Classical Self-Consistent Nuclear Model*

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The Thomas-Fermi method in simplest form is applied to find the radial distribution of nucleons in a spherical nucleus in the absence of Coulomb forces. Saturation is obtained by hypothesizing a two-body force quadratically dependent on relative momentum. The effective one-nucleon potential energy is therefore velocity dependent. Solving the basic integral equation and imposing generally accepted values for the average and Fermi kinetic energies in the nuclear matter limit ($A \to \infty$) gives a solution exhibiting surface and saturated interior regions. Fixing one more parameter (the force range, taken to be $\hbar/m_\pi c$) determines all numerical features (e.g., surface thickness, interaction strength) at reasonable values.

INTRODUCTION

THIS paper reports the results of a calculation of the density of nucleons as a function of the distance from the center of an isotropic nucleus of finite size, in which four nucleons can occupy a given momentum state, but in which there is no Coulomb repulsion. The idea is to take seriously and explore quantitatively the simplest form of the degenerate-fermion-gas (or Thomas-Fermi or "classical") nuclear model. The two-body force law is assumed to have a Yukawa space dependence. A quadratic dependence on relative momentum is inserted into the interaction to give saturation by roughly imitating the effect of a repulsive core in the true interaction.

We treat a nucleus in its ground state as a degenerate gas of nucleons. At each point of configuration space, momentum space is filled as densely as allowed by the Pauli principle up to the Fermi momentum p_F. We consider spherical nuclei, in which case the Fermi momentum will depend on only the distance $r(=|\mathbf{r}|)$ from the center of the nucleus: $p_F=p_F(r)$. The nucleon number-density is given by

$$n(r)=4\frac{(4\pi/3)[p_F(r)]^3}{(2\pi\hbar)^3}. \qquad (1)$$

Now we assume that each nucleon moves in a self-consistent potential arising from its interaction with the other nucleons. We assume further that the effective or average nucleon-nucleon interaction can be written as central but dependent on relative momentum. The potential energy of a single nucleon will then be momentum dependent as well as position dependent.[1]

To calculate the potential energy of a nucleon with momentum $\mathbf{p}$ at the position $\mathbf{r}$, one must *first* add the energy contributions of the nucleon's interactions with all those nucleons having momenta within the Fermi sphere, (FS), at $\mathbf{r}'$, and *second* add the contributions from different $\mathbf{r}'$ throughout the nuclear volume (NV).

$$U(\mathbf{r},\mathbf{p})=-U_0\int_{NV}d\mathbf{r}'\int_{FS}d\mathbf{p}'G\left(\left|\frac{\mathbf{p}'-\mathbf{p}}{p_D}\right|\right)F\left(\left|\frac{\mathbf{r}'-\mathbf{r}}{r_D}\right|\right)\frac{4}{(2\pi\hbar)^3}, \qquad (2)$$

where U_0 is a fixed positive energy giving the strength of the interaction. The function G expresses the momentum dependence of the interaction, F represents the spatial dependence, and r_D and p_D are, respectively, length and momentum parameters introduced to make the arguments of F and G dimensionless. Integration over the Fermi sphere means over all momenta whose magnitudes are less than or equal to the Fermi momentum at the particular configuration point. Integration over the nuclear volume implies summing the contributions from all configuration space volume elements which are within the nucleus.

We adopt, for F, the Yukawa function,

$$F(x)=e^{-x}/x, \qquad (3)$$

and for G a simple (from the computational point of view) quadratic,

$$G(x)=1-x^2. \qquad (4)$$

For the case of infinite nuclear matter, where p_F is independent of r, the assumption of a quadratic momentum dependence for the potential energy has been examined by Weisskopf[1] and Mittelstaedt[2] and is known as the effective-mass approximation. In the limit of infinite nuclear volume our equations should describe nuclear matter and will simplify to the effective-mass approximation.

THEORY

We seek the density distribution of the nucleons in the ground state of a nucleus. We consider only a spherically symmetric distribution. Our approach is to look for a density distribution which will minimize the total energy E_T of a nucleus, subject to the condition

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[1] V. Weisskopf, Nuclear Phys. 3, 423 (1957), has shown that the existence of nuclear matter and the independent particle description of its properties imply that the average potential energy is momentum dependent.

[2] P. Mittelstaedt, Nuclear Phys. 8, 171 (1958).

that the total number of nucleons A is fixed.

$$A=\int_V d\mathbf{r}\int_{FS} d\mathbf{p}\frac{4}{(2\pi\hbar)^3}, \tag{5}$$

$$E_T=\int_{NV} d\mathbf{r}\int_{FS} d\mathbf{p}\left\{\frac{p^2}{2M}+\frac{1}{2}U(r,p)\right\}\frac{4}{(2\pi\hbar)^3}, \tag{6}$$

where U is given by Eq. (2) and M is the nucleon mass.

We introduce the dimensionless quantities

$$x\equiv r/r_D,\quad x_0\equiv R/r_D, \tag{7a}$$

where R is the radius of the nucleon distribution,[3] and

$$\varphi(x)\equiv p_F(r)/p_D,\quad C\equiv\frac{4}{3\pi}\frac{U_0}{(p_D^2/2M)}\left(\frac{r_Dp_D}{\hbar}\right)^3. \tag{7b}$$

Carrying out the integrations in Eqs. (5) and (6) as far as we can for arbitrary φ, we find A and E_T as functionals of $\varphi(x)$:

$$A=A\{\varphi(x)\}=\int_0^R 4\pi r^2dr(4/3)\pi p_F^3(r)4/(2\pi\hbar)^3$$
$$=(8/(3\pi))(r_Dp_D/\hbar)^3\int_0^{x_0}\varphi^3(x)x^2dx, \tag{8}$$

and

$$E_T=E_T\{\varphi(x)\}=\int_0^R 4\pi r^2dr\int_0^{p_F(r)}4\pi p^2dp$$
$$\times\left\{p^2/2M-\tfrac{1}{2}U_0\int_{-1}^{+1}d\mu\int^R 2\pi r'^2dr'\int_{-1}^{+1}d\sigma\right.$$
$$\times\int_0^{p_F(r')}2\pi p'^2dp'\left\{1-\frac{(p^2+p'^2-2\sigma pp')}{p_D^2}\right\}$$
$$\times\exp\left\{-\frac{(r^2+r'^2-2\mu rr')^{\frac{1}{2}}}{r_D}\right\}\frac{r_D}{(r^2+r'^2-2\mu rr')^{\frac{1}{2}}}$$
$$\left.\times\frac{4}{(2\pi\hbar)^3}\right\}\frac{4}{(2\pi\hbar)^3}$$
$$=\frac{8}{5\pi}\left(\frac{r_Dp_D}{\hbar}\right)^3\left(\frac{p_D^2}{2M}\right)\left\{\int_0^{x_0}\varphi^5(x)x^2dx-(5C/6)\right.$$
$$\times\int_0^{x_0}xdx\int_0^{x_0}ydy(e^{-|x-y|}-e^{-(x+y)})\varphi^3(x)\varphi^3(y)$$
$$\left.\times[1-\tfrac{3}{5}(\varphi^2(x)+\varphi^2(y))]\right\}. \tag{9}$$

[3] We shall argue later that $\varphi(x)$ achieves the value zero for a finite value of x, which we call x_0 and identify as the nuclear radius.

To minimize E_T while holding A constant, we introduce a Lagrangian multiplier E_F and require that

$$\delta\{E_T[\varphi(x)]-E_FA[\varphi(x)]\}=0,$$

which is equivalent to requiring that

$$\frac{\partial}{\partial\epsilon}\{E_T[\varphi(x)-\epsilon F(x)]+E_FA[\varphi(x)-\epsilon F(x)]\}=0,$$

for arbitrary $F(x)$ in the limit $\epsilon\rightarrow 0$.

Carrying out the indicated operations, we deduce the following condition on $\varphi(x)$:

$$\varphi^2(x)=\frac{C}{x}\int_0^{x_0}(e^{-|x-y|}-e^{-(x+y)})\varphi^3(y)$$
$$\times[1-\tfrac{3}{5}\varphi^2(y)-\varphi^2(x)]ydy+\frac{E_F}{p_D^2/2M}. \tag{10}$$

Equation (10) can be written as

$$\varphi^2(x)=\frac{1}{p_D^2/2M}[-U(r,p_F)+E_F].$$

Solving for E_F and using Eq. (7), we obtain

$$E_F=\frac{p_F^2(r)}{2M}+U(r,p_F). \tag{11}$$

Thus E_F is just the Fermi energy, i.e., the total energy of a particle having momentum equal to $\mathbf{p}_F(r)$. [E_F should not be confused with the Fermi kinetic energy, T_F, which is equal to $p_F^2/(2M)$.] We define

$$\epsilon_F\equiv\frac{E_F}{p_D^2/2M}, \tag{12}$$

and rewrite Eq. (10) as

$$\varphi^2(x)=\frac{\frac{C}{x}\int_0^{x_0}(e^{-|x-y|}-e^{-(x+y)})\varphi^3(y)(1-\frac{3}{5}\varphi^2(y))ydy+\epsilon_F}{\frac{C}{x}\int_0^{x_0}(e^{-|x-y|}-e^{-(x+y)})\varphi^3(y)ydy+1}. \tag{13}$$

This equation serves as the basic working equation of our model. Its solution determines, through Eqs. (7) and (1), the nucleon density distribution.

Physically x_0 is the distance (measured in units of r_D) beyond which the nuclear density is zero. Thus to be physically reasonable $\varphi(x)$ must vanish for $x>x_0$. Therefore we proceed to seek a function which is (i) a solution of Eq. (13) for $0\leq x\leq x_0$, and (ii) identically zero for $x>x_0$.

It is seen that a function defined in this manner for all x still satisfies Eq. (13) for $x\leq x_0$. But since Eq. (13)

for $x>x_0$ is not satisfied by such a function, it would appear that we cannot conclude that the energy of the particular nucleus considered is minimized by this function. However, an examination of the energy variation procedure which led to Eq. (13) reveals that Eq. (13) occurs multiplied by some common factors, one of which is $\varphi(x)$. Thus this factor being zero also is a sufficient condition for an energy extremum. Thus we are assured that the density function obtained by the above described procedure will, for all x, minimize the energy of the nucleus being considered.

We now have a procedure [valid for all x_0 assuming we can solve Eq. (13) in the region $x<x_0$] for obtaining the desired nuclear density distribution and have the assurance that the density function, thus calculated, is physically meaningful (for the particular nucleus considered) in that it minimizes the energy of the nucleus and exhibits reasonable behavior for $x>x_0$.

We inject here an interesting point of comparison between this calculation and that of the Thomas-Fermi (TF) neutral atom. An investigation of Eq. (13) reveals that $[\varphi(x)]^2$, the nuclear density raised to the 2/3 power, approaches the point x_0 (where density actually becomes zero) linearly as $[(x_0+1)/(x_0+2)]\times\epsilon_F(x_0-x)$, while on the other hand, the TF atomic potential, the electron density raised to the 2/3 power, approaches zero only asymptotically. This feature in the TF atom reflects the infinite range of the Coulomb force and not quantum mechanical diffuseness. Thus the finite range nuclear force leads to the "classical" nucleus looking more "classical" than the "classical" atom, in that its edge is sharp.

In connection with the previous remarks concerning the physical meaningfulness of the calculated nuclear density function, the critical test of the proposed nuclear model entails the examination of the over-all behavior of the solutions for various values of x_0, more than a scrutiny of the details of the density function for a particular value of x_0. Specifically we demand that the solutions of our nuclear model must exhibit the saturation effect as is found experimentally, e.g., they must support the conclusion that the "volume" of a nucleus is proportional to the number of nucleons present.

APPLICATIONS

Infinite Nuclear Matter

The Coulomb energy of the protons has been neglected, thus the working Eq. (13) should be applicable to nuclei with arbitrarily large radii and uniform density, i.e., to nuclear matter. In this limit (φ being a constant φ_∞, and $x_0\rightarrow\infty$), Eq. (13) becomes

$$\epsilon_{F\infty}=\varphi_\infty^2-2C\varphi_\infty^3\{1-(8/5)\varphi_\infty^2\}, \tag{14}$$

and in the same limit the ratio of Eqs. (9) and (8) yields the equation.

$$\bar{\epsilon}_\infty=(3/5)\varphi_\infty^2-C\varphi_\infty^3\{1-(6/5)\varphi_\infty^2\}, \tag{15}$$

where the subscript ∞ is used to remind us that we are dealing with infinite nuclear matter, and where we have made use of a special case of the following definition.

$$\bar{\epsilon}\equiv\frac{\bar{E}}{p_D^2/2M}\equiv\frac{E_T/A}{p_D^2/2M}. \tag{16}$$

We define a dimensionless quantity α such that

$$\bar{\epsilon}_\infty\equiv\alpha\epsilon_{F\infty}. \tag{17}$$

According to the Hugenholtz-Van Hove theorem,[4] α should be equal to unity, and we make calculations for that case only in this work.

Substituting Eqs. (14) and (15) into (17), we find

$$C=\frac{\alpha-3/5}{\varphi_\infty[2\alpha-1-(2/5)\varphi_\infty^2(8\alpha-3)]}, \tag{18}$$

which can be resubstituted in Eq. (15) to give, after solving for φ_∞^2,

$$\varphi_\infty^2=\frac{1-5(\bar{E}_\infty/T_{F\infty})(2-1/\alpha)}{(18/5)-2(\bar{E}_\infty/T_{F\infty})(8-3/\alpha)}. \tag{19}$$

The volume term in the Weizsäcker semiempirical mass formula[5] gives $\bar{E}_\infty=-15.74$ Mev and Hofstadter's nuclear density experiments[6] suggest that $T_{F\infty}$ is about 38 Mev. Substituting these values and $\alpha=1$ into (19) gives the value 0.630 for φ_∞. Using this result in Eq. (18), we obtain $C=3.07$. The parameter C expresses primarily the strength of the interaction ($\propto U_0$), and we shall retain this value for subsequent calculation for finite nuclei.

We note that $\bar{E}_\infty$ and $T_{F\infty}$ enter into Eq. (19) only in the form of the ratio $\bar{E}_\infty/T_{F\infty}$ which can be treated as a single input parameter in the computation of a solution for $\varphi(x)$.

By evaluating Eq. (2) for nuclear matter (p_F independent of position), we can compare our results in the infinite nucleus case with the nuclear matter effective mass approximation of Weisskopf[1] and Mittelstaedt,[2] wherein it is assumed that the energy of a nucleon with momentum p can be represented by

$$E(p)=p^2/2M+V_0+V_1(p/p_F)^2. \tag{22}$$

Comparing with (22) we obtain

$$\begin{aligned}V_0&=-2CT_{F\infty}\varphi_\infty\{1-(3/5)\varphi_\infty^2\},\\ V_1&=2CT_{F\infty}\varphi_\infty^3,\end{aligned} \tag{23}$$

and thus the ratio of effective to free mass is

$$\frac{M^*}{M}\equiv\frac{T_{F\infty}}{T_{F\infty}+V_1}=\frac{1}{1+2C\varphi_\infty^3}. \tag{24}$$

[4] N. M. Hugenholtz and L. Van Hove, Physica 24, 363 (1958).
[5] A. E. S. Green, Phys. Rev. 95, 1001 (1954).
[6] R. Hofstadter, Revs. Modern Phys. 28, 214 (1956).

Using the value of φ_∞ given above, we find from (22) and (23) with the help of (18)

$$V_0=-112.0 \text{ Mev},\quad V_1=+58.3 \text{ Mev},\quad M^*/M=0.394, \tag{25}$$

which agree with the values given in references 1 and 2 for the case Δ (rearrangement energy)$=0$, i.e., $-E_F=S$ (separation energy).

The saturation property of infinite nuclear matter implies that the volume per nucleon is a constant. We write this constant as $(4/3)\pi r_0{}^3{}_\infty$, where $r_{0\infty}$ is a constant length. If we now let A be the number of nucleons in a sphere of radius R_{nm}, which is completely enclosed within nuclear matter, it follows that

$$R_{nm}=r_{0\infty}A^{\frac{1}{3}}. \tag{26}$$

Applying the reciprocal of Eq. (1) to nuclear matter $[p_F(r)=p_{F\infty}]$ gives a second expression for the volume per nucleon, which can be equated to the first to give

$$p_{F\infty}=(9\pi/8)^{\frac{1}{3}}\hbar/r_{0\infty}. \tag{27}$$

The Fermi kinetic energy of nuclear matter can then be related to $r_{0\infty}$ through the equation $T_{F\infty}=p_{F\infty}{}^2/2M$, where M is the nucleon mass, giving

$$T_{F\infty}=(1/2M)(9\pi/8)^{\frac{2}{3}}(\hbar/r_{0\infty})^2. \tag{28}$$

With the help of Eq. (27) and the definition of φ_∞, we can express Eq. (8) as

$$A=3[r_D/(r_{0\infty}\varphi_\infty)]^3\int_0^{x_0}\varphi^3(x)x^2dx. \tag{29}$$

Finite Nuclei

We turn now to the solving of Eq. (13) for finite x_0 and $x<x_0$. The spirit of the calculation will be to treat C as being independent of x_0 and therefore having the same value, $C=3.07$, as in the infinite nucleus, but to permit ϵ_F to depend on x_0. As $x_0\rightarrow\infty$, $\epsilon_F(x_0)$ should approach $\epsilon_{F\infty}$ whose numerical value, from Eq. (14), is -0.164.

Employing an iteration method of solution we rewrite Eq. (13), for a particular value of x_0, as

$$[\varphi^{(n+1)}(x)]^2=\frac{\dfrac{C}{x}\displaystyle\int_0^{x_0}(e^{-|x-y|}-e^{-(x+y)})[\varphi^{(n)}(y)]^3\{1-\tfrac{3}{5}[\varphi^{(n)}(y)]^2\}ydy+\epsilon_F{}^{(n)}(x_0)}{\dfrac{C}{x}\displaystyle\int_0^{x_0}(e^{-|x-y|}-e^{-(x+y)})[\varphi^{(n)}(y)]^3ydy+1}, \tag{30}$$

with the hope that as n becomes large (i.e., large number of iterations) $\varphi^{(n)}(x)$ and $\epsilon_F{}^{(n)}(x_0)$ approach definite limits, say $\varphi(x)$ and $\epsilon_F(x_0)$, respectively. To complete the definition of the iteration procedure we specify a first guess, $\varphi^{(0)}(x)$, which we insert into the right-hand side of Eq. (30), and require $\varphi^{(1)}(x)$ to be zero for $x=x_0$ which determines $\epsilon_F{}^{(0)}$. This procedure is iterated until two successive iterants are within an arbitrary prespecified tolerance of each other for all $x<x_0$.

Since we expect the interior of all except the very lightest nuclei to have properties similar to those of hypothetical "nuclear matter," we select the constant φ_∞ as our first guess, i.e., we set

$$\varphi^{(0)}(x)=\varphi_\infty\quad\text{for}\quad x<x_0.$$

The iterations were carried out on an IBM 650 digital computer, the necessary integrals being

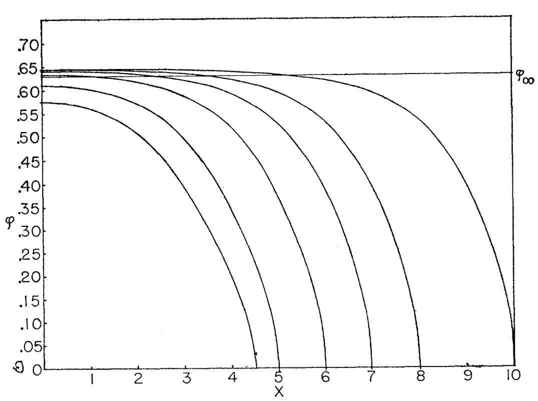

FIG. 1. The variation of φ (the Fermi momentum in units of p_D) as a function of x (the radial distance in units of r_D) for $x<x_0$. Curves are superposed for six different values of x_0, namely: 4.5, 5, 6, 7, 8, and 10. The value of φ in the limit $x_0\rightarrow\infty$ is indicated as φ_∞.

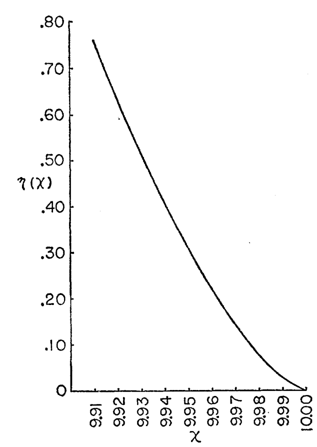

FIG. 2. The variation of $n(x)$ (density in 10^{-3} nucleons per cubic fermi) as a function of x near $x=x_0=10$. The functional dependence of $n(x)$ near x_0 can be given as

$$n(x)=0.668\times\{[(x_0+1)/(x_0+2)]\times\epsilon_F(x_0-x)\}^{\frac{3}{2}}$$

nucleons per cubic fermi.

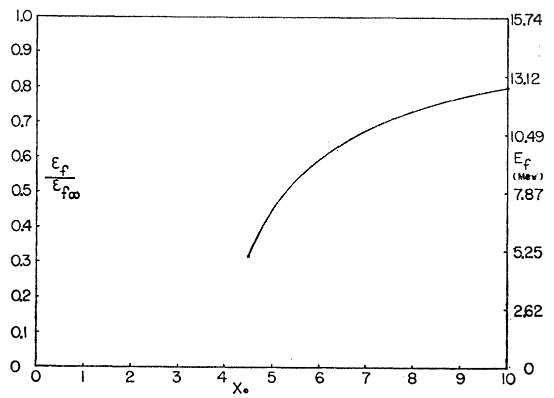

FIG. 3. The variation of the ratio $\epsilon_F/\epsilon_{F\infty}$ (the ratio of the Fermi energy of a finite nucleus of radius x_0 to the Fermi energy of an infinite nucleus) as a function of x_0. The right-hand legend gives the Fermi energy, E_F, in units of Mev, where $E_{F\infty}$ is taken as -15.74 Mev.

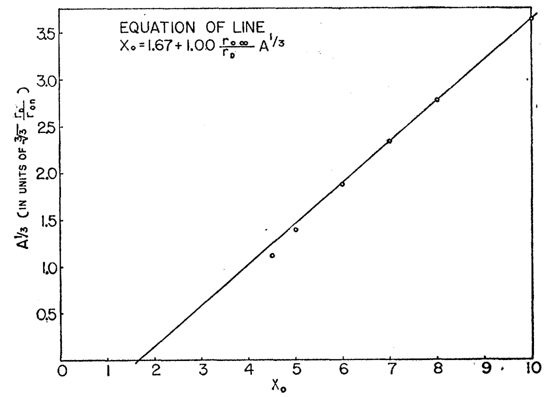

FIG. 4. The variation of $A^{\frac{1}{3}}$ (cube root of the number of nucleons) plotted in units of $\sqrt[3]{3}(r_D/r_{0\infty})$ as a function of x_0 (the nuclear radius in units of r_D). The higher values of $A^{\frac{1}{3}}$ can be connected by a line whose equation is given.

evaluated numerically by dividing the interval 0 to x_0 into 100 points.

The solutions for a half-dozen values of x_0 ranging from 4.5 to 10, (we shall show that this corresponds to the total number of nucleons, A, varying from approximately 30 to 1100) are exhibited in Fig. 1, from which it is seen that the inner density (proportional to φ^3) of a nucleus is a constant approximately equal to the density of nuclear matter and essentially (except for the very light nuclei) independent of x_0.

In order to exhibit the detailed behavior of the density in the vicinity of x_0 we have included Fig. 2 in which we have selected a particular value of $x_0(=10)$, blown up the x scale near $x=10$, and made use of Eq. (1) in the form $n(x)=0.6676[\varphi(x)]^3$ in nucleons per cubic fermi.

In Fig. 3 we present graphically the dependence of ϵ_F on x_0. The infinite nuclear matter value, $\epsilon_{F\infty}$, (-0.164) is used as a standard for comparison. The right-hand legend gives E_F in Mev with $E_{F\infty}$ taken as -15.74 Mev. ϵ_F approaches the nuclear matter value, $\epsilon_{F\infty}$, in the limit $x_0\rightarrow\infty$. The dependence of ϵ_F (or E_F) on A is weak (particularly for higher A) as expected.

Figure 4 exhibits the dependence of x_0 on $A^{\frac{1}{3}}$ (plotted in units of $\sqrt[3]{3}r_D/r_{0\infty}$). The calculated points at the higher values of x_0 lie on a straight line the equation of which is graphically determined to be

$$x_0=1.67+1.00(r_{0\infty}/r_D)A^{\frac{1}{3}}. \tag{31}$$

Upon multiplying by r_D we have for the nuclear radius

$$R=1.67r_D+1.00r_{0\infty}A^{\frac{1}{3}}. \tag{32}$$

The following remarks apply to those nuclei lying on the line in Fig. 4, i.e., those nuclei whose radii are given by Eq. (32). (Only the very light nuclei fail to fall into this category.) The first term in Eq. (32), $1.67r_D$, is independent of A and can be interpreted as the constant "surface thickness" of the nuclei. (We define the "interior" of a nucleus as the region remaining when the surface region is imagined to be removed.) From Eq. (32) the radius of the region interior to the surface region is proportional to $A^{\frac{1}{3}}$ and consequently its volume is proportional to A.

Furthermore, upon comparing Eqs. (26) and (32), we see that the coefficient of $A^{\frac{1}{3}}$ given by the present calculation agrees to within 1% with the coefficient deduced for nuclear matter, thus enabling us to conclude that the interior of actual (excluding Coulomb effect) nuclei is made up of nuclear matter.

All foregoing results have been deduced as a result of assigning numerical values to only *two* input parameters, α and the ratio $\bar{E}_\infty/T_{F\infty}$.

In order to determine the value of A corresponding to a particular value of x_0, it is necessary, as seen by Eq. (29), to know a third parameter, $r_D/r_{0\infty}$, which is, by Eq. (28), proportional to $r_D(T_{F\infty})^{\frac{1}{2}}$. And finally, to assign numbers to quantities like the interaction strength U_0 (averaged over the two-body spin and i-spin states occurring in nuclei) and the surface thickness, we must specify one additional constant, for example r_D itself.

In other words the specification of the four constants α, $\bar{E}_\infty$, $T_{F\infty}$, and r_D is sufficient for determining completely all other quantities but is not necessary for deducing the qualitative results of the model. Using for these four constant the values $\alpha=1$, $\bar{E}_\infty=-15.7$ Mev, $T_{F\infty}=38$ Mev, and r_D equal to the Compton wavelength of the π-meson $\hbar/m_\pi c=1.4$ fermi, we find the following values for other derived quantities:

Surface thickness $(1.67r_D)=2.34$ fermi,

$U_0=24.8$ Mev,

$p_D=2.26\times10^{-14}$ g cm/sec,

$(p_D^2/2M=95.7$ Mev),

$r_{0\infty}=1.13$ fermi.

Conversely, given the single model parameter C (proportional to the product $U_0 r_0^3 p_D$), one needs only to solve Eq. (13) for enough successively larger values of x_0 so as to be able to predict the limits φ_∞ and $\epsilon_{F\infty}$ [which can be checked by Eq. (14)] in order to determine α, the ratio $\bar{E}_\infty / T_{F\infty}$, and the qualitative saturation properties. The additional knowledge of p_D (or equivalently $U_0 r_D^3$) is necessary to determine the individual quantities $\bar{E}_\infty$, $T_{F\infty}$ and the slope of the R versus $A^{\frac{1}{3}}$ curve. Finally we need r_D (or equivalently U_0) to find the value of A corresponding to a particular value of x_0 and to find the value of the surface thickness.

CONCLUSION

Treating the ground state of a nucleus as a degenerate gas of nucleons and assuming simple expressions for the space- and momentum-dependent potential energy between two nucleons (averaged over spin or more precisely, between two differential nuclear volume elements) we have deduced a nonlinear integral equation for the self-consistent nucleon spatial density, through the use of a variational procedure. The solutions for all but the very light nuclei show saturation (e.g., a linear $A^{\frac{1}{3}}$ versus R relation).

We have shown that in the limit of very large nuclei ($A \to \infty$) the spatial densities describe nuclear matter and moreover, two of the model parameters p_D and the combination $U_0 r_D^3$ can be fixed so as to give, in this limit, (i) the usual nuclear matter values for the Fermi kinetic energy (38 Mev) and the average energy (-15.7 Mev) and (ii), agreement with the Hugenholtz-Van Hove theorem (in our notation $\alpha = 1$). For finite nuclei it is possible to define a surface region, whose thickness is independent of the size of the nucleons, such that the interior (remainder of the nucleus) is just nuclear matter.

Since r_D represents the range of the Yukawa spatial interaction, it is natural to evaluate it as the Compton wavelength of the π meson. This choice for r_D fixes (i) the relationship between A and R, (ii) the magnitude of the surface thickness (2.3 fermis), and (iii) the strength of the interaction, U_0 (-25 Mev).

The possibility of exhibiting saturation with reasonable parameter values does not constitute a verification of the details of the especially simple nuclear force law which we have used.

We are currently investigating the inclusion into the model of the Coulomb repulsion energy of the protons.

Note added in proof. The present paper shows that the gross features of nuclear structure can be exhibited by a "classical" model with velocity-dependent interactions. The treatment above is rather schematic, in the neglect of Coulomb effects and in the discussion of the surface thickness. The further work including Coulomb effects promised above has now been done. Coupled nonlinear integral equations for separate neutron and proton densities, with Coulomb interactions among the protons, have been solved. A Yukawa space dependence of the nucleon-nucleon interaction is again assumed. The force-range r_D is now not a free parameter, but is determined by the solution procedure; and it takes a value substantially smaller than the $\hbar/(m_\pi c)$ arbitrarily assigned in the earlier work. Nevertheless, the surface thickness, now defined as the 90% to 10% dropoff distance in nucleon density, retains a value close to the experimental value. Quite accurate values of binding energies and neutron-proton ratios are found. This work will be submitted for publication shortly.

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