Analysis of the Transition Region between a Plasma and its Confining Magnetic Field*

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The problem considered here is the transition region separating a uniform plasma from its confining magnetic field. The geometry chosen is one-dimensional. A self-consistent solution of Maxwell's equations and the equations of motion of the particles is obtained. This problem was formulated and solved numerically by Tonks. It is the purpose of this paper to present an analytic solution. The most interesting feature of the analysis is that the transition region is finite, the transition taking place entirely within the orbit of the particle of deepest penetration into the confining field.

INTRODUCTION

In a recent paper,[1] Tonks considered the problem of the transition region between a plasma and a confining magnetic field. We shall not carry through the details of the formulation and the assumptions on which it is based, as these are given in Tonks' paper. We shall only present the results and define the notation.

We define a class η particle to be one which crosses the area $d\eta dz$ (lying on the $x-z$ plane at $x=\eta$) within a time dt and whose velocity vector lies within $\pm\alpha/2$ of the vertical. We let

$$\sigma(\eta)\alpha d\eta dz dt$$

be the number of class η particles crossing the surface $d\eta dz$ in a time dt. The current arising from all classes is

$$j_y(x)=\frac{2e}{v}\int_{\eta(x)}^{x}\frac{\sigma(\eta)\omega(\eta)\dot{y}(\eta,x)}{|\dot{x}(\eta,x)|}d\eta, \tag{1}$$

where the integration is over all classes of particles which have trajectories through the point x. The lower limit characterizes that trajectory which just reaches x. The field must satisfy Maxwell's equations so that

$$\frac{dB}{dx}=\frac{-4\pi}{c}j_y=\frac{-8\pi e}{cv}\int_{\eta(x)}^{x}\frac{\sigma(\eta)\omega(\eta)\dot{y}(\eta,x)}{|\dot{x}(\eta,x)|}d\eta. \tag{2}$$

This field equation together with the equations of motion of the particles in the field

$$\ddot{x}=\omega\dot{y},\qquad \ddot{y}=-\omega\dot{x}, \tag{3}$$

determine the solution, provided $\sigma(\eta)$ is known. This density distribution must be determined by a statistical study of the thermal equilibrium and is beyond the scope of this analysis. Since we are investigating a plasma which is to be uniform at $x=-\infty$, we choose $\sigma(\eta)$ to be a constant.

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[1] L. Tonks, Phys. Rev. **113**, 400 (1959).

The above system of equations was developed and solved numerically by Tonks. We would like to present an analytic solution of the problem.

First we introduce, following Tonks, the dimensionless variables

$$\Omega=\omega mc/eB_0,$$
$$T=\omega_0 t,$$
$$(X,Y,\mathrm{H})=(\omega/v)(x,y,\eta),$$

and

$$S=8\pi mv\sigma/B_0^2.$$

Then

$$\frac{d\Omega}{dX}=-S\int\frac{\Omega\dot{Y}}{|\dot{X}|}d\mathrm{H}, \tag{4}$$

$$d\dot{X}/dT=\Omega\dot{Y},\qquad d\dot{Y}/dT=-\Omega\dot{X}, \tag{5}$$

with $\dot{X}=dX/dT$, etc.

ANALYSIS

We shall replace the integration variable H by the angle between the velocity vector and the x axis (see Fig. 1). First we integrate the equations of motion to give

$$\dot{X}^2+\dot{Y}^2=1, \tag{6}$$

and

$$\dot{Y}=-A(X)+\text{const},$$

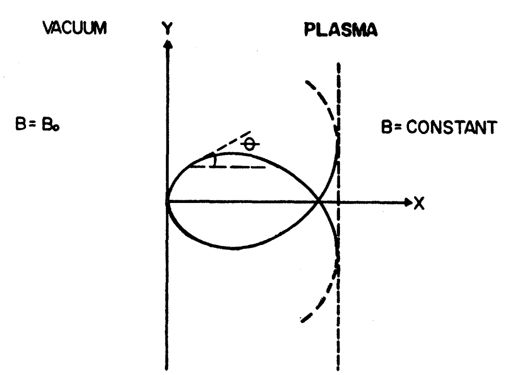

FIG. 1. Trajectory of the particle of deepest penetration into the confining magnetic field.

TABLE I. Transition layer thickness.

	L		D
S	Eq. (12)	Tonks	Eq. (13)
0	2.00	2.0	2.00
0.01	2.02	2.0	2.02
0.05	2.09	2.1	2.08
0.10	2.19	2.2	2.19
0.15	2.32	2.35	2.31
0.20	2.50	2.5	2.47
0.25	2.79	2.75	2.71
0.30	3.43		3.23
$1/\pi$	6.39		4.00

where $\Omega = dA/dX$. These equations represent energy and momentum conservation. Since $\dot{Y}=1$ when $X=\mathrm{H}$,

$$\dot{Y} = -A(X) + A(\mathrm{H}) + 1. \tag{7}$$

From Eq. (6) we see that we may set

$$\dot{X} = \cos\theta, \quad \dot{Y} = \sin\theta. \tag{8}$$

Differentiating Eq. (7)

$$\frac{d\dot{Y}}{d\mathrm{H}} = \frac{dA(\mathrm{H})}{d\mathrm{H}} = \Omega(\mathrm{H}) = \cos\theta \frac{d\theta}{d\mathrm{H}},$$

or

$$\Omega d\mathrm{H} = \cos\theta d\theta,$$

and substituting into Eq. (4)

$$\frac{d\Omega}{dX} = S \int \sin\theta \, \mathrm{sign}(\cos\theta) d\theta.$$

We see immediately that if X lies outside the orbit of the particle of deepest penetration ($\mathrm{H}=0$) the limits of integration are $\pi/2$ and $-\pi/2$ and the integral vanishes. [We integrate over the outgoing trajectories only, since the factor 2 in Eq. (1) takes account of the incoming trajectories.] We find then that the transition takes place entirely within this orbit and the field is strictly constant on both sides.

Inside this orbit

$$\frac{d\Omega}{dX} = S \int_{\pi/2}^{\theta} \sin\theta d\theta = -S\cos\theta, \tag{9}$$

where $\tan\theta$ is the slope of the velocity vector of the particle of deepest penetration into the confining field ($\mathrm{H}=0$) at the point X. X and θ are related by Eqs. (7) and (8):

$$\sin\theta = 1 - A(X),$$

where we have set $A(0)=0$. Using this relation to eliminate θ from the field equation [Eq. (9)], we find

$$d\Omega/dX = d^2A/dX^2 = -S[1-(1-A)^2]^{\frac{1}{2}}, \tag{10}$$

which is an ordinary differential equation.

Perhaps the most significant feature of this analysis is that the transition region is finite. We would like to obtain an expression for this thickness as a function of S. We may do so without obtaining an explicit solution for the field [Eq. (10)]. Let L be the transition thickness. Then $L=\int dX$, where the integral is taken over the outgoing part of that particle trajectory for which $\mathrm{H}=0$. For this trajectory, $\sin\theta = 1-A(X)$. Differentiating,

$$\cos\theta d\theta = -\Omega(X) dX,$$

so that

$$L = -\int \frac{\cos\theta}{\Omega} d\theta. \tag{11}$$

We may integrate Eq. (10) once to obtain Ω as a function of θ. This gives

$$\Omega^2 = S(\theta + \sin\theta\cos\theta) + 1 - \pi S/2.$$

Inserting this in Eq. (11), we have

$$L = \int_0^{\pi} \frac{\sin\varphi d\varphi}{[1-S(\varphi - \sin\varphi\cos\varphi)]^{\frac{1}{2}}}, \tag{12}$$

where $\varphi = \pi/2 - \theta$. Although we cannot perform the integral in terms of elementary functions, we can integrate numerically to obtain L for any given S. In Table I we list several values of the transition thickness for different S. We have included in the table the values computed by Tonks and the diameter of the orbit of a particle moving in a uniform field which is the average of $\Omega(0)$ and $\Omega(\infty)$. [See Eq. (38) of reference 1.]

$$D = 4/[1+(1-\pi S)^{\frac{1}{2}}] \tag{13}$$

We have tabulated only as far as $S=1/\pi$ since the integral becomes imaginary for $S>1/\pi$. The physical reason for this is that the confining field must withstand the pressure of the plasma and the pressure of the backbone field (B_∞). Thus for a fixed field the plasma density may not increase indefinitely. The basis for these ideas is contained in Eq. (37) of Tonks' paper

$$\Omega(0)^2 - \Omega(\infty)^2 = 1 - \Omega(\infty)^2 = \pi S.$$

A certain amount of care must be exercised in interpreting L for $S=0$ and $S=1/\pi$. When $S=0$ there is no plasma and a transition layer thickness becomes meaningless. Also when $S=1/\pi$, $\Omega(\infty)=0$ and the particle trajectories are not cyclic as we have assumed. Thus L must be interpreted as the limiting value of the transition thickness as $S\to 0$ or $1/\pi$. The special case $\Omega(\infty)=0$ has been treated, using an entirely different technique, by Grad[2] and Schmidt.[3]

ACKNOWLEDGMENT

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[2] H. Grad, Atomic Energy Commission Report NYO-9491, Institute of Mathematical Sciences, New York University, 1960 (unpublished).

[3] G. Schmidt, Bull, Am. Phys. Soc. 6, 290 (1961).