tions of He^3 in He^4 would not separate into two phases at 0°K. In mixtures containing enough He^3 for phase separation to occur, the equilibrium concentration of He^3 in the lower He^4-rich phase at 0°K would then be determined by higher order terms in the free energies which have not been included in (6).

On the other hand, the experimental values of NE_3 appear to vary approximately linearly with concentration and are not inconsistent with the hypothesis that $NE_3 \rightarrow L_3^0$ as $X \rightarrow 0$. This hypothesis removes any unusual behavior from the separation line near absolute zero, but it is difficult to accept from another viewpoint: It implies that the binding energy of He^3 does not depend at all on whether its neighbors are He^3 or He^4.

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Dissipation in Quantum Mechanics. The Harmonic Oscillator. II

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The development of a quantum-mechanical formalism for systems with dissipation that was presented in an earlier article, and intended mainly for application to the electromagnetic field in a cavity, is extended. The problem of the harmonic oscillator with dissipation is shown to be the same as that of a harmonic oscillator coupled to a thermal reservoir, and the need of the formalism to contain the appropriate statistical mechanics is discussed. The derivation of relationships which permit the calculation of all moments of the oscillator coordinate and momentum provides the necessary extension of the theory. The formal resemblance of the completed theory to that of classical Brownian motion, some differences due to quantum mechanics, and the fact that certain fundamental relationships which are assumed in the latter are derived in the present analysis, are pointed out. The application of the theory is illustrated by the consideration of three problems: the proof of Ott's formula, and the derivation of both the probability density and energy distribution of the oscillator in equilibrium with a thermal reservoir.

INTRODUCTION

IN the first article on the present subject[1] (hereafter referred to as I), a quantum-mechanical theory of the harmonic oscillator with dissipation was developed. The motivation behind this theory was its application to the electromagnetic radiation field in a resonant cavity, and for this purpose, the analysis was carried sufficiently far. Recently, the theory has been applied to an entirely different subject,[2] and some questions arose which were not treated in I. In view of this and, possibly, other applications not yet envisaged, it is the purpose of present article tq extend the above theory.

The results of I will be summarized for the sake of intelligibility. They will be presented in a modified notation, the modification having no other significance than the simplification of the appearance of analytic expressions.

The new notation is defined, essentially, by the statement that the part of the total Hamiltonian which refers to the harmonic oscillator alone (and not to the coupling between it and the loss mechanism or possible driving mechanism) is given by

$$H_{\text{osc}}=\tfrac{1}{2}\hbar\omega(q^2+p^2), \tag{1}$$

with

$$[q,p]=i. \tag{2}$$

We will refer to q and p as coordinate and momentum, respectively. Comparing Eq. (1) with the analogous expression for the electromagnetic field of a mode of (angular) frequency ω in a resonant cavity, or with that for a mechanical oscillator, we can obtain immediately expressions for the electric and magnetic field strengths or for the coordinate and momentum of the mechanical harmonic oscillator.[3] The loss of the oscillator is described, as in I, by the constant β which is the reciprocal relaxation time of the oscillator. [Energy other than thermal or zero-point energy decays as $\exp(-\beta t)$. For the resonant cavity, $\beta=\omega/Q$, Q being the quality factor.] It is assumed throughout that

$$\beta/\omega\ll 1. \tag{3}$$

The results of I are contained, essentially, in the following two relationships:

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[1] I. R. Senitzky, Phys. Rev. **119**, 670 (1960).
[2] T. J. Krieger, Phys. Rev. **121**, 1388 (1961).
[3] The electromagnetic field of a cavity mode is given by $\mathbf{E}=-(4\pi\hbar\omega)^{\frac{1}{2}}p(t)\mathbf{u}(\mathbf{r})$, $\mathbf{H}=(4\pi c^2\hbar/\omega)^{\frac{1}{2}}q(t)\nabla\times\mathbf{u}(\mathbf{r})$, where $\mathbf{u}(\mathbf{r})$ is a normalized function describing the spatial dependence of the cavity field. The coordinate and momentum of a mechanical oscillator of mass m are given by $q_m=(\hbar/m\omega)^{\frac{1}{2}}q$, $p_m=(\hbar\omega m)^{\frac{1}{2}}p$.

$$q=q^{(0)}e^{-\frac{1}{2}\beta t}+\int_0^t dt_1\, W(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\cos\omega(t-t_1), \quad (4)$$

$$p=p^{(0)}e^{-\frac{1}{2}\beta t}-\int_0^t dt_1\, W(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\sin\omega(t-t_1), \quad (5)$$

where $q^{(0)}$ and $p^{(0)}$ are the coordinate and momentum of the lossless uncoupled oscillator, and obviously contain the initial values of q and p. For W we have

$$W=F+w,$$

where F is an operator that refers to the loss mechanism only and describes the effects of thermal and zero-point fluctuations of the loss mechanism, and w refers to classical and/or quantum-mechanical systems coupled to the oscillator. The properties of F are given by two expectation values,[4]

$$\langle F\rangle=0, \quad (6)$$

$$\langle F(t_1)F(t_2)\rangle =\frac{\beta}{\pi}\left\{i\frac{\mathcal{P}}{t_2-t_1}+2\pi\delta(t_1-t_2)\left[\frac{1}{2}+\frac{1}{e^{\hbar\omega/kT}-1}\right]\right\}. \quad (7)$$

As discussed in I, the process of taking expectation values with respect to the loss mechanism involves two types of averaging. One type is the usual quantum mechanical averaging and the other is an averaging over a thermodynamic (canonical) ensemble at temperature T. The nature of the systems which are coupled to the oscillator determine w, but for present purposes we have no interest in this aspect of the theory, and will not define w further.[5]

Before turning to the main questions of the present article, we consider a slight modification of Eqs. (6) and (7). In deriving the above results, it was assumed that the coupling to the lossless oscillator of the loss mechanism and the other systems are turned on at $t=0$. It may be more convenient (and is actually more realistic) to consider the loss mechanism to be coupled to the oscillator from $t=-\infty$. We therefore transfer the time origin to $-\infty$. The transient terms in Eqs. (4) and (5) drop out, and we have

$$q=\int_{-\infty}^t dt_1\, W(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\cos\omega(t-t_1), \quad (8)$$

$$p=-\int_{-\infty}^t dt_1\, W(t_1)e^{-\frac{1}{2}\beta(t-t_1)}\sin\omega(t-t_1). \quad (9)$$

If the coupling to the systems other than the loss mechanism begins at some prescribed time, we let $w(t)$ vanish prior to that time. For purposes of the present article, we consider $w(t)$ to be zero for all t, and W will be assumed to be replaced by F in all further discussions.

In I we considered linear, bilinear, and quadratic expressions in q and p; we calculated the expectation values of q and p, their commutation relationships, and also the expectation values of the energy. All the physically meaningful results, even if they were in operator form as far as the oscillator is concerned, were expectation values in the loss mechanism space. Thus Eqs. (6) and (7) were completely sufficient for our purposes, since $F(t)$ occurred only in linear and bilinear expressions.

Let us shift our point of view somewhat and consider a lossless oscillator in contact with a thermal reservoir. Equations (4) and (5) apply equally well in this case. The reservoir absorbs energy from the oscillator (as well as transmits thermal energy to it), and acts as the dissipation. One does not usually think of a dissipation constant in connection with a thermal reservoir; there exists, however, a thermal relaxation time, determined by the coupling between oscillator and reservoir, and this serves equally well to define β.[2] The thermal reservoir, therefore, is the dissipation mechanism, and alternately, the dissipation mechanism may be regarded as a thermal reservoir.

Since Eqs. (4) and (5) [or (8) and (9)] are explicit expressions for the coordinate and momentum operators of the oscillator coupled to a thermal reservoir, they must contain the statistical mechanics of the harmonic oscillator as well as its quantum mechanics. We should therefore expect to be able to obtain directly from these equations a thermal (simultaneously with a quantum-mechanical) distribution function for position, energy, or any other function of the coordinates. It is evident that this calculation will involve the evaluation of moments of q and p higher than the second, which in turn will require expressions for

$$X_n\equiv\langle F(t_1)F(t_2)\cdots F(t_n)\rangle, \quad (10)$$

with $n>2$. Thus, to describe completely all aspects of the behavior of the oscillator, we must add an expression for (10) to Eqs. (6) and (7). It is the purpose of the present article to do so, and then to illustrate the application of the theory by considering several specific problems. Expressions for X_n will be derived in Part I, and the specific problems will be considered in Part II.

I

As mentioned previously, $F(t)$ is an operator referring to the loss mechanism. It corresponds to the dynamical variable through which the loss mechanism couples to the oscillator, but it is the expression for the uncoupled operator, describing the fluctuations of the loss mechanism in its free state. [In the case of the resonant cavity,

[4] Equation (7) corresponds to Eq. (I, 74) with $F\equiv(4\pi c^2/\hbar\omega^3)^{\frac{1}{2}}D$. The sign of the principal value term in the latter equation is wrong due to a regrettable misprint.

[5] For a treatment of specific problems in which both classical and quantum-mechanical systems are coupled to the harmonic oscillator, see, for instance, I. R. Senitzky, Phys. Rev. **121**, 171 (1961).

for instance, it may be the operator corresponding to the current (in suitable units) in the cavity walls.] In general, each X_n has a value determined by the dynamic properties of the specific loss mechanism under consideration and there need not be any simple relationship between the X_n's for various n. Without knowing the microscopic details of the loss mechanism, we would therefore need a different parameter to specify each X_n. We know from the discussion in I, however, that the concept of dissipation (or thermal reservoir, for that matter) involves sufficient assumptions, approximation, and averaging (over the thermodynamic ensemble) in the description of the loss mechanism so that this description is contained completely, at a given temperature, in a single loss constant. We conclude, therefore, that the simplifications involved must permit the specification of the X_n's in terms of X_2, and that such an approximate specification exists in general. As soon as we assume the existence of a relationship between X_n and X_2 which is independent of the individual characteristics of the loss mechanism, it becomes easy to find this relationship. We merely consider a loss mechanism which is specialized to a sufficient extent to make this relationship obvious.

We show first, quite generally, that when n is odd, averaging over a thermodynamic ensemble of loss mechanisms gives $X_n=0$. Since the initial state of the loss mechanism is described (see I) by a diagonal density matrix, only diagonal matrix elements of the product $F(t_1)\cdots F(t_n)$ occur in X_n. For n odd, we must have at least one factor in each term of the sums that constitute these diagonal matrix elements which is either a diagonal matrix element of F, and therefore vanishes, or is an off-diagonal matrix element of F not multiplied by its complex conjugate. Now, the phase of an off-diagonal matrix element is determined by the phases entering into the description of the initial state of the loss mechanism, and is a random variable in the thermodynamic ensemble.[6] Averaging will therefore yield $X_n=0$ for n odd.

We proceed now to consider X_n for n even. Consider a loss mechanism which may be regarded as being composed of N (macroscopic) subsystems, N being large compared to unity, where the coupling between subsystems is negligible when the loss mechanism is free.[7] We can then set

$$F=\sum_j F^{(j)}, \tag{11}$$

where $F^{(j)}$ refers only to the jth subsystem, and where the $F^{(j)}$'s are independent of each other. (In the case of the cavity where the loss mechanism is the cavity wall, the subsystems may be small wall areas.) The same reasoning which led to the relationship $X_n=0$ for n odd likewise yields

$$\langle F^{(j)}(t_1)\cdots F^{(j)}(t_n)\rangle=0 \tag{12}$$

for n odd.

Let us consider first the expression for X_4. We have

$$\begin{aligned}X_4=&\sum_{ijkl}\langle F^{(i)}(t_1)F^{(j)}(t_2)F^{(k)}(t_3)F^{(l)}(t_4)\rangle\\ =&\sum_{il}\langle F^{(i)}(t_1)F^{(i)}(t_2)\rangle\langle F^{(l)}(t_3)F^{(l)}(t_4)\rangle\\ &+\sum_{il}\langle F^{(i)}(t_1)F^{(i)}(t_3)\rangle\langle F^{(l)}(t_2)F^{(l)}(t_4)\rangle\\ &+\sum_{il}\langle F^{(i)}(t_1)F^{(i)}(t_4)\rangle\langle F^{(l)}(t_2)F^{(l)}(t_3)\rangle\\ &+\sum_{i}\langle F^{(i)}(t_1)F^{(i)}(t_2)F^{(i)}(t_3)F^{(i)}(t_4)\rangle.\end{aligned} \tag{13}$$

Use has been made of the fact that $F^{(i)}$ commutes with $F^{(j)}$ for $i\neq j$; we must bear in mind however that $F^{(i)}(t_1)$ does not commute with $F^{(i)}(t_2)$. Since the number of terms in each summation is large compared to unity, we can neglect the single summation compared to the double summations. Noting that

$$\sum_i\langle F^{(i)}(t_1)F^{(i)}(t_2)\rangle=\langle F(t_1)F(t_2)\rangle, \tag{14}$$

we see that

$$\begin{aligned}X_4\approx&\langle F(t_1)F(t_2)\rangle\langle F(t_3)F(t_4)\rangle+\langle F(t_1)F(t_3)\rangle\langle F(t_2)F(t_4)\rangle\\ &+\langle F(t_1)F(t_4)\rangle\langle F(t_2)F(t_3)\rangle.\end{aligned} \tag{15}$$

We thus have an expression for the expectation value of a fourfold product as a sum of products of the expectation value of twofold products. Each term in the sum is obtained by pairing the factors in X_4, the order within each pair being the same as in X_4, and then replacing each pair by its expectation value. The terms of the sum correspond to the different ways of pairing. It is obvious that the above reasoning leads to the same rule for any X_n, n being even. Thus, for even n

$$\begin{aligned}&\langle F(t_1)F(t_2)\cdots F(t_n)\rangle\\ &\quad=\sum\langle F(t_{j1})F(t_{j2})\rangle\cdots\langle F(t_{jn-1})F(t_{jn})\rangle,\end{aligned} \tag{16}$$

where $j_{2k-1}<j_{2k}$, and where the summation is taken over all the different arrangements into pairs; and, as we have shown previously, for odd n

$$\langle F(t_1)F(t_2)\cdots F(t_n)\rangle=0. \tag{17}$$

It can readily be seen that the number of terms on the right side of Eq. (16) is

$$\frac{n!}{(\frac{1}{2}n)!2^{\frac{1}{2}n}}. \tag{18}$$

[6] If the initial state is represented as a superposition of energy states, then the phases of the superposition constants are random variables in the thermodynamic ensemble. See, for instance, R. C. Tolman, *Principles of Statistical Mechanics* (Oxford University Press, New York, 1938).

[7] This is the specialization previously mentioned. It is not readily apparent to what extent we are restricting the class of loss mechanisms by this requirement, since it is difficult to envisage a specific loss mechanism for which this requirement does not hold. After all, a tightly coupled system could hardly behave like an average loss mechanism described by a single dissipation constant. However, this question will not be pursued further.

Equations (16) and (17) thus supplement Eq. (7) so as to permit the calculation of any physically meaningful result. It is to be noted that Eq. (6) is a special case of Eq. (17).[8]

II

We apply the above theory to several specific problems. The results to be obtained are not new, but the method of solution will given an insight into the significance and potentialities of the theory.

The first problem we consider is the proof of Ott's formula,[9] which states that for a thermal distribution of harmonic oscillators

$$\langle \exp(i\lambda q)\rangle = \exp(-\tfrac{1}{2}\lambda^2\langle q^2\rangle). \tag{19}$$

This theorem is important in the theory of x-ray and neutron diffraction, and will be used to show that the eigenvalues of q have a Gaussian distribution. We have

$$\langle \exp(i\lambda q)\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(i\lambda)^n\langle q^n\rangle. \tag{20}$$

Equation (8) gives us q for a harmonic oscillator in thermal equilibrium with the loss mechanism (or heat reservoir). It yields

$$\langle q^n\rangle = \int_{-\infty}^{t} dt_1\cdots\int_{-\infty}^{t} dt_n\langle F(t_1)\cdots F(t_n)\rangle e^{-\frac{1}{2}\beta(t-t_1)}\cdots \times e^{-\frac{1}{2}\beta(t-t_n)}\cos\omega(t-t_1)\cdots\cos\omega(t-t_n). \tag{21}$$

From Eqs. (16) and (17),

$$\langle q^n\rangle = 0 \tag{22}$$

for n odd, and

$$\langle q^n\rangle = \sum\int_{-\infty}^{t} dt_1\cdots\int_{-\infty}^{t} dt_n\langle F(t_{j1})F(t_{j2})\rangle\cdots \times\langle F(t_{jn-1})F(t_{jn})\rangle e^{-\frac{1}{2}\beta(t-t_1)}\cdots e^{-\frac{1}{2}\beta(t-t_n)} \times\cos\omega(t-t_1)\cdots\cos\omega(t-t_n) \tag{23}$$

for n even. Now

$$\int_{-\infty}^{t} dt_{jr}\int_{-\infty}^{t} dt_{jr+1}\langle F(t_{jr})F(t_{jr+1})\rangle \times\exp[-\tfrac{1}{2}\beta(t-t_{jr})]\exp[-\tfrac{1}{2}\beta(t-t_{jr+1})] \times\cos\omega(t-t_{jr})\cos\omega(t-t_{jr+1}) = \langle q^2\rangle. \tag{24}$$

Each term in the summation of Eq. (23) is therefore $\langle q^2\rangle^{\frac{1}{2}n}$, and using (18), we have

$$\langle q^n\rangle = \frac{n!}{(\frac{1}{2}n)!2^{\frac{1}{2}n}}\langle q^2\rangle^{\frac{1}{2}n}. \tag{25}$$

Substituting from Eqs. (22) and (25) into Eq. (20), we obtain

$$\begin{aligned}\langle \exp(i\lambda q)\rangle &= \sum_{n\ \text{even}}\frac{(i\lambda)^n}{(\frac{1}{2}n)!2^{\frac{1}{2}n}}\langle q^2\rangle^{\frac{1}{2}n} \\ &= \sum_p \frac{(-\lambda^2)^p}{p!2^p}\langle q^2\rangle^p \\ &= \exp(-\tfrac{1}{2}\lambda^2\langle q^2\rangle),\end{aligned} \tag{26}$$

which is Ott's formula. An alternative proof, based directly on Eq. (11) rather than on Eqs. (16) and (17) [which were derived from Eq. (11)] is given in Appendix A.

We consider next the following question: What is the distribution of the oscillator coordinate in thermal equilibrium? Let us denote the eigenvalue of q by q', and the probability of finding this eigenvalue by $D(q')$. Then

$$\langle \exp(i\lambda q)\rangle = \int_{-\infty}^{\infty} dq' \exp(i\lambda q')D(q'), \tag{27}$$

and

$$\begin{aligned}D(q') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\exp(-i\lambda q')\langle \exp(i\lambda q)\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\exp(-i\lambda q')\exp(-\tfrac{1}{2}\lambda^2\langle q^2\rangle).\end{aligned} \tag{28}$$

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[8] Equations (16) and (17) are identical in appearance to relationships found in the classical theory of Brownian motion [see, for instance, the review article by Ming Chen Wang and G. E. Uhlenbeck, Revs. Modern Phys. **17**, 393 (1945)] and, indeed, the present theory may be regarded as a study of certain aspects of the Brownian motion of a quantum-mechanical harmonic oscillator. It is therefore appropriate to place in perspective, at this point, the relationship of the present analysis to other work on Brownian motion. Equation (I, 72), which is equivalent to Eq. (5), is the quantum mechanical version of Langevin's equation in the theory of Brownian motion. The latter is always *assumed* in the classical theory. Likewise assumed are the properties of $F(t)$, which, in the classical theory is a random variable defined by Eqs. (16), (17), and the additional equation,

$$\langle F(t_1)F(t_2)\rangle = \text{const}\times\delta(t_1-t_2), \tag{7a}$$

instead of our Eq. (7). Equations (7a), (16), and (17) are equivalent to the statement that $F(t)$ is a Gaussian random variable. [See, also, N. Wiener, *Cybernetics* (John Wiley & Sons, Inc., New York, 1948)]. The essential difference between the present work and the classical theory is not only the consideration of a quantum-mechanical rather than a classical system [thus, $F(t)$ is an operator, $F(t_1)$ and $F(t_2)$ do not commute, and therefore Eq. (7) has an antisymmetric part in contrast to Eq. (7a)] but also the fact that we are deriving, rather than assuming, the quantum-mechanical version of both Langevin's equation and the properties of $F(t)$.

Since the publication of I, two articles have appeared on the Brownian motion of a quantum mechanical oscillator: C. George, Physica, **26**, 453 (1960); and Julian Schwinger, J. Math. Phys. **2**, 407 (1961). Both of these treatments likewise start from fundamental principles, rather than from the assumptions made in the classical theory, but the methods used are entirely different from that of I and the present article. An interesting characteristic of the present methods is their formal resemblance to the classical theory.

[9] Equation (19) is referred to as Ott's formula by Born and Sarginson (reference below). Independent proofs of this equation have been given by: H. Ott, Ann. Physik **23**, 169 (1935); M. Born and K. Sarginson, Proc. Roy. Soc. (London) **A179**, 69 (1941/42); A. C. Zemach and R. J. Glauber, Phys. Rev. **101**, 118 (1956); Julian Schwinger, reference 8.

The value of $\langle q^2 \rangle$ may be derived easily from Eqs. (24) and (7), and is shown in Appendix B to be given by

$$\langle q^2 \rangle = \tfrac{1}{2} + (e^{\hbar\omega/kT} - 1)^{-1} \equiv \tfrac{1}{2}\varphi(T), \tag{29}$$

so that

$$D(q) = [\pi\varphi(T)]^{-\frac{1}{2}} \exp[-\varphi(T)q^2]. \tag{30}$$

We see that this is a Gaussian distribution function that coincides with the ground-state harmonic oscillator wave function at absolute zero, and widens as the temperature increases.[10]

The last problem to be considered is that of the oscillator energy distribution. For this purpose we define the non-Hermitian operators,

$$a = 2^{-\frac{1}{2}}(q + ip), \quad a^* = 2^{-\frac{1}{2}}(q - ip). \tag{31}$$

The Hamiltonian of Eq. (1) is then

$$H_{\text{osc}} = (a^*a + \tfrac{1}{2})\hbar\omega. \tag{32}$$

From Eqs. (8) and (9) we obtain

$$a = 2^{-\frac{1}{2}} \int_{-\infty}^{t} dt_1\, F(t_1) e^{-\frac{1}{2}\beta(t-t_1)} e^{-i\omega(t-t_1)}, \tag{33}$$

$$a^* = 2^{-\frac{1}{2}} \int_{-\infty}^{t} dt_1\, F(t_1) e^{-\frac{1}{2}\beta(t-t_1)} e^{i\omega(t-t_1)}, \tag{34}$$

and from Eqs. (31) we have

$$[a, a^*] = 1. \tag{35}$$

The expectation value of a^*a is shown in Appendix C to be given by

$$\langle a^*a \rangle = [\exp(\hbar\omega/kT) - 1]^{-1}. \tag{36}$$

For later use, we also write this expression in the form

$$\langle a^*a \rangle = z(1-z)^{-1}, \quad z = e^x, \quad x = -\hbar\omega/kT. \tag{36a}$$

Just as in the case of the probability distribution for the coordinate, we have for the (discrete) probability distribution of the energy $D(E_n)$,

$$\langle \exp(i\lambda H) \rangle = \textstyle\sum_n \exp(i\lambda E_n) D(E_n), \tag{37}$$

where E_n is the nth energy eigenvalue of the lossless harmonic oscillator:

$$E_n = (n + \tfrac{1}{2})\hbar\omega. \tag{38}$$

From Eqs. (37) and (38) we obtain

$$D(E_n) = \frac{\hbar\omega}{2\pi} \int_{-\pi/\hbar\omega}^{\pi/\hbar\omega} d\lambda \langle \exp(i\lambda H) \rangle e^{-i\lambda E_n},$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \langle \exp(i\lambda a^*a) \rangle e^{-i\lambda n}. \tag{39}$$

[10] Equation (30) has been derived by the following: F. Bloch, Z. Physik **74**, 295 (1932); R. Kronig, Physica **9**, 113 (1942); Julian Schwinger, reference 8.

We must now evaluate

$$\langle e^{i\lambda a^*a} \rangle = \textstyle\sum_n \frac{1}{n!} (i\lambda)^n \langle (a^*a)^n \rangle. \tag{40}$$

A straightforward evaluation of $\langle (a^*a)^n \rangle$ based on Eqs. (33) and (34) together with the rule for evaluating X_n is difficult. We proceed therefore in a somewhat indirect manner. We show first that

$$\langle a^{*p} a^p \rangle = p! \langle a^*a \rangle^p. \tag{41}$$

Equations (33) and (34) yield

$$\langle a^{*p} a^p \rangle = (\tfrac{1}{2})^p \int_0^t dt_1 \cdots \int_0^t dt_{2p} \langle F(t_1) \cdots F(t_{2p}) \rangle$$
$$\times \exp[-\tfrac{1}{2}\beta(2pt - t_1 - \cdots - t_{2p})]$$
$$\times \exp[-i\omega(t_1 + \cdots + t_p - t_{p+1} - \cdots - t_{2p})]. \tag{42}$$

Using the rule of Eq. (16) for expanding the expectation value of the $2p$-fold product in the integrand into a sum of products of expectation values of twofold products, we have

$$\langle a^{*p} a^p \rangle = \sum (\tfrac{1}{2})^p \int_0^t dt_1 \cdots \int_0^t dt_{2p} \langle F(t_{j1}) F(t_{j2}) \rangle \cdots$$
$$\times \langle F(t_{j_{2p-1}}) F(t_{j_p}) \rangle \exp[-\tfrac{1}{2}\beta(2pt - t_1 - \cdots - t_{2p})]$$
$$\times \exp[-i\omega(t_1 + \cdots + t_p - t_{p+1} - \cdots - t_{2p})]. \tag{43}$$

Each term in this sum can be written as a product of p twofold integrals, each integral being

$$\tfrac{1}{2} \int_0^t dt_m \int_0^t dt_n \langle F(t_m) F(t_n) \rangle$$
$$\times e^{-\frac{1}{2}\beta(2t - t_m - t_n)} e^{-i\omega(\epsilon_m t_m + \epsilon_n t_n)}, \tag{44}$$

where ϵ_m and ϵ_n each stand for ± 1, the sign depending on the index. If ϵ_m is positive and ϵ_n negative, then expression (44) is equal to $\langle a^*a \rangle$. If ϵ_m and ϵ_n both have the same sign, the integrand is oscillatory and the integral is negligible. (There cannot be a situation in which ϵ_m is negative and ϵ_n positive, since our rule for expanding X_{2p} requires $n > m$.) Thus, the only significant terms in Eq. (43) will be those for which $1 \leqslant m \leqslant p$, $p+1 \leqslant n \leqslant 2p$. There are $p!$ such terms. We therefore obtain Eq. (41), which was to be proved.

From Eq. (36a) it follows that $\langle a^*a \rangle$ may be expressed as

$$\langle a^*a \rangle = (1-z)(zd/dz)(1-z)^{-1}, \tag{45}$$

and Eq. (41) shows that

$$\langle a^{*p} a^p \rangle = p! z^p (1-z)^{-p},$$
$$= (1-z)(z^p d^p/dz^p)(1-z)^{-1}. \tag{46}$$

Now, a product containing a's and a^*'s in any order may be reduced, by means of the commutation rule for a and a^* [Eq. (35)], to a sum of terms in each of which the a^*'s appear to the left of the a's. Likewise, a product containing (d/dz)'s and z's in any order may be reduced, by means of the commutation rule for d/dz and z, to a sum of terms in each of which the z's appear to the left of the (d/dz)'s. Since

$$[a,a^*]=[(d/dz),z], \tag{47}$$

it follows from Eqs. (45) and (46) that

$$\langle(a^*a)^p\rangle=(1-z)(zd/dz)^p(1-z)^{-1}, \tag{48}$$

and

$$\langle\exp(i\lambda a^*a)\rangle=(1-z)\exp(i\lambda zd/dz)(1-z)^{-1}. \tag{49}$$

From the definition of z in Eq. (36a), we have

$$z\frac{d}{dz}f(z)=\frac{d}{dx}f[z(x)], \tag{50}$$

where $f(z)$ is an arbitrary function of z. We therefore have

$$\begin{aligned}\langle\exp(i\lambda a^*a)\rangle&=(1-z)\exp\left(i\lambda\frac{d}{dx}\right)(1-z)^{-1}\\&=\frac{1-\exp(x)}{1-\exp(x+i\lambda)}\\&=[1-\exp(x)]\sum_{n=0}^{\infty}e^{n(x+i\lambda)},\end{aligned} \tag{51}$$

where note has been taken of the operator form of Taylor's expansion. Substituting from Eq. (51) into Eq. (39), we obtain

$$\begin{aligned}D(E_m)&=[1-\exp(x)]\textstyle\sum_n\exp(nx)\delta_{nm}\\&=[1-\exp(x)]\exp(mx),\end{aligned} \tag{52}$$

where, it is recalled,

$$x=-\hbar\omega/kT.$$

This is precisely the normalized Boltzmann distribution for the harmonic oscillator. It is interesting to observe that Eqs. (7) and (16) contain sufficient statistical mechanics to give us this result.

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APPENDIX A

We prove Ott's formula directly from Eq. (11). We first set

$$F^{(j)}=\epsilon_j f^{(j)}, \tag{A1}$$

so that

$$F=\textstyle\sum_j\epsilon_j f^{(j)}, \tag{A2}$$

where ϵ_j is a c number that is small compared to unity. This change merely has the purpose of bringing out explicitly the smallness of $F^{(j)}$ compared to F. From Eq. (8), we have

$$\begin{aligned}q&=\int_{-\infty}^{t}dt_1\,F(t_1)\psi(t-t_1)\\&=\sum_j\epsilon_j\int_{-\infty}^{t}dt_1\,f^{(j)}(t_1)\psi(t-t_1),\end{aligned} \tag{A3}$$

where

$$\psi(\tau)\equiv\exp(-\tfrac{1}{2}\beta\tau)\cos\omega\tau.$$

Substituting from Eq. (A3), we obtain

$$\langle e^{i\lambda q}\rangle=\left\langle\exp\left[i\lambda\sum_j\epsilon_j\int_{-\infty}^{t}dt_1\,f^{(j)}(t_1)\psi(t-t_1)\right]\right\rangle. \tag{A4}$$

Since the $f^{(j)}$'s commute with each other, we can write

$$\langle e^{i\lambda q}\rangle=\left\langle\prod_j\exp\left[i\lambda\epsilon_j\int_{-\infty}^{t}dt_1\,f^{(j)}(t_1)\psi(t-t_1)\right]\right\rangle, \tag{A5}$$

and since the $f^{(j)}$'s refer to different systems, we have

$$\langle e^{i\lambda q}\rangle=\prod_j\left\langle\exp\left[i\lambda\epsilon_j\int_{-\infty}^{t}dt_1\,f^{(j)}(t_1)\psi(t-t_1)\right]\right\rangle. \tag{A6}$$

In view of the fact that $\epsilon_j\ll1$, we expand each exponential and retain only the lowest order terms. From Eq. (12) we have

$$\langle f^{(j)}\rangle=0. \tag{A7}$$

We therefore obtain

$$\begin{aligned}\langle e^{i\lambda q}\rangle=\prod_j\Big[1-\tfrac{1}{2}\lambda^2\epsilon_j^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\\\times\langle f^{(j)}(t_1)f^{(j)}(t_2)\rangle\psi(t-t_1)\psi(t-t_2)\Big],\end{aligned} \tag{A8}$$

and, to the lowest order in ϵ,

$$\begin{aligned}\langle e^{i\lambda q}\rangle&=\prod_j\exp\Big[-\tfrac{1}{2}\lambda^2\epsilon_j^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\\&\qquad\times\langle f^{(j)}(t_1)f^{(j)}(t_2)\rangle\psi(t-t_1)\psi(t-t_2)\Big]\\&=\exp\Big[-\tfrac{1}{2}\lambda^2\sum_j\epsilon_j^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\\&\qquad\times\langle f^{(j)}(t_1)f^{(j)}(t_2)\rangle\psi(t-t_1)\psi(t-t_2)\Big].\end{aligned} \tag{A9}$$

Now, because of Eq. (A7) and the independence of the $f^{(j)}$'s,

$$\textstyle\sum_j\epsilon_j^2\langle f^{(j)}(t_1)f^{(j)}(t_2)\rangle=\langle F(t_1)F(t_2)\rangle. \tag{A10}$$

We have, thus,

$$\langle e^{i\lambda q}\rangle=\exp\biggl[-\tfrac{1}{2}\lambda^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2 \times\langle F(t_1)F(t_2)\rangle\psi(t-t_1)\psi(t-t_2)\biggr]$$

$$=\exp(-\tfrac{1}{2}\lambda^2\langle q^2\rangle), \qquad \text{(A11)}$$

which was to be proved.

APPENDIX B

We derive here Eq. (29). From Eq. (24), we have

$$\langle q^2\rangle=\tfrac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\langle\{F(t_1),F(t_2)\}\rangle \times\psi(t-t_1)\psi(t-t_2), \qquad \text{(A12)}$$

where the symmetrized product $[\{A,B\}=AB+BA]$ may be used, since t_1 and t_2 are variables of integration with the same limits. From Eq. (7) we obtain

$$\tfrac{1}{2}\langle\{F(t_1),F(t_2)\}\rangle=2\beta\delta(t_1-t_2)[\tfrac{1}{2}+(e^{\hbar\omega/kT}-1)^{-1}],$$
$$\equiv\beta\delta(t_1-t_2)\varphi(T). \qquad \text{(A13)}$$

Substituting into Eq. (A12), we have

$$\langle q^2\rangle=\beta\varphi(T)\int_{-\infty}^{t}dt_1\, e^{-\beta(t-t_1)}\cos^2\omega(t-t_1)$$

$$=\tfrac{1}{2}\varphi(T), \qquad \text{(A14)}$$

where we have dropped an oscillatory term in the integrand. It is easy to see, by comparing Eqs. (8) and (9), that the same result is obtained for $\langle p^2\rangle$, so that

$$\langle p^2\rangle=\langle q^2\rangle. \qquad \text{(A15)}$$

APPENDIX C

The value of $\langle a^*a\rangle$ may be obtained directly from Eqs. (33) and (34) together with Eq. (7). However, it is simpler to use the relationship

$$a^*a=\tfrac{1}{2}(q^2+p^2-1), \qquad \text{(A16)}$$

together with Eqs. (A14) and (A15), to obtain

$$\langle a^*a\rangle=\tfrac{1}{2}[\varphi(T)-1]=(e^{\hbar\omega/kT}-1)^{-1}. \qquad \text{(A17)}$$