Analytical Formula for Continuous Absorption Coefficient of the Hydrogen Negative Ion

T. TIETZ

Department of Theoretical Physics, University Łódź, Łódź, Poland

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In this paper we derive an approximate analytical formula for the continuous absorption coefficient of the hydrogen negative ion. This formula gives results comparable to those obtained by Chandrasekhar, who used an eleven-parameter ground-state function and the dipole velocity matrix element. Our numerical results are compared with the corresponding numerical calculations of Geltman.

IT is well known that the continuous absorption coefficient of H$^-$ is of great astrophysical interest. This coefficient has been calculated by several authors,[1-12] the most accurate numerical calculations being those of Chandrasekhar and of John. The previous theoretical calculations have been carried out with many-parameter ground-state wave functions: e.g., Chandrasekhar and Elbert used a Hart and Herzberg 20-parameter ground-state wave function and a Hartree approximation for the continuum wave function. The simplest calculation concerning the total continuous absorption coefficient κ_ν for H$^-$ are given by Geltman,[8] but Geltman's considerations do not give an analytical formula for κ_ν. The purpose of this note is to derive without much trouble, by a simple assumption, an approximate analytical formula for κ_ν. It is known that the total continuous absorption coefficient κ_ν for H$^-$, according to Geltman, can be written in the form

$$\kappa_\nu = \frac{32\pi^2}{3}\alpha a_0 \frac{k_0^2+k^2}{k}\left|\int_0^\infty \varphi_0 \chi_1 r^2 dr\right|^2,\tag{1}$$

where a_0 is the Bohr radius, α is the fine structure constant, and φ_0 is the normalized ground state function which fulfills the following Schrödinger equation:

$$[\nabla^2 - 2V(r) + 2E_0]\varphi_0 = 0,\tag{2}$$

where $V(r)$ is the spherical potential and $2E_0 = k_0^2$; E_0 is the bound state having an energy equal in magnitude to the accepted electron affinity of the hydrogen atom. k^2 appearing in Eq. (1) is $k^2 = 2E_k$, where E_k is the continuum state energy.

Between the frequency ν, E_0, and E_k, as is known, there exists the relation $E_0 - E_k = h\nu$. The Schrödinger equation given by (2) is written in atomic units for length (Bohr radius) and energy (2 rydberg units). The continuum wave function φ_k, as we know, may be expanded in terms of Legendre polynomials as

$$\varphi_k = \sum_{l=0}^\infty \frac{i^l(2l+1)}{kr}P_l(\cos\theta)\chi_l(r,k).\tag{3}$$

The partial wave χ_1 in Eq. (1) is the wave which contributes to the dipole matrix element in the p wave. In order to obtain an analytical expression for κ_ν we must have an analytical function for φ_0 and χ_1. Therefore, we adopt for $V(r)$ the following potential

$$V(r) = -b\frac{e^{-ar}}{1-e^{-ar}},\tag{4}$$

where b and a are constants. Our potential is compared in Table I with the Hartree field,

$$V_H(r) = -e^{-2r}(1+1/r).\tag{5}$$

Table I shows that our resultant potential (4), if we adopt for the constants b, a the values $a=2.3$ and $b=2.5$, agrees well with $V_H(r)$ if r is not too large.

Our approximate potential $V(r)$ has the advantage that with it one can solve exactly the Schrödinger equation (2) and obtain the ground-state wave function φ_0 and its eigenvalue E_0. For φ_0 we obtain

$$\varphi_0 = (N_0/2\sqrt{\pi})e^{-k_0 r}[1-e^{-ar}]/r,\tag{6}$$

where N_0 is the radial normalization constant,

$$N_0 = [2k_0(k_0+a)(2k_0+a)]^{\frac{1}{2}}/a.\tag{7}$$

The eigenfunction φ_0 for the ground state corresponds to an eigenvalue E_0, where

$$k_0^2 = 2|E_0| = \left(\frac{2b-a^2}{2a}\right)^2,\tag{8}$$

The best value for the electron affinity of the hydrogen atom is the Pekeris[13] value. As we want to compare our considerations with the corresponding results of

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Geltman, we adopt for the electron affinity of the hydrogen atom Henrich's[5] value 0.747 ev. The Pekeris value for the electron affinity of the hydrogen atom will change our absorption coefficient negligibly. Using Henrich's value as the binding energy of the $1s$ level in the Schrödinger Eq. (2) for our approximate potential (4) we see from Eq. (8) that if $b=2.5$ the constant a must change its value slightly, from 2.3 to $a=2.01407$. If we adopt the values $a=2.01407$ and $b=2.5$, we have the advantage that we obtain exactly Henrich's value for the electron affinity of the hydrogen atom, 0.747 ev.

This small change of the constant a does not change appreciably the values of our potential given in Table I. As k_0^2 is very small, the $1p$ phase shifts η_l will be very small, so we do not make a great error if we adopt for χ_1 the solution of the Schrödinger equation for $V=0$, which is known and can be written in the following form:

$$\chi = \frac{\sin kr}{kr} - \cos kr. \tag{9}$$

Substituting φ_0 given by Eq. (6) and χ_1 given by the last formula in the expression for the total continuous absorption coefficient κ_ν, Eq. (1), we obtain after simple calculations the following analytical formula:

$$\kappa_\nu = \frac{64\pi}{3}\alpha\left(\frac{a_0}{a}\right)^2 k_0(k_0+a)(2k_0+a)\left(\frac{k}{k_0^2+k^2}\right)^3$$

$$\times\left[1-\left(\frac{k_0^2+k^2}{(k_0+a)^2+k^2}\right)^2\right]^2. \tag{10}$$

Since in atomic units the unit of energy is $e^2/a_0=27.23$ ev, in our case $k_0^2=0.054866$. Taking into consideration the corresponding values of our constants, we can write κ_ν as follows:

$$\kappa_\nu = (10^{-17} \text{ cm}^2)\times 0.44137\left(\frac{k}{0.054866+k^2}\right)^3$$

$$\times\left[1-\left(\frac{0.054866+k^2}{5.05488+k^2}\right)^2\right]^2. \tag{11}$$

TABLE I. Comparison of the potential given by Eq. (4) with the Hartree potential of the hydrogen atom.

r	$e^{-2r}(1+1/r)$	$be^{-ar}/(1-e^{-ar})$
0	$+\infty$	$+\infty$
0.1	9.006	12.354
0.3	2.378	2.511
0.5	1.104	1.157
0.8	0.454	0.472
1.0	0.271	0.279
1.2	0.166	0.169
1.5	0.0830	0.0862
1.8	0.0425	0.0405
2.0	0.0275	0.0254
2.5	0.00943	0.00798
3.0	0.00330	0.00252

TABLE II. Comparison of our continuous absorption coefficient of H⁻, Eq. (11), with the corresponding numerical results of Geltman.

k^2	$\lambda(\text{A})$	κ_λ $(10^{-17}$ cm$^2)$ Our values	κ_λ $(10^{-17}$ cm$^2)$ Geltman's values
0	16533	0	0
0.005	15158	0.727	0.69
0.01	13994	1.617	1.54
0.015	12996	2.378	2.26
0.02	12131	2.975	2.83
0.03	10706	3.750	3.57
0.04	9581	4.134	3.95
0.05	8669	4.242	4.09
0.06	7916	4.278	4.10
0.07	...	4.195	...
0.08	6774	4.066	3.91
0.10	5875	3.758	3.62
0.12	5204	3.420	3.31
0.16	4236	2.839	2.76
0.20	3572	2.373	2.31
0.30	2566	1.607	1.58
0.40	2002	1.177	1.15
0.60	1391	0.714	0.69
1.00	864	0.354	0.33
∞	0	0	0

In Table II we have a comparison of our κ_ν with the values of κ_ν obtained numerically by Geltman, with the corresponding wavelengths given. Table II shows that our values for κ_λ agree well with the corresponding results of Geltman. Our results are a little larger than Geltman's results and therefore correspond better to Chandrasekhar's results as shown in Fig. 2 of Geltman's[8] paper.

Smith and Burch[11] have shown in Fig. 3 of their paper that Geltman's numerical results are in good agreement with the experimental results. The results of Geltman agree better with the experimental results the longer the wavelength, but deviate at shorter wavelength. The same sort of accuracy holds for our formula (11). If we take into consideration the more recent experimental and theoretical results of John,[12] given in Table I and Fig. 1 of his paper, we see that

TABLE III. Comparison of the values of $\varphi_0(r)$ in our Eq. (6) with Geltman's numerical values.

r	$\varphi_0(r)$ Our results	$\varphi_0(r)$ Geltman's results
0	0.4562	0.428
0.5	0.2546	0.266
1.0	0.1553	0.171
1.5	0.1007	0.108
2.0	0.06964	0.0689
3.0	0.03722	0.0364
4.0	0.02218	0.0216
6.0	0.009260	0.00903
8.0	0.004347	0.00425
10.0	0.002343	0.00210
12.0	0.001136	0.00109
14.0	0.0006139	0.000584
16.0	0.0003337	0.000321
20.0	0.0001046	0.000100

Geltman's maximum of κ_λ is more accurate than our corresponding maximum given by Eq. (11).

In Table III we have a comparison of our bound-state function φ_0, Eq. (6), with the corresponding function of Geltman. Table III shows that crudely the results are similar. We see that the above given consideration gives just the value of the electron affinity of the hydrogen atom given by Henrich, while the analytical formula obtained for the total continuous absorption coefficient is very simple and its degree of accuracy in comparison with Geltman's results is a good one.

Geltman has simplified extensively the calculation of Chandrasekhar by adopting the cut-off Coulomb potential but the solutions obtained for the discrete and continuous spectrum do not give an analytical expression for the total continuous absorption coefficient.

Concerning Eq. (1) we must make it clear that this equation implies the use of a special form for the bound and continuum two-electron wave functions, as well as the dipole length form of the matrix element. Since the continuum functions used in this paper do not satisfy the Schrödinger equation with $V(r)$ given by Eq. (4), different results will be obtained with the dipole velocity and acceleration forms of the matrix element. A check on the self-consistency of this calculation would be the spread in results with all three forms of the dipole matrix element. The other two could also be obtained analytically without much trouble.

Theory of Bound-State Beta Decay*

John N. Bahcall
Indiana University, Bloomington, Indiana
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The theory of beta-decay processes in which an electron is created in a bound atomic state is developed in the allowed approximation. The correlations and total decay rate are calculated with the renormalized $V-A$ theory and the results are valid for atoms of arbitrary electronic configuration. The relative probability of bound-state to continuum-state decay is shown to be independent of nuclear matrix elements; some bound-state decay rates are presented that were calculated by making use of this fact. The possibility of experimentally detecting bound-state decay is also briefly examined. The beta decay of nuclei in stellar interiors is discussed and a number of examples are presented for which bound-state decay is more likely than continuum-state decay under the conditions that obtain in stellar interiors.

I. INTRODUCTION

THE usual theory of beta decay assumes that the transformation of a neutron into a proton is accompanied by the creation, in continuum states, of an electron and an antineutrino. This assumption ignores decays in which an electron is created in a previously unoccupied bound atomic state.

We shall develop, in the allowed approximation, the theory of the usually ignored decays in which a neutron transforms into a proton, an antineutrino is produced in a free state, and an electron is created in a bound atomic state.[1] It is important to realize that the bound-state decay process does not take place through the capture into an atomic orbit of an electron initially created in a continuum state; the direct creation of an electron in a bound state is more probable than the capture process.

The relative frequency of bound-state to continuum-state decays can be estimated with a phase-space argument that does not depend on the formal theory of weak interactions. The phase-space volume available for continuum state decays is represented by the function $f(Z,W_0)$,[2] where the dependence on Z indicates that the Coulomb correction has been included. For bound-state decays, the analogous corrected phase space volume is the square of the neutrino's momentum times the square of the modulus of the electron's wave function evaluated at the nuclear surface. The relative frequency of bound-state to continuum-state decays is approximately equal to the ratio of the phase-space volumes when these volumes are corrected for the electron density at the nuclear surface. Thus the relative frequency of bound-state to continuum-state decays is

$$\Gamma_B/\Gamma_C \sim q^2 |\Psi(R)|^2 / f(Z,W_0)$$
$$\sim (W_0-1)^2(\alpha Z)^2/n^3 f(Z,W_0). \qquad (1)$$

This ratio depends sensitively on the nuclear energy release, W_0; the atomic number, Z, of the daughter nucleus; and the principal quantum number, n, of the lowest unoccupied atomic orbit.

The possibility of bound-state decays was first

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[1] In terrestrial experiments, the daughter atoms are almost always neutral and hence difficult to detect.

[2] We use here units in which $\hbar=m=c=1$. The function f is most familiar in the combination $\log ft$.