On the Proof of Parametric Dispersion Relations*

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Parametric dispersion relations for the connected time-ordered Green's functions, conjectured by Nishijima on the basis of perturbation theory, are shown to exist for the special case of decay processes. Extension of the proof to the more interesting situations of scattering and production depends upon the possibility of continuing one or more of the energy variables to negative values.

1. INTRODUCTION

NISHIJIMA[1] has shown on the basis of perturbation theory that there exist parametric dispersion relations (PDR) for the connected time-ordered Green's functions, and he has conjectured that such relations are valid in an exact field theory. We would like to point out that this conjecture is true for at least the special case of decay processes. An extension of the proof to include scattering and production amplitudes requires a statement concerning analytic continuation of the energy variables appearing in these functions to negative values.

The method employed consists of first finding a representation for the connected part of a time-ordered Green's function in terms of the corresponding retarded Green's function; this can be accomplished with a special choice of timelike momenta as the argument of both functions. This is a useful procedure because it is not difficult to derive a PDR for the special retarded function, and in this way obtain a PDR for the special time-ordered function. There then remains only the problem of generalizing, or continuing, the result to an arbitrary connected time-ordered function.

2. THE SPECIAL FUNCTIONS

We begin by briefly stating the notation used.[2] The operators $\mathcal{T}$ and R_x are the spin zero, scalar field time-ordered, and retarded functionals, respectively, where $\mathcal{T}_x \equiv \delta\mathcal{T}/\delta j(x)$ and R_x may be defined as $-i\mathcal{T}^\dagger\mathcal{T}_x$. The vacuum expectation value of the result of calculating $n-1$ functional derivatives of $\mathcal{T}_x$ and R_x yields the general n-point time-ordered function $\langle \mathcal{T}_{xy_1\cdots y_{n-1}}\rangle_0$, and the corresponding retarded function $\langle R_{x,y_1\cdots y_{n-1}}\rangle_0$; the subscript zero indicates that the source functions $j(z)$ have been set equal to zero after the differentiation is performed. The time-ordered function is symmetric in all n indices; the retarded function is symmetric in the $n-1$ y_i indices, each of which refers to a time earlier than x_0. We introduce the Fourier transforms

$$\delta(q+p_1+\cdots+p_{n-1})r(p_1,\cdots p_{n-1}) = (2\pi)^{-4}\int dxdy_1\cdots dy_{n-1} \times\exp(iq\cdot x+ip_1\cdot y_1+\cdots+ip_{n-1}\cdot y_{n-1}) \times\langle R_{x,y_1\cdots y_{n-1}}\rangle_0, \quad (1)$$

and

$$\delta(q+p_1+\cdots+p_{n-1})\tau(p_1,\cdots p_{n-1}) = -i(2\pi)^{-4}\int dxdy_1\cdots dy_{n-1} \times\exp(iq\cdot x+ip_1\cdot y_1+\cdots+ip_{n-1}\cdot y_{n-1}) \times\langle \mathcal{T}_{xy_1\cdots y_{n-1}}\rangle_0. \quad (2)$$

It will be convenient later on to write the essential part of Eq. (1) in the form

$$r(p_1,\cdots p_{n-1}) = \int du_1\cdots du_{n-1} \times\exp[-i(p_1\cdot u_1+\cdots+p_{n-1}\cdot u_{n-1})] \times\bar{r}(u_1,\cdots u_{n-1}), \quad (3)$$

where $\bar{r}(u_1,\cdots u_{n-1})$ is a real symmetric function of the timelike vectors u_i, each of which lies inside its forward light cone (each $u_{i0}>0$).

By inverting the defining equation for R_x, we can write

$$\mathcal{T}_x = i\mathcal{T}R_x, \quad (4)$$

which serves to provide a useful connection between the retarded and time-ordered n-point functions; that is, by functional differentiation of both sides, the repeated application of Eq. (4), and the condition $\mathcal{T}[j=0]=1$, it is possible to express $\langle \mathcal{T}_{xy_1\cdots y_{n-1}}\rangle_0$ in terms of the vacuum expectation value of products of R functionals. For example,

$$\langle \mathcal{T}_{xy}\rangle_0 = i\langle R_{x,y}\rangle_0 + i^2\langle R_yR_x\rangle_0$$

is the appropriate statement relating the 2-point

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[1] K. Nishijima, Phys. Rev. **119**, 485 (1960); M. Muraskin and K. Nishijima, *ibid.* **122**, 331 (1961).

[2] We follow the definitions of the time-ordered and retarded functionals given by K. Symanzik, J. Math. Phys. **1**, 249 (1960).

functions; the general structure is of the form

$$\begin{aligned}\langle \mathcal{T}xy_1\cdots y_{n-1}\rangle_0 &= i\langle R_{x,y_1}\cdots y_{n-1}\rangle_0+i^2\textstyle\sum_p \{\langle R_{y_1,y_2}\cdots y_{n-1}R_x\rangle_0\\ &\quad+\langle R_{y_1,y_2}\cdots y_{n-2}R_{x,y_{n-1}}\rangle_0+\cdots+\langle R_{y_1}R_{x,y_2}\cdots y_{n-1}\rangle_0\}\\ &\quad+\cdots+i^n\langle R_{y_{n-1}}\cdots R_{y_1}R_x\rangle_0, \quad (5)\end{aligned}$$

where $\sum_p$ stands for the sum of all permutations of the $n-1$ y_i indices.

We now compare the Fourier transform of both sides of Eq. (5) for the special case of timelike momenta, where each $p_{i0}>0$ and $q_0<0$; this restriction removes all disconnected τ functions, since the latter must have at least two momenta with negative energies. We denote by $\tau_+{}^c(p_1,\cdots p_{n-1})$ the connected time-ordered n-point function with all $n-1$ momenta positive timelike; the corresponding retarded function will be called $r_+(p_1,\cdots p_{n-1})$. In perturbation theory, $\tau_+{}^c$ has a continuation to negative energy values (generalized crossing), but whether or not such continuations exist in general for an exact theory is as yet unknown; similar statements can be made for r_+. The point of interest here is that for this special choice of positive energies it is not difficult to see, by inserting intermediate states in the terms on the right-hand side of Eq. (5), that

$$\tau_+{}^c(p_1,\cdots p_{n-1})=r_+(p_1,\cdots p_{n-1}). \quad (6)$$

The contribution of all terms other than $\langle R_{x,y_1}\cdots y_{n-1}\rangle_0$ to the integrand of Eq. (2) vanishes; this is an implicit statement of conservation of energy and appears as a consequence of the positive energy spectrum of all (on-the-mass-shell) intermediate states.

3. SCALING ANALYTICITY

To obtain a PDR for r_+ we return to Eq. (3) and consider the scaling transformation obtained by replacing each p_i by[3] λp_i. Because the vectors u_i are positive timelike, the special choice of all $p_{i0}>0$ means that the invariants[4] $p_i\cdot u_i$ are all negative; hence the retarded function $r_+(\lambda p_1,\cdots\lambda p_{n-1})\equiv r_+(\lambda p)$ exists for all complex λ with $\text{Im}\lambda>0$. If we consider the mapping $\xi=\lambda^2$, then the function $r_+(\xi^{\frac12}p)$ is analytic in the entire cut ξ plane. We take the branch cut defining the function $\xi^{\frac12}$ to lie along the positive real axis and write a Cauchy integral

$$r_+(\xi^{\frac12}p)=\frac{1}{2\pi i}\oint_C \frac{d\xi'}{\xi'-\xi}r_+(\xi'^{\frac12}p),$$

where the contour C encompasses the entire upper half ξ' plane, enclosing the complex point ξ. As ξ approaches the positive real axis from above, $\xi\rightarrow|\xi|+i\epsilon$, we obtain

$$r_+(|\xi|^{\frac12}p)=-\frac{1}{\pi i}\int_0^\infty \frac{d\xi'}{\xi'+|\xi|}r_+(i|\xi'|^{\frac12}p)+\frac{\text{P}}{\pi i}\int_0^\infty \frac{d\xi'}{\xi'-|\xi|}r_+(|\xi'|^{\frac12}p), \quad (7)$$

where we have disregarded the possible need for subtractions. Noting that $\text{Im}r_+(i|\xi|^{\frac12}p)\equiv 0$, we may take the real part of Eq. (7) and obtain the PDR[5]

$$\text{Re}r_+(|\xi|^{\frac12}p)=\frac{\text{P}}{\pi}\int_0^\infty \frac{d\xi'}{\xi'-|\xi|}\,\text{Im}r_+(|\xi'|^{\frac12}p), \quad (8)$$

which, with Eq. (6), implies

$$\text{Re}\tau_+{}^c(\xi p^2)=\frac{\text{P}}{\pi}\int_0^\infty \frac{d\xi'}{\xi'-\xi}\,\text{Im}\tau_+{}^c(\xi' p^2). \quad (9)$$

The expression ξp^2 represents all the bilinear momentum combinations which are the arguments of $\tau(p)$, scaled by the positive real factor ξ.

4. DISCUSSION

For the special case of timelike momenta, Eq. (9) is equivalent to the PDR introduced by Nishijima. The integral over negative values of the scaling parameter, which appears in Nishijima's PDR, cannot contribute when none of the momentum variables are spacelike. As has been emphasized, this relation has been established in general only for decay-type processes. For the 2-point function, however, there is no difficulty in removing the subscript on $\tau_+{}^c$; here, $\tau_+{}^c(p)=\tau(p^2)$, and by a change of variable Eq. (9) reduces to the familiar dispersion relation in the variable $-p^2$.

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[3] This is equivalent, for real λ, to the replacement of

$$\langle R_{x,y_1}\cdots y_{n-1}\rangle_0$$

by

$$\langle R_{\lambda^{-1}x,\lambda^{-1}y_1}\cdots\lambda^{-1}y_{n-1}\rangle_0=\exp\{-(\ln\lambda)[x\cdot\partial_x+y_1\cdot\partial_1+\cdots y_{n-1}\cdot\partial_{n-1}]\}\times\langle R_{x,y_1}\cdots y_{n-1}\rangle_0.$$

Analogous formal representations of the scaling transformation may be derived from a time-ordered operator $\mathcal{T}^\lambda$, obtained from the original $\mathcal{T}$ by replacing every source function $j(z)$ by $j(\lambda z)$; $\mathcal{T}^\lambda$ may be expressed in terms of $\mathcal{T}$ by the formal relation $\mathcal{T}^\lambda=\exp(-Q\cdot\ln\lambda)\mathcal{T}$, where $Q=\int dz\,j(z)(z_\mu\partial/\partial z_\mu)[\delta/\delta j(z)]$. This suggests that *all* the Green's functions will have regions of analyticity in the scaling parameter, since all functionals formally satisfy Cauchy-Riemann equations in the complex variable λ.

[4] We use the relativistic notation: $a\cdot b=\mathbf{a}\cdot\mathbf{b}-a_0b_0$.

[5] The choice of all $p_{i0}<0$ defines a function $r_-(\xi^{\frac12}p)$ analytic in the entire second sheet of the cut ξ plane. Integration of this function in the lower half plane on the second sheet yields a PDR for the complex conjugate of Eq. (8).