented, and that it represents the analog in the boson case of the first term in the correlation energy of the charged fermion gas at high densities as computed by Gell-Mann and Brueckner, and by Sawada. In the boson case, there is of course no Fermi energy nor exchange energy, so that the energy computed above may be regarded as correlation energy.

Turning finally to the second term in Eq. (11), which describes the elementary excitations of the system, it is clear from the excitation energies as given in Eq. (8) that for small $\mathbf{k}$ these must be of the nature of plasma oscillations or plasma waves with a specific dispersion relation. The fact that there is a finite energy gap, at least in the approximation here considered, suggests that a charged boson gas at high densities is both a superfluid and a superconductor. It is perhaps interesting that there is a continuous change in the energy of the elementary excitations, as the momentum of the excitations increases, from that appropriate to plasma waves to that appropriate to single-particle excitation.

We remark finally that in the low-density limit, a gas of charged particles has a behavior independent of the statistics and hence is the same for a gas of fermions and a gas of bosons. As Wigner[6] has pointed out, in this limit the particles crystallize in a body-centered cubic lattice in the uniform matrix of opposite charge. The energy of the system is given by the Coulomb energy of such a crystalline array, and the next-order correction term is simply the zero-point vibrational energy of the lattice. The energy is then given by

$$u_0 = -1.792 r_s^{-1} + (2.55 \pm 0.30) r_s^{-\frac{3}{2}}, \tag{23}$$

in the same notation as that employed above. The coefficient of the second term has here been determined by estimating the mean vibrational frequency of the lattice from the mean square and mean fourth power of the frequency which can easily be calculated. Since the low-lying excitations consist of phonons, which, when transverse, can have arbitrarily low energy, one sees that the energy gap must disappear as the boson gas is expanded.

ACKNOWLEDGMENT

The aid of Dr. W. F. Ford in checking our calculations is gratefully acknowledged.

[6] E. Wigner, Phys. Rev. **46**, 1002 (1938); Trans. Faraday Soc. (London) **34**, 678 (1938). See also P. Nozières and D. Pines, Phys. Rev. **111**, 442 (1958) for a more general review of the high- and low-density behavior of a charged fermion gas.