Rigorous Calculation of the Nucleation Field in a Ferromagnetic Film or Plate*

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The rigorous methods of micromagnetics have been used to calculate nucleation fields and corresponding incipient-domain widths for a ferromagnetic film or plate with the applied field along its faces and with a direction of easy magnetization perpendicular to them. The calculation requires finding the largest zero of a fourth-order determinant whose elements depend on the roots of a cubic equation, and then maximizing the result by trial-and-error variation of the parameters that determine the period and direction of the sinusoidal oscillation in the plane of the film. The calculation was therefore programmed for an electronic digital computer; and to narrow the area of search for zeros, preliminary calculations were done by use of two types of approximation, one of which set an upper and the other a lower limit to the correct zero. The results show that Muller's approximate method of solving the same problem is satisfactory in the range of parameters for which it was designed. Experimental data of Huber and Smith agree satisfactorily with the theory.

1. INTRODUCTION

RIGOROUS calculations in micromagnetics[1,2] encounter two obstacles. The first is that the general equations are nonlinear. This obstacle is not present when attention is concentrated on the determination of the nucleation field (defined as the value to which the applied field must be decreased in order that an initial uniform magnetization may become unstable). The second obstacle is the importance of "imperfections," be they of geometry, of structure, or of homogeneity. The easy nucleation-field calculations relate to ideal specimens, difficult to realize experimentally; the easy experiments involve specimens subject to unknown random perturbations, difficult to include in the theory. An exception is the single-domain (or almost single-domain) particle; here imperfections apparently play a minor role, and nucleation-field theory and experiment agree satisfactorily.[3,4] In bulk material the difficulties mentioned have not yet been surmounted.

The calculation to be presented relates to another case in which we expect the effect of imperfections on the nucleation field to be minor. This is the case of a film with a direction of easy magnetization perpendicular to the film faces, and with the applied field parallel to them. Some experimental data[5] have been taken on such films. In contrast to the usual films, where an easy direction is parallel to the faces and the nucleation field is negative, here the nucleation field is positive and may be quite large; it should therefore be comparatively insensitive to imperfections.[6]

2. GENERAL THEORY

The theory[3,4] when applied to a film takes the following form. We take the film surfaces in xz planes at $y=\pm b$, and the applied field H_0 along the z axis. The z axis is assumed to be a direction of (not necessarily stable) equilibrium of the anisotropy forces by themselves.

Starting with a large H_0, we decrease H_0. At any value of H_0, we consider the possibility of a small deviation (α,β) from the original uniform magnetization along z; here α and β are the x and y direction cosines of the spontaneous magnetization M_s. The free energy associated with such a deviation is, to the second order in α and β,

$$W=\tfrac{1}{2}C\int[(\nabla\alpha)^2+(\nabla\beta)^2]d\tau + \frac{1}{2}\int(g_{11}\alpha^2+2g_{12}\alpha\beta+g_{22}\beta^2)d\tau + \tfrac{1}{2}H_0M_s\int(\alpha^2+\beta^2)d\tau+\frac{1}{8\pi}\int_{\text{space}}\mathbf{h}^2d\tau, \quad (1)$$

in the notation of reference 3 (except that J_s is now written M_s). The four terms are, in order, the exchange, anisotropy, external magnetic, and internal magnetic energies. The first three integrals extend over the volume of the specimen, which must be supposed finite until the later variational procedures have been completed. The internal magnetizing force $\mathbf{h}=h_x\mathbf{i}+h_y\mathbf{j}+h_z\mathbf{k}=-\nabla U$ must be computed from the transverse magnetization,

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[3] W. F. Brown, Jr., Phys. Rev. **105**, 1479–1482 (1957).
[4] E. H. Frei, S. Shtrikman, and D. Treves, Phys. Rev. **106**, 446–455 (1957).
[5] E. E. Huber, Jr., and D. O. Smith, J. Appl. Phys. **30**, 267S–269S (1959).
[6] Additional calculations were originally planned. After the film, other geometries were to be investigated, by modification of parts of the digital-computer program; and after determination of the nucleation field, the subsequent nonlinear process was to be investigated. It proved impossible to carry out these plans because the available computer, unlike that used in previously reported calculations [W. F. Brown, Jr., J. Appl. Phys. **29**, 470–471 (1958)], was not provided with flexible symbolic-programming routines or with fast input-output devices. Programming even of the calculation reported here proved almost prohibitively laborious and time-consuming; transformation to a structurally similar program, differing only in the particular mathematical functions used, would have been almost as laborious as the writing of the original program; and attempting any calculations beyond those reported was out of the question with the time, funds, and staff available.

$M_s\mathbf{u}=M_s(\alpha\mathbf{i}+\beta\mathbf{j})$, by the methods of potential theory. In the numerical calculations to be discussed, $g_{11}=g_{12}=0$ and $g_{22}=-g$, with $g>0$.

If W is positive for all vector functions $\mathbf{u}(x,y,z)$, the original state is stable. This is certainly the case if H_0 is so large that the quadratic form $\frac{1}{2}(g_{11}+M_sH_0)\alpha^2+g_{12}\alpha\beta+\frac{1}{2}(g_{22}+M_sH_0)\beta^2$ is positive definite; for the exchange and internal magnetic terms in W are non-negative. In the special case under consideration, this sufficient condition for stability is satisfied if $M_sH_0>g$.

If W is negative for some vector function $\mathbf{u}$, the original state is unstable with respect to a deviation of the form $\mathbf{u}$. This is certainly the case if H_0 is sufficiently small (algebraically). For corresponding to any particular function $\mathbf{u}$, there is a critical value of H_0, say H_c, for which $W=0$; and if $H_0<H_c$, this particular $\mathbf{u}$ makes W negative. From Eq. (1),

$$M_sH_c=\left\{-\tfrac{1}{2}C\int[(\nabla\alpha)^2+(\nabla\beta)^2]d\tau - \frac{1}{2}\int[g_{11}\alpha^2+2g_{12}\alpha\beta+g_{22}\beta^2]d\tau - \frac{1}{8\pi}\int_{\text{space}}\mathbf{h}^2d\tau\right\}\Big/\left\{\frac{1}{2}\int(\alpha^2+\beta^2)d\tau\right\}. \quad (2)$$

Our problem is to maximize H_c with respect to the vector function $\mathbf{u}$, i.e., with respect to the scalar functions α and β; the value thus obtained is the value of H_0 at which the original uniform state becomes unstable, and the corresponding functions α and β describe the initial deviation from uniform magnetization (either a uniform rotation or an incipient domain structure).

Standard variational procedures lead to the following pair of linear partial differential equations in the region inside the film, together with the boundary conditions $\partial\alpha/\partial n=\partial\beta/\partial n=0$ on the surface:

$$-C\nabla^2\alpha+(g_{11}-M_sH_0)\alpha+g_{12}\beta+M_s\partial U/\partial x=0, \quad (3)$$

$$-C\nabla^2\beta+g_{12}\alpha+(g_{22}-M_sH_0)\beta+M_s\partial U/\partial y=0. \quad (4)$$

These differential equations also involve the magnetostatic potential U, which is related to α and β by Poisson's equation $\nabla^2U=4\pi M_s\nabla\cdot\mathbf{u}$. The potential U' in the external region satisfies Laplace's equation; there are the usual two boundary conditions of potential theory at the surface and the usual finiteness conditions at infinity. One solution of this system of equations is $\alpha=\beta=0$, $\nabla U=\nabla U'=0$; and if $M_sH_0>g$ (more generally, if H_0 is so large that the quadratic form mentioned above is positive definite), this solution is unique. Nonvanishing solutions in general exist only when H_0 has one of a set of characteristic values or eigenvalues; our problem is to find the largest of these eigenvalues.

For the film, we may now allow the x and z dimensions to become infinite. We may then assume a dependence on x and z through a factor $e^{i(\lambda x+\nu z)}$. The partial differential equations thereupon reduce to second-order ordinary differential equations in y. For any given λ and ν, solutions exist only for certain discrete values of H_0; but λ and ν may have any real values. Our problem is to find the largest eigenvalue of H_0 as a function of λ and ν, and then to maximize it with respect to λ and ν.

To solve the differential equations of the internal region, we may assume a dependence on y through a factor $e^{i\mu y}$; then

$$\alpha=Ae^{i(\lambda x+\mu y+\nu z)},\ \beta=Be^{i(\lambda x+\mu y+\nu z)},\ U=C'e^{i(\lambda x+\mu y+\nu z)}. \quad (5)$$

The constant μ need not be real. The three differential equations reduce to three homogeneous linear equations in A, B, and C'; permissible values of μ are found by setting the determinant of the system equal to zero. This gives, in general, a cubic equation in μ^2 and therefore six values of μ: μ_1, μ_2, μ_3, $-\mu_1$, $-\mu_2$, $-\mu_3$. The general solution for U in the internal region is then of the form

$$U=\sum_{j=1}^{3}[C_j^+e^{i\mu_jy}+C_j^-e^{-i\mu_jy}]e^{i(\lambda x+\nu z)}; \quad (6)$$

the constant coefficients that appear in the equations for α and β can be expressed in terms of the C's. The solution in the external region $y>b$ is of the form

$$U'=D^+\exp[-(\lambda^2+\nu^2)^{\frac{1}{2}}y+i(\lambda x+\nu z)]; \quad (6a)$$

in the external region $y<-b$, y is replaced by $-y$ and D^+ by another constant D^-. Thus there are eight constants to be evaluated by use of the four boundary conditions at $y=+b$ and of the four at $y=-b$. The resulting system of eight linear homogeneous equations has a solution only if its determinant vanishes; this gives the secular equation that determines the permissible values of H_0 for given λ and ν.

The problem is considerably simplified by noticing that the solutions can be classified into two groups: solutions for which U is even in y (then α is even and β odd), and solutions for which U is odd in y (then α is odd and β even). Separate treatment of even and odd solutions halves the number of constants and boundary conditions involved. The secular equation in each case is obtained by setting a fourth-order determinant equal to zero; it involves trigonometric functions whose arguments must be presumed, in general, to be complex.

A particular problem is specified by giving the values of C, M_s, and g (properties of the material) and the value of the half-thickness b (property of the specimen). By going over to dimensionless quantities, the number of parameters necessary to specify a problem is reduced to two: $\rho=4\pi M_s^2b^2/C$ and $\bar{g}=gb^2/C$. When values of these two parameters have been given, the problem is: For given $l(=\lambda b)$ and $n(=\nu b)$, find the largest eigenvalue of $h(=H_0M_sb^2/C$, the dimensionless applied field); then maximize this h with respect to l and n. The

numerical procedure for finding the eigenvalue, for given l and n, must consist essentially in computing $\Delta(h)$, the determinant which is to vanish, as a function of h over a sufficient range of values to locate the largest zero. For each h, the computation of $\Delta(h)$ requires first solving the cubic equation to find the permissible values m_1, m_2, and m_3 of $m(=\mu b)$, then substituting these in the formula for $\Delta(h)$.

The solutions of interest turn out to be those with U odd in y. In dimensionless form, the cubic equation is

$$(m^2+n^2)(m^2-s^2)(m^2-s^2+\bar{g}) +\rho m^2(m^2-s^2+\bar{g})+\rho l^2(m^2-s^2)=0, \quad (7)$$

where[7]

$$s^2=\bar{g}-h-\eta^2\geq 0,\ \eta^2=l^2+n^2. \quad (8)$$

The determinant $\Delta(h)$ has elements Δ_{ij} ($i, j=1$ to 4), where for $j=1$ to 3

$$\Delta_{1j}=\sin m_j, \quad (9)$$

$$\Delta_{2j}=(m_j^2-s^2+\rho)m_j(\cos m_j)/(m_j^2-s^2), \quad (10)$$

$$\Delta_{3j}=m_j^2(\sin m_j)/(m_j^2-s^2), \quad (11)$$

$$\Delta_{4j}=m_j(\cos m_j)/(m_j^2-s^2+\bar{g}), \quad (12)$$

and where

$$\Delta_{14}=1,\ \Delta_{24}=-\eta,\ \Delta_{34}=\Delta_{44}=0. \quad (13)$$

The procedure outlined is laborious, time-consuming, and costly. Therefore it seemed desirable to find preliminary calculation procedures that would locate the solutions approximately, and that preferably would place them between definite upper and lower limits.

3. APPROXIMATIONS

If the maximization of H_c as given by Eq. (2) is performed under an additional constraint, the nucleation field so obtained will be equal to or less than the correct value. If, on the other hand, it is performed after a nonpositive term has been dropped from the right member, the nucleation field so obtained will be equal to or greater than the correct value. Approximate solutions obtained by these two methods may be called overconstrained and underconstrained solutions, respectively.

TABLE I. Overconstrained solutions for film with $g_{11}=g_{12}=0$, $g_{22}=-g<0$.

	$\alpha=0$, $\partial\beta/\partial y=0$		$\alpha=0$	
ρ	η	$\bar{g}-h$	η	$\bar{g}-h$
1	0.327	0.8409	0.3272	0.8409
2	0.518	1.514	0.5178	1.5135
10	1.20	5.229	1.2128	5.1964
20	1.61	8.555	1.6364	8.3971
100	2.90	25.60	2.9860	23.200
200	3.68	40.70	3.7609	34.802
1000	6.30	119.1	6.1471	85.286
2000	7.94	189.0	7.4798	123.86

TABLE II. Underconstrained solution for film with $g_{11}=g_{12}=0$, $g_{22}=-g<0$.

$\bar{g}$	ρ	$\eta(=l)$	h	$\bar{g}-h$
1	1	0.40068	0.23890	0.76110
	2	0.54763	0.09031	0.90969
	10		<0	
2	1	0.35010	1.17554	0.82446
	2	0.58241	0.66360	1.33640
	10		<0	
3	1		≈2.166	≈0.834
	2	0.55929	1.56844	1.43156
	10	0.94143	0.48346	9.51654
5	1	0.33404	4.16367	0.83633
	2	0.53955	3.52404	1.47596
	10	1.10221	1.55820	3.44180
10	1	0.33082	9.16118	0.83882
	2	0.52814	8.50223	1.49777
	10	1.20711	5.61732	4.38268
20	1	0.32925	19.16009	0.83991
	2	0.52312	18.49375	1.50625
	10	1.22313	15.16750	4.83250
100	1	0.32808	99.15929	0.84071
	2	0.51948	98.48785	1.51215
	10	1.21894	94.86850	5.13150

A number of such approximate solutions may be found, but not all are useful. For instance, if β is constrained to be zero, the resulting problem is easily solved, but all the nucleation fields so obtained are negative; this solution is too severely overconstrained to be useful.

The double constraint $\alpha=0$, $\partial\beta/\partial y=0$ leads to a problem that requires only solution of the potential problem and evaluation of the average magnetizing force through the thickness of the film. The single constraint $\alpha=0$ leads to a problem similar to the rigorous one except that only a quadratic equation has to be solved, instead of a cubic; in this approximation, the parameters l and n occur only in the combination $\eta^2=l^2+n^2$, and the eigenvalues of $\bar{g}-h$ are independent of $\bar{g}$. Thus this approximation requires specification of fewer parameters than does the rigorous calculation. Calculations were made of both of these overconstrained solutions; the results of the two (with, for the second, U odd and β even in y) agree quite closely with each other until ρ becomes very large, i.e., until the exchange forces become very small. Representative results are given in Table I.

Dropping of the nonpositive term in $(\nabla\alpha)^2$ in Eq. (2) leads to another problem that requires only a quadratic equation; in this case, however, unless $l=0$, $\bar{g}$ and the value of l (as well as the value of η) enter explicitly. For

[7] Physically, the inequality given in (8) states that there are no solutions sinusoidal in x and z for $M_sH_0>g-C(\lambda^2+\nu^2)$. To prove it, assume such sinusoidality in Eq. (1) and repeat the argument previously used to show that there are no solutions for $M_sH_0>g$. The integrations are now extended over the film thickness and over an integral number of periods along x and z.

TABLE III. Rigorous solutions for film with $g_{11}=g_{12}=0$, $g_{22}=-g<0$.

$\bar{g}$	ρ	$\eta(=l)$	h	$\bar{g}-h$
1	1	0.33694	0.16595	0.83405
	2		<0	
2	1	0.33451	1.16411	0.83589
	2	0.53668	0.52609	1.47391
	10		<0	
10	1	0.32961	9.16072	0.83928
	2	0.52482	8.49831	1.50170
	10	1.18961	5.34745	4.65255
	20	1.45142	3.53959	6.46041
20	1	0.32905	19.15995	0.84005
	2	0.52242	18.49273	1.50727
	10	1.21193	15.09718	4.90282
	20	1.56624	12.74659	7.25341
	100	2.04740	8.08268	11.91732
100	1	0.32753	99.15927	0.84073
	2	0.51864	98.48779	1.51221
	10	1.21386	94.86556	5.13444
	20	1.62949	91.86411	8.13589
	100	2.72636	81.29878	18.70122
	200	3.09995	76.73827	23.26173
	1000	3.51827	70.72039	29.27961
	2000	3.58080	69.71406	30.28594
200	20	1.63417	191.73545	8.26455
	200	3.38437	172.59086	27.40914
1000	100	2.96277	977.38088	22.61912
	1000	5.41001	935.44763	64.55237
2000	200	3.72236	1966.18	33.82000
	2000	6.55185	1907.38	92.62000

$l=0$, this underconstrained solution coincides with the overconstrained solution $\alpha=0$. This fact implies (as is easily verified directly) that for $l=0$, the rigorous solution coincides with these, and the cubic equation reduces to a quadratic. Results of calculations of this underconstrained solution are given in Table II. From physical considerations one expects (and the numerical results confirm) that when $\bar{g}$ is large, this underconstrained solution and the overconstrained solution $\alpha=0$ will agree well; but that when $\bar{g}$ is of order unity or less, they will differ considerably.

The over- and underconstrained solutions were first explored by hand calculation and then programmed for the computer (a Remington–Rand Univac Scientific, model 1103).

The rigorous calculation was then programmed. In it, selection of parameter values and of starting values was guided by the previous results of the approximate methods.

4. NUMERICAL RESULTS

Results by the overconstrained solutions are tabulated in Table I, by the underconstrained solution in Table II, and by the rigorous solution in Tables III and IV. The rigorous results of Table III are also plotted in Figs. 1 and 2. As was predicted, the overconstrained and underconstrained values of h bracket the rigorous values, for given ρ and $\bar{g}$.

Muller[8] has studied this problem, but with emphasis on a thick slab rather than a film, and with approximations appropriate to that case. To obtain dimensionless quantities, he uses the magnetic energy-density parameter M_s^2 rather than the exchange energy-density parameter C/b^2 as denominator, and he assumes that the latter is relatively small. Table V shows a compar-

TABLE IV. Calculated values of h/ρ as a function of $\rho^{\frac{1}{2}}$ at constant $\bar{g}/\rho$. (Values marked * are from Table III; the rest were calculated independently.)

$\bar{g}/\rho$	$\rho^{\frac{1}{2}}$	h	h/ρ	$\eta(=l)$
0.001	100.00	0.096	0.0000096	1.585
	141.42	5.9827	0.0002991	2.136
	173.21	12.825	0.0004275	2.477
	223.61	27.818	0.0005564	2.939
	316.23	68.619	0.0006862	3.638
0.002	70.711	0.1130	0.0000226	1.587
	100.00	6.0018	0.0006002	2.136
	122.48	12.846	0.0008564	2.477
	158.11	27.841	0.0011136	2.938
	223.61	68.652	0.0013730	≈3.65
0.005	44.721	0.1522	0.000076	1.592
	63.246	6.0495	0.001512	2.136
	77.460	12.898	0.002150	2.476
	100.00	27.901	0.002790	2.936
	141.42	68.721	0.003436	3.633
0.01	31.623	0.2067	0.000207	1.597
	44.721	6.1188	0.003059	2.136
	54.772	12.977	0.004326	2.473
	70.711	27.994	0.005599	2.931
	100.00	68.839	0.006884	3.627
0.02	22.361	0.3030	0.000606	1.604
	31.623	6.2449	0.006245	2.134
	38.730	13.122	0.008748	2.456
	50.000	28.169	0.011268	2.922
	70.711	69.067	0.013813	3.614
0.05	14.142	0.5624	0.002812	1.618
	20.000	6.5895	0.016474	2.125
	24.495	13.525	0.022542	2.450
	31.623	28.660	0.028660	2.895
	*44.721	69.714	0.034857	3.581
0.10	10.000	0.9661	0.009661	1.627
	14.142	7.1195	0.035598	2.103
	17.321	14.1456	0.047152	2.416
	22.361	29.4182	0.058836	2.849
	*31.623	70.7204	0.070720	3.518
0.20	7.0711	1.7217	0.034434	1.609
	*10.0000	8.0827	0.080827	2.047
	12.2475	15.2653	0.101769	2.342
	15.8114	30.7823	0.123129	2.756
	22.3607	72.5261	0.145052	3.401
0.50	3.1623	0.7304	0.07304	1.127
	*4.4721	3.5396	0.17698	1.451
	6.3246	10.3652	0.25913	1.824
	7.7460	17.9025	0.29838	2.117
	10.0000	33.9784	0.33978	2.499
	*14.1421	76.7383	0.38369	3.100

[8] M. W. Muller, Phys. Rev. **122**, 1485–1489 (1961).

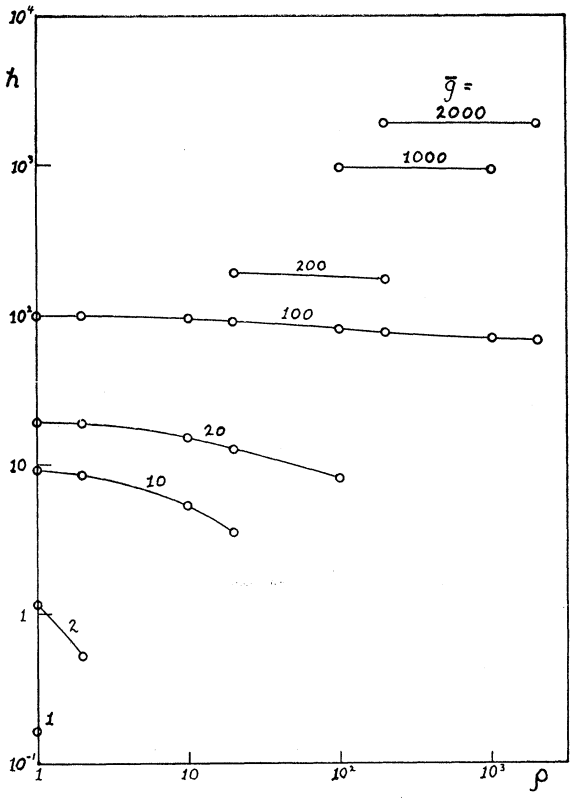

FIG. 1. Calculated values of reduced nucleation field $h=H_0M_s b^2/C$ as a function of reduced magnetic energy $\rho=4\pi M_s^2 b^2/C$, for various values of reduced anisotropy energy $\bar{g}=gb^2/C$ (C=exchange constant, b=half thickness, M_s=spontaneous magnetization, g=anisotropy constant, H_0=applied field intensity at nucleation). Nucleation field has been maximized with respect to wave number and orientation of incipient domain structure. From Table III. Reduced wave number ($\eta/2\pi$) increases from left to right along each curve.

ison of the rigorous values with the approximate values calculated with Muller's formula.

For the overconstrained and underconstrained solutions the function $\beta(y)$ (with the factor $e^{i(\lambda x+\nu z)}$ omitted) was calculated in some cases; it showed no sudden changes or other unusual features. For the rigorous solution, this calculation was not programmed for lack of time and manpower. The variation of β in the xz plane, in this incipient domain structure characteristic of nucleation, is sinusoidal; the more abrupt changes characteristic of a fully developed domain structure evidently depend in an essential way on the nonlinearity of the later stages of magnetization reversal.

For given $\bar{g}$, ρ, and η, the maximum nucleation field occurs at $l=\eta$, $n=0$ over the range of parameters explored. Physically this means that the incipient domain walls are parallel to the applied field. Examination of Eq. (2) shows that only the term containing $\mathbf{h}^2$ is sensitive to the angle $\theta=\tan^{-1}(l/n)$ and that the orientation $\theta=\pi/2$ is favored if $|\mathbf{h}_\beta|\gg|\mathbf{h}_\alpha|$, where $\mathbf{h}_\beta$ is the part of $\mathbf{h}$ due to β and $\mathbf{h}_\alpha$ the part due to α. In the range $g>0$ explored, the anisotropy favors a relatively large β and therefore a relatively large $\mathbf{h}_\beta$.

TABLE V. Test of Muller's approximation.

$\bar{g}$	ρ	h Rigorous	h Approximate
1	1	0.16595	0.09130
2	2	0.52609	0.32936
10	10	5.34745	4.47584
20	20	12.74659	11.50888
100	100	81.29878	79.1204
200	200	172.59086	169.88
1000	1000	935.44763	931.02
2000	2000	1907.38	1901.89
10	20	3.53959	4.16637
100	200	76.73822	76.61758
2	1	1.16411	0.36899
20	10	15.09718	12.58416

5. COMPARISON WITH EXPERIMENT

It is possible to compare the predictions of the theory with a few experimental values of nucleation field given in Fig. 2 of Huber and Smith.[5] In the experiments, the thickness $2b$ was varied, and the nucleation field H_0 (H_s in Huber and Smith's notation) was observed; presumably the film properties, including the anisotropy constant g, remained constant during the variation of thickness. In terms of our dimensionless quantities, the experiment measures h/ρ as a function of $\rho^{\frac{1}{2}}$ at constant

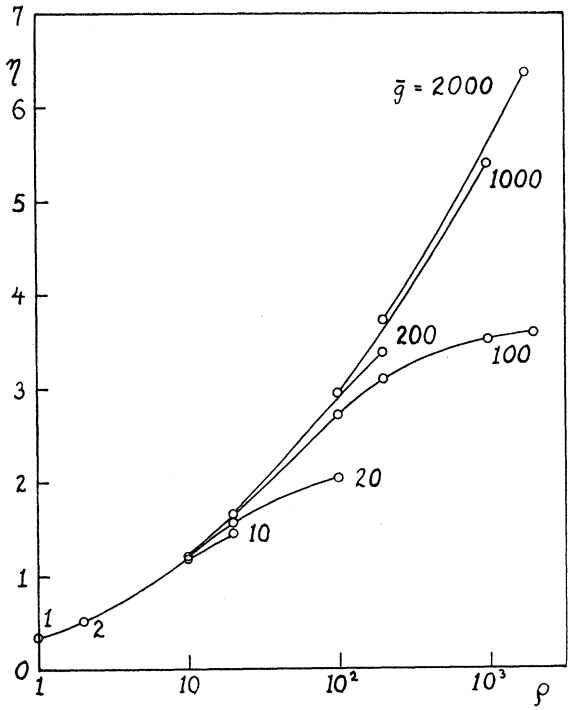

FIG. 2. Values of $\eta(\eta/2\pi$=reduced wave number=wave number×half thickness) corresponding to the reduced nucleation fields shown in Fig. 1.

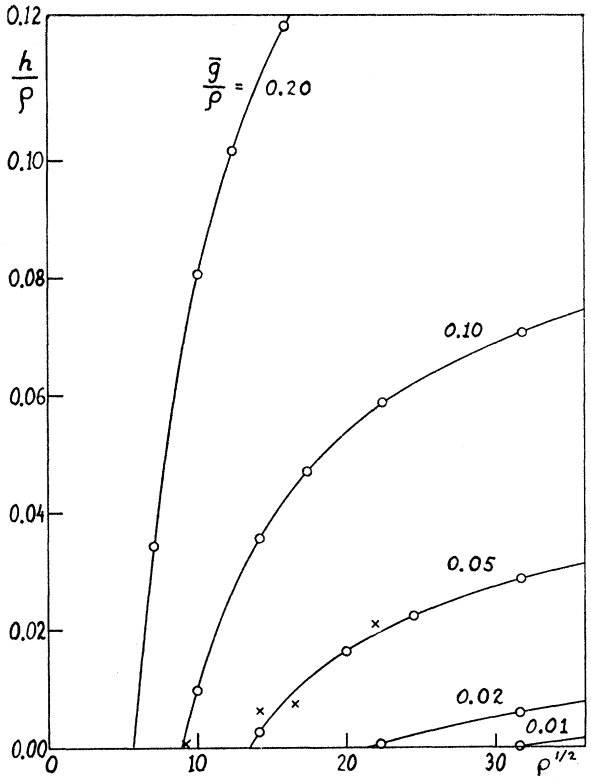

FIG. 3. Theoretical values of $h/\rho=H_0/4\pi M_s$ as a function of $\rho^{\frac{1}{2}}=2bM_s(\pi/C)^{\frac{1}{2}}$ for several values of $\bar{g}/\rho=g/4\pi M_s^2$. The calculated points (circles) have been connected by curves. The crosses are experimental points of Huber and Smith, reference 5.

$\bar{g}/\rho$; for

$$h/\rho=H_0/4\pi M_s,\ \rho^{\frac{1}{2}}=2bM_s(\pi/C)^{\frac{1}{2}},\qquad \bar{g}/\rho=g/4\pi M_s^2. \tag{14}$$

Table IV gives values of h/ρ vs $\rho^{\frac{1}{2}}$ for various values of $\bar{g}/\rho$. Some of these sets of values are plotted in Fig. 3. Also plotted are experimental points of Huber and Smith; these have been converted by taking $4\pi M_s=9.2\times10^3$ gauss, $C=3\times10^{-6}$ erg cm^{-1} (C is twice the exchange constant A used by some authors). Agreement of experiment with theory is not expected at small or negative nucleation fields because of the tendency of imperfections to nucleate domains. The observed positive nucleation fields agree reasonably well with the theoretical curve for $\bar{g}/\rho=0.05$ and deviate widely from the curves for $\bar{g}/\rho=0.02$ and 0.10. From $\bar{g}/\rho=0.05$ we get $g=3.4\times10^5$ erg cm^{-3}.

Huber and Smith attribute the anisotropy to tension isotropic in the xz plane. For simplicity we assume, in the usual elastic notation, $e_{xx}=e_{zz}=\text{const}=e$, $Y_y=0$; then the magnetostrictive contribution to the anisotropy energy is of the form $\text{const}+Be\beta^2$, where B is the same as the B of Huber and Smith. Therefore $Be=-\frac{1}{2}g=-1.7\times10^5$ erg cm^{-3}. The estimate of $|Be|$ given by Huber and Smith, and based on direct observation of the strain, is 10^4 to 10^5 erg cm^{-3}. The agreement is satisfactory.

By the Bitter pattern technique, Huber and Smith obtained a clearly developed domain structure in a field perpendicular to the film, but only a mottled appearance when the field remained parallel to the faces. This qualitative observation is consistent with the theory, which predicts a sinusoidal variation with a period of order 10^{-5} cm, about a thousandth the width of the domains observed in a perpendicular field.

Kaczér and Gemperle[9] have observed domain structures in magnetoplumbite after application of fields at various angles to the specimen surface, which is a basal plane and is normal to a direction of minimum uniaxial crystalline anisotropy energy. The present theory should be capable of predicting nucleation-field intensities and incipient-domain widths in such experiments. The published data include neither but do include detailed observations on the fully developed domain structure characteristic of the remanent state. Unfortunately, the values of ρ (10^4 to 10^{12}) lie in a range in which the computer routines gave large-number overflow alarms. Muller's approximations, however, should be usable in this range. Calculations by this method and by rough extrapolation of the rigorous numerical results both give domain widths that agree in order of magnitude with those observed. Observations of nucleation fields and of the structure at the instant of nucleation would provide a direct test of the theory.

6. CONCLUDING REMARKS

Micromagnetics can be important in two ways. First, it can produce some rigorous results that are of interest in themselves; I hope that the present results will be of some interest as further relevant film and plate data are obtained. Second, it can provide systematic methods of obtaining and assessing approximate calculation methods. The latter function may well be the more important of the two. In this respect, the method of overconstrained and underconstrained solutions, illustrated above for the film, may have general usefulness. No great originality is claimed for this; it goes back at least to Rayleigh. But it seems not to have been applied consciously and systematically in this field. Such systematic procedures should lead to more reliable approximations, and to more reliable estimates of their reliability, than can be achieved by the rather haphazard procedures of present domain theory.

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[9] J. Kaczér and R. Gemperle, Czech. J. Phys. **B10**, 505–510 (1960); **B11**, 510–522 (1961).