Limit on High-Energy Cross Section from Analyticity in Lehmann Ellipses*

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High-energy limits on the scattering amplitude and the total cross section follow from that analyticity of the scattering amplitude in ellipses in $\cos\theta_{\mathrm{c.m.}}$ which has been proved rigorously by Lehmann, together with unitarity. The limits for the scattering amplitude are $|T| < (\mathrm{const})W^4 \ln^2 W$, for the forward and backward directions, and $|T| < (\mathrm{const})W^3 \ln^{\frac{3}{2}} W$ for other fixed directions, where W is the total center-of-mass (c.m.) energy. For the total cross section the limit is $\sigma < (\mathrm{const})W^2 \ln^2 W$.

FROISSART[1] has obtained bounds on the high-energy behavior of the scattering amplitude T and the cross section σ for an amplitude which is given by the Mandelstam representation[2] by using two ideas. First, analyticity of $T(W,z)$ in z, where W is the center-of-mass (c.m.) energy, and $z = \cos\theta$, where θ is the c.m. scattering angle, leads to a geometric decrease of the partial wave amplitudes, $a_l(W)$, with increasing l. Secondly, unitarity requires $|a_l(W)| \leqslant 1$ for all l, W.

In this note we use Froissart's ideas to obtain bounds on the high-energy behavior of T and σ which follow from that analyticity of $T(W,z)$ in ellipses in the z plane which has been proved rigorously by Lehmann.[3]

Lehmann's result is that the real and imaginary parts of the elastic scattering amplitude $T(W,z)$ are analytic functions of z in ellipses with center at the origin and major and minor axes x_0, $(x_0^2-1)^{\frac{1}{2}}$ for the real part and $2x_0^2-1$, $2x_0(x_0^2-1)^{\frac{1}{2}}$ for the imaginary part. Here

$$x_0(W) = \left\{1 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{K^2[W^2-(m_1-m_2)^2]}\right\}^{\frac{1}{2}},$$

and

$$K = (1/2W)[W^2-(m+\mu)^2]^{\frac{1}{2}}[W^2-(m-\mu)^2]^{\frac{1}{2}}$$

is the magnitude of the c.m. momentum of either particle, μ and m are the masses of the particles, and m_1 and m_2 are the lowest masses of states $|n\rangle$ for which $\langle 0|j|n\rangle\langle n|f|p+k,\gamma\rangle \neq 0$ and $\langle 0|f|n\rangle\langle n|j|p+k,\gamma\rangle \neq 0$, respectively. [The operators j and f are the currents associated with the particles of masses μ and m, respectively, and $|p+k,\gamma\rangle$ is a state of total momentum $p+k$, internal quantum numbers γ, and total mass squared not less than $(m+\mu)^2$.]

We will show that Lehmann's analyticity leads to geometric decrease of $a_l(W)$ with l by using Cauchy's theorem. To treat both ReT and ImT together we choose as Cauchy contour the ellipse E with center at the origin and major and minor axes a, $(a^2-1)^{\frac{1}{2}}$ where

$$a = \left[1 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{2K^2[W^2-(m_1-m_2)^2]}\right]^{\frac{1}{2}}.$$

This ellipse E is always inside both Lehmann ellipses. Using this ellipse, Cauchy's theorem states

$$T(W,z) = \frac{1}{2\pi i}\oint_E \frac{T(W,z')}{z'-z}dz', \quad z \text{ in } E.$$

Using the expansion of $(z'-z)^{-1}$ in Legendre polynomials,[4]

$$\frac{1}{z'-z} = \sum_{l=0}^{\infty} (2l+1)P_l(z)Q_l(z'),$$

and the expansion of T in terms of partial wave amplitudes $a_l(W)$,

$$T(W,z) = \frac{1}{\pi^2}\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)a_l(W)P_l(z), \qquad (1)$$

we find

$$a_l(W) = \frac{1}{\pi^2}\frac{W}{K}\frac{1}{2\pi i}\oint_E T(W,z')Q_l(z')dz'. \qquad (2)$$

From Eq. (2) we can bound $a_l(W)$:

$$|a_l(W)| \leqslant \frac{1}{2\pi^3}\frac{W}{K}|T(W,z)|_{\max}|Q_l(z)|_{\max}L, \qquad (3)$$

where the maxima are taken for z on E, and L is the arc length of the ellipse. We now state upper bounds for the right-hand side of Eq. (3):

$$|T(W,z)|_{\max} \leqslant R_1(W), \qquad (4a)$$

where $R_1(W)$ is a fixed polynomial,

$$|Q_l(z)|_{\max} \leqslant \left(\frac{\pi}{l}\right)^{\frac{1}{2}}\left[1-\frac{1}{u^2}\right]^{-\frac{1}{2}}\frac{1}{u^{l+1}} \qquad (4b)$$

where

$$u = a + (a^2-1)^{\frac{1}{2}},$$

and

$$L < 2[+(a^2a-1)^{\frac{1}{2}}]. \qquad (4c)$$

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1 M. Froissart, Phys. Rev. **123**, 1053 (1961). We thank Dr. Froissart for sending us a copy of his paper prior to publication.

2 S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

3 H. Lehmann, Nuovo cimento **10**, 579 (1958).

4 E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1927), 4th ed., p. 322.

Equation (4a) follows from the temperateness of the (Jost-Lehmann)-Dyson-Lehmann representation for T, which we assume. Equation (4b) follows from a standard bound[5] on $Q_l(z)$, and the fact that $|z+(z^2-1)^{\frac{1}{2}}|=u$ on E. Equation (4c) states that the arc length of the ellipse is less than the perimeter of the circumscribed rectangle. We have thus obtained the bound

$$|a_l(W)|<\frac{1}{\pi^{\frac{3}{2}}}\frac{1}{l^{\frac{1}{2}}}\left[1-\frac{1}{u^2}\right]^{-\frac{1}{2}}\frac{W}{K}\frac{R_1(W)}{u^l}, \tag{5a}$$

or, sufficient for our present purposes,

$$|a_l(W)|<R(W)/l^{\frac{1}{2}}u^l, \tag{5b}$$

where inessential factors have been absorbed in $R(W)$. Equation (5a) or (5b) exhibits the geometric decrease of $a_l(W)$ with l.

Following Froissart, we now choose l_0 so that for $l\geqslant l_0$, $|a_l(W)|<1$ follows from Eq. (5b); we use the unitarity bound $|a_l(W)|\leqslant 1$ for $l<l_0$. Then using the partial wave expansion, Eq. (1), we find

$$|T(W,1)|<\frac{1}{\pi^2}\frac{W}{K}\left[\sum_{l=0}^{l_0-1}(2l+1)+\sum_{l=l_0}^{\infty}\frac{2l+1}{l^{\frac{1}{2}}}\frac{R(W)}{u^l}\right]. \tag{6}$$

The two sums in Eq. (6) are bonded by

$$\sum_{l=0}^{l_0=1}(2l+1)=l_0^2,$$

and

$$\sum_{l=l_0}^{\infty}(2l+1)\frac{1}{l^{\frac{1}{2}}}\frac{R(W)}{u^l}<R(W)\sum_{l=l_0}^{\infty}\frac{2l+1}{u^l}$$
$$=\frac{R(W)}{u^{l_0}}\left[\frac{2u^2}{(u-1)^2}+\frac{(2l_0-1)u}{u-1}\right].$$

We choose

$$l_0=[\ln R(W)]/(\ln u),$$

which leads to the bound

$$|T(W,1)|<(\text{const})W^4\ln^2 W,$$

where we have used the approximations

$$\left.\begin{array}{l} a(W)=1+c/W^4, \\ u\approx 1+(2c)^{\frac{1}{2}}/W^2, \end{array}\right\} \begin{array}{l} W \text{ large}, \\ c=(m_1^2-\mu^2)(m_2^2-m^2). \end{array}$$

In terms of q, the laboratory momentum of the incident particle, this bound is

$$|T(q,1)|<(\text{const})q^2\ln^2 q.$$

Using the optical theorem, the total cross section σ is bounded by

$$\sigma(W)<(\text{const})W^2\ln^2 W,$$

or

$$\sigma(q)<(\text{const})q\ln^2 q.$$

In a similar way, using the standard bound[5] on $P_l(\cos\theta)$, $\theta\neq 0$ or π,

$$|P_l(\cos\theta)|<2/(l\pi\sin\theta)^{\frac{1}{2}},$$

we find

$$|T(W,\cos\theta)|<\frac{(\text{const})}{(\sin\theta)^{\frac{1}{2}}}W^3\ln^{\frac{3}{2}}W,\quad 0<\theta<\pi,$$

or

$$|T(q,\cos\theta)|<\frac{(\text{const})}{(\sin\theta)^{\frac{1}{2}}}q^{\frac{3}{2}}\ln^{\frac{3}{2}}q,\quad 0<\theta<\pi.$$

We have obtained our bounds using analyticity in the small Lehmann ellipse. The bound obtained depends on the rate with which the major axis approaches the value one for high energies. Since for both Lehmann ellipses, $a(W)\approx 1+(\text{const})/W^4$ for large W, use of the larger ellipse would not improve the result. For the same reason, the somewhat larger domain of analyticity obtained rigorously by Mandelstam[6] would not lead to a stronger result. Froissart assumes the validity of the Mandelstam representation, in which the intersection of the boundary of analyticity and the real axis goes as $1+(\text{const})/W^2$ for large W, and obtains a correspondingly stronger result. It is interesting that analyticity in an ellipse with this larger major axis would yield the same high-energy bound as the Mandelstam analyticity in the cut plane.

Finally we remark that the bound on T which we have obtained is not strong enough to guarantee the validity of dispersion relations with one subtraction.

[5] E. W. Hobson, *Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, New York, 1931), pp. 61 and 309.

[6] S. Mandelstam, Nuovo cimento **15**, 658 (1960).