Lifetime of the Σ^0†

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A dispersion-theoretic calculation of the lifetime of the Σ^0 is carried out along lines suggested by previous treatments of the nucleon electromagnetic form factors. In terms of the coupling constants of the Σ, Λ, and N baryons to the pion (g_Λ, g_Σ, and g), the Σ^0 lifetime is predicted to be $\tau=(g_\Sigma g_\Lambda/g^2)^{-2}\times 1.1\times 10^{-19}$ sec, irrespective of the relative (Λ,Σ) parity, the validity of this expression depending on the dominance of the two-pion resonating state contribution to the $\Sigma^0\to\Lambda+\gamma$ transition magnetic moment.

INTRODUCTION

A MODERATE amount of success has been encountered in the study of the three-point (vertex) function where at least one of the particles entering the vertex is strongly interacting. By applying dispersion-theoretic techniques, a sequence of investigations—those of Chew *et al.*,[1] of Federbush *et al.*,[2] and of Frazer and Fulco[3]—have shed some light on the origin of the electromagnetic structure of nucleons. In particular, the nucleon isovector magnetic moment form factor stands as the candidate for being the relatively best understood.

The task we set ourselves here is the calculation of the Σ^0-Λ^0-γ vertex function for low momentum transferred into the vertex. At zero momentum transfer, the value of this vertex determines the Σ^0 lifetime. Although the procedure and approximations involved are very similar to schemes employed in the study of the nucleon magnetic moment isovector form factor, differences enter because the masses of the three particles forming the vertex all differ, a circumstance which leads to appreciable numerical deviations from the equal baryon mass approximation.

On the experimental side, the very brief time of existence of the Σ^0 is not known quantitatively and only a very crude bracketing of the lifetime is available (10^{-10} sec$>\tau>10^{-22}$ sec). However, proposals for determining the Σ^0 radiative transition rate have been put forward[4] and, hopefully, experimental data on this rate will eventually be gathered.

If a conserved vector current of baryons in the form proposed by Feynman and Gell-Mann[5] is coupled to the weak interaction lepton "current" and if the minimal weak coupling scheme[4] is realized, then the vertex function, which concerns us here, is also proper to the study of the Σ^+ and Σ^- disintegrations into Λ^0 and the lepton pair ($e\nu$). In fact, if we neglect the Σ^+, Σ^- beta decay induced by the primitive axial vector coupling which we estimate as negligibly small,[4] the whole scheme of Σ multiplet decay into Λ^0 is completely predictable.

In Sec. I, the Σ-Λ-γ vertex is dispersed and the dispersion relations for the form factors are recorded. The two-pion contribution to the absorptive part of the vertex is then analyzed. Sections II and III treat, in particular, the dispersion relations satisfied by the $J=1$, $T=1$ partial wave amplitude for $\bar{\Lambda}+\Sigma\to\pi+\pi$ on the basis of the Mandelstam representation. The singularities of this amplitude are exhibited for complex as well as real values of the energy variable associated with the channel. It is then shown (Sec. IV) to be reasonable to construct a Frazer-Fulco approximation (dominance of a pion-pion resonance) for the absorptive part of the form factor. Explicit expressions for the form factors are then obtained and numerical predictions given.

Units with $\hbar=c=1$ will be used in the following. The relative (Λ,Σ) parity will be assumed to be even throughout this paper. The results for odd relative (Λ,Σ) parity are summarized in the last section.

I. DISPERSION RELATIONS FOR THE $\Sigma-\Lambda-\gamma$ VERTEX

With the requirement of Lorentz and gauge invariance, the most general structure of the vertex of interest $\langle\Lambda|j_\mu(0)|\Sigma^0\rangle$ is, assuming the relative (Λ,Σ) parity to be even,

$$(E_\Lambda/m_\Lambda)^{\frac{1}{2}}(E_\Sigma/m_\Sigma)^{\frac{1}{2}}\langle\Lambda|j_\mu(0)|\Sigma\rangle$$
$$=F_1(-\xi^2)i\bar{u}_\Lambda\sigma_{\mu\nu}\xi_\nu u_\Sigma+F_2(-\xi^2)\bar{u}_\Lambda i\gamma_\mu u_\Sigma$$
$$+F_3(-\xi^2)\xi_\mu\bar{u}_\Lambda u_\Sigma;\quad \xi_\mu=(p_\Sigma-p_\Lambda)_\mu. \tag{I.1}$$

Here we are treating the vertex function in lowest order in e; with this proviso, Σ^0 may be treated as if it were stable. Only the isovector part of j_μ has a nonvanishing matrix element since Σ is an isovector and Λ an isoscalar. The $F_1(-\xi^2)$ and $F_2(-\xi^2)$ of Eq. (I.1) are related [because of the gauge condition $\langle\Lambda|\xi_\mu j_\mu(0)|\Sigma^0\rangle=0$] by

$$F_3(-\xi^2)=(\Delta/\xi^2)F_2(-\xi^2), \tag{I.2}$$
$$\Delta=m_\Sigma-m_\Lambda.$$

It proves convenient[2] to consider instead of (I.1) the

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[1] G. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958).
[2] P. Federbush, M. Goldberger, and S. Treiman, Phys. Rev. **112**, 642 (1958).
[3] W. Frazer and J. Fulco, Phys. Rev. **117**, 1604 (1960); **117**, 1609 (1960).
[4] J. Dreitlein and H. Primakoff, Phys. Rev. **124**, 268 (1961).
[5] R. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

related quantity:

$$
\begin{aligned}
(E_{\bar{\Lambda}}/m_\Lambda)^{\frac{1}{2}}(E_\Sigma/m_\Sigma)^{\frac{1}{2}}\langle 0|j_\mu|\bar{\Lambda}\Sigma^0\rangle
&= -F_1(t)\bar{v}_\Lambda i\sigma_{\mu\nu}\xi_\nu u_\Sigma - F_2(t)\bar{v}_\Lambda i\gamma_\mu u_\Sigma \\
&\quad - F_3(t)\xi_\mu \bar{v}_\Lambda u_\Sigma, \qquad \text{(I.3)}\\
\xi = p_\Sigma + p_{\bar{\Lambda}}; \quad t &= -(p_\Sigma+p_{\bar{\Lambda}})^2 = -\xi^2,
\end{aligned}
$$

where the identity of the F_i in Eqs. (I.1) and (I.3) follow from the substitution rule.[6]

By contracting out the $\bar{\Lambda}$ in Eq. (I.3), we are led heuristically to dispersion relations for the form factors $F_1(t)$ and $F_2(t)$:

$$F_1(t) = \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} dt' \frac{\mathrm{Im}F_1(t')}{t'-t}, \qquad \text{(I.4a)}$$

$$F_2(t) = F_2(0) + \frac{t}{\pi}\int_{(2m_\pi)^2}^{\infty} dt' \frac{\mathrm{Im}F_2(t')}{(t'-t)t'}. \qquad \text{(I.4b)}$$

In order that $F_3(\xi^2)$ of Eq. (I.2) be finite for $\xi^2=0$, $F_2(\xi^2)$ must vanish as $\xi^2\to 0$. This condition determines the subtraction constant $F_2(0)$ of Eq. (I.4b) to be zero. $F_1(t)$ is assumed to obey an unsubtracted dispersion relation.

The absorptive part of the form factors $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ is found from the absorptive amplitudes obtained by dispersing $\langle 0|j_\mu|\bar{\Lambda}\Sigma^0\rangle$ (Fig. 1):

$$
\begin{aligned}
A_\mu = -8\pi^4 \textstyle\sum_n \langle 0|j_\mu|n\rangle\langle n|f_\Lambda(0)|\Sigma^0\rangle \\
\times \delta^4(p_n - p_\Sigma - p_{\bar{\Lambda}})(E_\Sigma/n_\Sigma)^{\frac{1}{2}}, \qquad \text{(I.5)}
\end{aligned}
$$

$$f_\Lambda(x) = (\gamma_\mu\partial^\mu + m_\Lambda)\psi_\Lambda(x) = \text{source of } \Lambda^0 \text{ field,}$$

which must be of the form

$$
\begin{aligned}
A_\mu = -\mathrm{Im}F_1(t)\bar{v}_\Lambda i\sigma_{\mu\nu}\xi_\nu u_\Sigma \\
-\mathrm{Im}F_2(t)\bar{v}_\Lambda(i\gamma_\mu - \xi_\mu\Delta/t)u_\Sigma. \qquad \text{(I.6)}
\end{aligned}
$$

The lower limits of integration in Eq. (I.4) are a consequence of the two-pion state being the lowest mass state generating a nonvanishing contribution to the A_μ of Eq. (I.5). No anomalous threshold is associated with the two-pion contribution unlike the case of the electromagnetic structure of the $\Sigma^\pm$.[7] Since the generalized charge conjugation operator[8] G prevents three-pion states from contributing to A_μ, $(Gj_\mu{}^V G^{-1} = j_\mu{}^V)$, the next higher mass states are the four-pion state, the six-pion state, and then the K-meson pair with which an anomalous threshold is associated. Using the prescription of Karplus, Summerfield and Wichmann,[9] we find that the threshold is lowered from the expected value of $t_{\mathrm{normal}} = (2m_K)^2 = 50.1\ m_\pi^2$ to $t_{\mathrm{anomalous}} = 49\ m_\pi^2$.

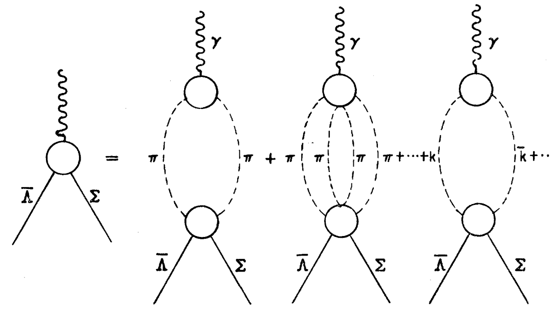

FIG. 1. Dispersion of the Σ-Λ-γ vertex.

Just as in the treatment of the isovector part of the nucleon form factors, the seemingly reasonable assumption is made that the lowest mass state, viz., the two-pion state, dominates the contribution to the $F_i(t)$ since such states are weighed most heavily in the dispersion integrals of Eq. (I.4).

The matrix element $\langle 0|j_\mu(0)|2\pi\rangle$ has been calculated[2]:

$$
\begin{aligned}
\langle 0|j_\mu|q,i;q',j(\mathrm{out})\rangle \\
= (4q_0q_0')^{-\frac{1}{2}} ie\epsilon_{3ij}(q-q')_\mu M^*(t), \qquad \text{(I.7)}
\end{aligned}
$$

where q, q' and i, j are the pion four-momenta and isobaric spin indices, respectively. $M^*(t)$ is the pion form factor which we assume has a sharp and strong peak at $t=t_r$, the Frazer-Fulco resonance energy for the two-pion system.

The other matrix element which must be computed, $\langle 2\pi|f_\Lambda|\Sigma_0\rangle$, is related to the process $\Sigma^0+\bar{\Lambda}\to\pi+\pi$ to which we now turn.

II. THE PROCESS: $\Lambda+\Sigma^0\to\pi+\pi$

The second essential ingredient in the absorptive amplitude is the matrix element $\langle\pi\pi(\mathrm{out})|f_\Lambda|\Sigma^0\rangle$ which is related to the S matrix for the process $\bar{\Lambda}+\Sigma\to\pi+\pi$:

$$
\begin{aligned}
\langle\pi\pi(\mathrm{out})|S|\bar{\Lambda}\Sigma(\mathrm{out})\rangle \\
= -i(m_\Lambda/E_\Lambda)^{\frac{1}{2}}(2\pi)^4\delta(q+q'-p_{\bar{\Lambda}}-p_\Sigma) \\
\times\langle q,i;q',j(\mathrm{out})|f_\Lambda(0)|\Sigma\rangle \qquad \text{(II.1)}\\
\equiv (2\pi)^4\delta(q+q'-p_{\bar{\Lambda}}-p_\Sigma)\langle\pi\pi|s|\bar{\Lambda}\Sigma\rangle.
\end{aligned}
$$

Since the absorptive amplitude $A_\mu(t)$ of Eq. (I.5) is Lorentz covariant, it may be evaluated in any convenient Lorentz frame, in particular in the barycentric frame of the baryon pair $\bar{\Lambda}\Sigma$. Only the $J=1$ even parity states of the $\bar{\Lambda}\Sigma$ contribute to the absorptive amplitude of Eq. (I.5). Although the ${}^3S_1$ and ${}^3D_1$ amplitudes could be discussed,[2] it proves more useful to use the helicity amplitudes[10] employed by Frazer and Fulco.[3]

Keeping only the $T=1$, $J=1$ projection of the S-matrix element in Eq. (II.1), we find for the two

[6] J. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison Wesley Publishing Company, Reading, Massachusetts, 1955), p. 161.

[7] R. Marr, L. Landovitz, and R. Blankenbecler, Bull. Am. Phys. Soc. **6**, 80 (1961); we thank Professor Blankenbecler for showing us the manuscript prior to publication.

[8] T. Lee and C. Yang, Nuovo cimento **3**, 749 (1956).

[9] R. Karplus, C. Summerfield, and E. Wichmann, Phys. Rev. **114**, 376 (1959).

[10] M. Jacob and G. Wick, Ann. Phys. **7**, 404 (1959).

helicity states[11]:

$$i\langle 2\pi|s^{T=J=1}|\bar{\Lambda}(+)\Sigma(+)\rangle = 4\pi\frac{(E_{\bar{\Lambda}}+E_\Sigma)}{(4E_{\bar{\Lambda}}E_\Sigma q_0 q_0')^{\frac{1}{2}}}[-i\epsilon_{3ij}]^{\frac{3}{2}}T^+\hat{p}\cdot\hat{q},$$
$$i\langle 2\pi|s^{T=J=1}|\bar{\Lambda}(+)\Sigma(-)\rangle = -4\pi\frac{(E_{\bar{\Lambda}}+E_\Sigma)}{(4E_{\bar{\Lambda}}E_\Sigma q_0 q_0')^{\frac{1}{2}}}[-i\epsilon_{3ij}]^{\frac{3}{2}}T^-(\hat{p}\times\hat{q})e^{i\varphi};\qquad\text{(II.2)}$$

where $\mathbf{p}=(\mathbf{p}_\Sigma-\mathbf{p}_\Lambda)/2$, $\mathbf{q}=(\mathbf{q}-\mathbf{q}')/2$ in the c.m. frame, and φ is the azimuthal angle of $\mathbf{q}$. Here $\bar{\Lambda}(+)$ signifies a positive-helicity state for the $\bar{\Lambda}$.

When the S-matrix elements of Eq. (II.2) are substituted into Eq. (I.5), we find

$$\mathbf{A}=-8\pi^4\sum_{ij}\int\frac{d^3q d^3q'}{2(2\pi)^6}\left[\langle 0|\mathbf{j}(0)|q,i;q',j\rangle\frac{i}{(2\pi)^4}\times\langle q,i;q',j|S|\bar{\Lambda}\Sigma\rangle\right]\left(\frac{E_{\bar{\Lambda}}E_\Sigma}{m_\Lambda m_\Sigma}\right)^{\frac{1}{2}},$$
$$\langle 0|\mathbf{j}|2\pi\rangle=(4q_0^2)^{-\frac{1}{2}}ie\epsilon_{3ij}2\mathbf{q}M^*(t),\qquad\text{(II.3)}$$
$$A_3=\hat{p}\cdot\mathbf{A}=-\frac{e}{2}\frac{M^*(t)}{(m_\Lambda m_\Sigma)^{\frac{1}{2}}}T^+q,$$
$$A_1=\frac{e}{2}\frac{M^*(t)}{(m_\Lambda m_\Sigma)^{\frac{1}{2}}}\frac{T^-}{\sqrt{2}}q.$$

To relate T^+ and T^- to the form factors, we simply evaluate the alternative expression for $\hat{p}\cdot\mathbf{A}$ and A_1 by using Eq. (I.6). The result is

$$\mathrm{Im}F_1=\frac{1}{t^{\frac{1}{2}}}\frac{(m_\Lambda+E_{\bar{\Lambda}})^{\frac{1}{2}}(m_\Sigma+E_\Sigma)^{\frac{1}{2}}}{(2M+t^{\frac{1}{2}})}eM^*(t)\frac{q}{p^2}\left[ET^++M\frac{T^-}{\sqrt{2}}\right],$$
$$\mathrm{Im}F_2=-\frac{(m_\Lambda+E_{\bar{\Lambda}})^{\frac{1}{2}}(m_\Sigma+E_\Sigma)^{\frac{1}{2}}}{(2M+t^{\frac{1}{2}})}eM^*(t)\times\frac{q}{p^2}\left[-MT^++E\frac{T^-}{\sqrt{2}}\right],\qquad\text{(II.4)}$$

with

$$q=\tfrac{1}{2}(t-4m_\pi)^{\frac{1}{2}},\quad p=\tfrac{1}{2}(t-4M^2)^{\frac{1}{2}}(1-\Delta^2/t)^{\frac{1}{2}},$$
$$M=\tfrac{1}{2}(m_\Sigma+m_\Lambda),\quad \Delta=m_\Sigma-m_\Lambda,$$

and

$$E=\tfrac{1}{2}(E_{\bar{\Lambda}}+E_\Sigma).$$

We shall show below that it is useful to define new energy dependent amplitudes f_+ and f_- instead of T^+ and T^-:

$$f_+(t)=\frac{(E_{\bar{\Lambda}}+m_\Lambda)^{\frac{1}{2}}(E_\Sigma+m_\Sigma)^{\frac{1}{2}}}{(2n+t^{\frac{1}{2}})}t^{\frac{1}{2}}\frac{p}{q}\frac{1}{pq}\frac{1}{1-\Delta^2/t}T^+,$$
$$f_-(t)=2\frac{(E_{\bar{\Lambda}}+m_\Lambda)^{\frac{1}{2}}(E_\Sigma+m_\Sigma)^{\frac{1}{2}}}{2M+t^{\frac{1}{2}}}\frac{p}{q}\frac{1}{pq}\frac{1}{1-\Delta^2/t}T^-,\qquad\text{(II.5)}$$

in terms of which Eq. (II.4) takes the form:

$$\mathrm{Im}F_i=-\frac{eM^*(t)}{t^{\frac{1}{2}}}q^3\Gamma_i(t),$$
$$\Gamma_1(t)=-\frac{(1-\Delta^2/t)}{2p^2}\left(f_++\frac{M}{\sqrt{2}}f_-\right),\qquad\text{(II.6)}$$
$$\Gamma_2(t)=-\frac{M}{p^2}\left(1-\frac{\Delta^2}{t}\right)\left(-\frac{E^2}{\sqrt{2}M}f_--f_+\right).$$

In order to explore the analytic properties of the partial wave amplitudes, the Mandelstam representation of the singularities of the invariant amplitude for $\bar{\Lambda}+\Sigma\to\pi+\pi$ is needed. The expressions for which we postulate the Mandelstam representation are the A and B defined by

$$\left(\frac{m_\Lambda}{E_{\bar{\Lambda}}}\right)^{\frac{1}{2}}\langle q,i;q',j|f_\Lambda(0)|\Sigma\rangle=\left(\frac{m_\Lambda m_\Sigma}{4q_0q_0'E_\Sigma E_{\bar{\Lambda}}}\right)^{\frac{1}{2}}[-A+\tfrac{1}{2}i\gamma\cdot(q-q')B]u_\Sigma,$$
$$A=A(t,s,\bar{s});\quad B=B(t,s,\bar{s}),\qquad\text{(II.7)}$$
$$t=-(q+q')^2=4(q_\pi^2+m_\pi^2)=4E^2,$$
$$\bar{s}=-(q_{\bar{\Lambda}}-q)^2=(E_{\bar{\Lambda}}-q_0)^2-p^2-q^2+2pq\cos\theta,$$
$$s=-(p_{\bar{\Lambda}}-q')^2=(E_{\bar{\Lambda}}-q_0')^2-p^2-q^2-2pq\cos\theta,$$

t, s, and $\bar{s}$ are expressed in the barycentric system. s and $\bar{s}$ can in turn be expressed in terms of t and $\cos\theta$:

$$s=-\frac{t}{2}+M^2+\frac{\Delta^2}{4}+m_\pi^2+2\cos\theta\left\{\left(\frac{t}{4}-M^2\right)^{\frac{1}{2}}\left(1-\frac{\Delta^2}{t}\right)^{\frac{1}{2}}\left(\frac{t}{4}-m_\pi^2\right)^{\frac{1}{2}}\right\}.\qquad\text{(II.8)}$$

Because only the $T=1$ amplitudes occur, the final two-pion state is spatially antisymmetric, thus:

$$A(t,s,\bar{s})=-A(t,\bar{s},s),$$
$$B(t,s,\bar{s})=B(t,\bar{s},s).\qquad\text{(II.9)}$$

[11] The consequences of the indistinguishability of the two pions have not been included here. However, in Eq. (II.3), it will be noted that the integration is carried out over half the phase space available. It is here that indistinguishability of the two pions is taken into account.

By again expressing Eq. (II.7) in partial-wave helicity amplitudes and then comparing with Eq. (II.4), T^+ and T^- can be expressed in terms of A_J and B_J defined by

$$A_J \equiv \int_{-1}^{+1} P_J(x) A(t,x) dx, \qquad B_J \equiv \int_{-1}^{+1} P_J(x) B(t,x) dx. \tag{II.10}$$

The explicit expressions for T^+ and T^- are

$$T^+ = \frac{1}{8\pi t^{\frac{1}{2}}} \frac{q}{p} \frac{(2M+t)^{\frac{1}{2}}(1-\Delta^2/t)}{(m_\Sigma+E_\Sigma)^{\frac{1}{2}}(m_\Lambda+E_\Lambda)^{\frac{1}{2}}} \times\left[\frac{p^2}{1-\Delta^2/t}A_1 + pqM(\tfrac{2}{3}B_2+\tfrac{1}{3}B_0)\right],$$
$$T^- = \frac{1}{8\pi} \frac{q^2}{\sqrt{2}} \frac{(2M+t)^{\frac{1}{2}}(1-\Delta^2/t)}{(m_\Sigma+E_\Sigma)^{\frac{1}{2}}(m_\Lambda+E_\Lambda)^{\frac{1}{2}}} \tfrac{1}{3}[A_0 - A_2],$$
$$\frac{p^2}{1-\Delta^2/t} = \left(\frac{t}{4} - M^2\right), \tag{II.11}$$

and for f_+ and f_-:

$$f_+ = \frac{1}{8\pi} \frac{1}{pq}\left[\frac{p^2}{1-\Delta^2/t}A_1 + pqM(\tfrac{2}{3}B_2+\tfrac{1}{3}B_0)\right],$$
$$f_- = -\frac{1}{8\pi} \frac{\sqrt{2}}{3}[B_0 - B_2]. \tag{II.12}$$

All the notation used in the above expressions has been chosen so as to yield the results of Frazer and Fulco in the limit $\Delta \to 0$ except for insignificant changes in sign convention. The following two identities were found useful in reducing the above formulas to the form given:

$$(m_\Lambda+E_\Lambda)(m_\Sigma+E_\Sigma) - p^2 = \tfrac{1}{2}(m_\Sigma+m_\Lambda)(E_\Lambda+m_\Lambda+E_\Sigma+m_\Sigma)(1-\Delta^2/t),$$
$$(m_\Lambda+E_\Lambda)(m_\Sigma+E_\Sigma) + p^2 = \tfrac{1}{2}(E_\Sigma+E_\Lambda)(E_\Lambda+m_\Lambda+E_\Sigma+m_\Sigma)(1-\Delta^2/t). \tag{II.13}$$

III. PROPERTIES OF THE $\bar{\Lambda}+\Sigma \to \pi+\pi$ PARTIAL-WAVE AMPLITUDE

Accepting the prescription of Mandelstam[12] for locating the singularities of scattering amplitudes, we can write down a representation for the A and B amplitudes introduced in Eq. (II.7):

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \Delta/2 \\ 1 \end{pmatrix} g_\Lambda g_\Sigma \left[\frac{1}{m_\Sigma^2 - s} \mp \frac{1}{m_\Sigma^2 - \bar{s}}\right]$$
$$+ \frac{1}{\pi^2}\int_{(m_\Lambda+m_\pi)^2}^{\infty} ds' \int_{(m_\Lambda+m_\pi)^2}^{\infty} d\bar{s}' \frac{\begin{pmatrix} a_{12}(s',\bar{s}') \\ b_{12}(s',\bar{s}') \end{pmatrix}}{(s'-s)(\bar{s}'-s)}$$
$$+ \frac{1}{\pi^2}\int_{(m_\Lambda+m_\pi)^2}^{\infty} ds' \int_{(2m_\pi)^2}^{\infty} dt' \frac{\begin{pmatrix} a_{13}(s',t') \\ b_{13}(s',t') \end{pmatrix}}{(s'-s)(t'-t)}$$
$$+ \frac{1}{\pi^2}\int_{(m_\Lambda+m_\pi)^2}^{\infty} d\bar{s}' \int_{(2m_\pi)^2}^{\infty} dt' \frac{\begin{pmatrix} a_{23}(\bar{s}',t') \\ b_{23}(\bar{s}',t') \end{pmatrix}}{(\bar{s}'-s)(t'-t)},$$
$$\begin{pmatrix} a_{12}(x,y) \\ b_{12}(x,y) \end{pmatrix} = \begin{pmatrix} -a_{12}(y,x) \\ +b_{12}(y,x) \end{pmatrix},$$
$$\begin{pmatrix} a_{13}(x,y) \\ b_{13}(x,y) \end{pmatrix} = \begin{pmatrix} -a_{23}(x,y) \\ +b_{23}(x,y) \end{pmatrix}. \tag{III.1}$$

Starting from the representation, the singularities of the partial-wave amplitudes can now be found by forming A_J and B_J:

$$\frac{\begin{pmatrix} A_J(t) \\ B_J(t) \end{pmatrix}}{(pq)^J} = \frac{1}{\pi}\int ds' \begin{pmatrix} a(s',t) \\ b(s',t) \end{pmatrix} \frac{1}{(pq)^J} \times \int_{-1}^{+1}\left[\frac{1}{s'-s(t,x)} \mp \frac{1}{s'-\bar{s}(t,x)}\right] P_J(x) dx,$$

where

$$\begin{pmatrix} a(s',t) \\ b(s',t) \end{pmatrix} = \begin{pmatrix} \Delta/2 \\ 1 \end{pmatrix} \pi g_\Lambda g_\Sigma \delta(s'-m_\Sigma^2) + \theta[s' - (m_\Lambda+m_\pi)^2]$$
$$\times\left[\int_{(2m_\pi)^2}^{\infty} dt' \frac{\begin{pmatrix} a_{13}(s',t') \\ b_{13}(s',t') \end{pmatrix}}{t'-t} + \int_{(m_\Lambda+m_\pi)^2}^{\infty} d\bar{s}' \frac{\begin{pmatrix} a_{12}(s',\bar{s}') \\ b_{12}(s',\bar{s}') \end{pmatrix}}{t+\bar{s}'+s'-2m_\pi^2-m_\Sigma^2-m_\Lambda^2}\right]. \tag{III.2}$$

If we neglect rescattering effects, the terms in Eq. (III.2) involving a_{12} and b_{12} are dropped. Physically such a neglect is tantamount to assuming that the only important force in the process $\bar{\Lambda}+\Sigma \to \pi+\pi$ for the t range of interest arises from the singularities associated

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[12] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 (1959); **115**, 1752 (1959).

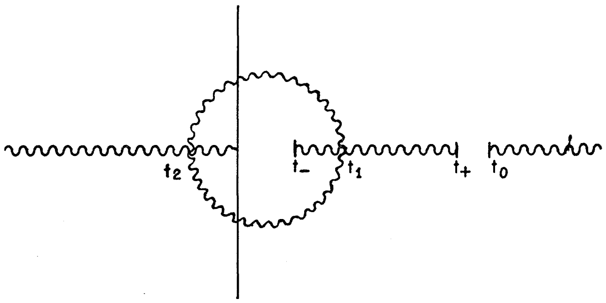

FIG. 2. Singularities of the partial-wave amplitude associated with the pole term. $t_+=3.57m_\pi^2$, $t_-=0.30m_\pi^2$, $t_1=0.91m_\pi^2$, $t_2=0.63m_\pi^2$; $t_0=4m_\pi^2$.

with the pole term of the Born approximation. To make such an approximation plausible, we must show that for the t values for which $A_J(t)$ are required, i.e., for $(2m\pi)^2<t<\infty$, the absorptive parts of the Born approximation are located in such a way as to give a large and hopefully dominant contribution to the dispersion relations which $A_J(t)$ and $B_J(t)$ satisfy.

It may be verified that both $f_+(t)$ and $f_-(t)$ and $\Gamma_1(t)$ and $\Gamma_2(t)$ as defined have the same singularities as $(pq)^{-J}A_J(t)$ and $(pq)^{-J}B_J(t)$. Indeed, these functions have been defined so as to have just such a property. We may therefore limit our explorations of singularities to those of $(pq)^{-J}A_J(t)$ whose singularities are located just where those of $(pq)^{-J}B_J(t)$ are.

The singularities of $A_1(t)/pq$ arise from two sources. First, the singularities of $a(s',t)$ of Eq. (III.2) produce a branch cut running from $t=(2m\pi)^2$ to $t=\infty$. Secondly, the integral

$$I_J=\frac{1}{(pq)^J}\int_{-1}^{+1}\left[\frac{1}{s'-s(t,x)}-\frac{1}{s'-\bar{s}(t,x)}\right]P_J(x)dx,\quad J\text{ odd},$$

produces singularities for those values of t for which $s'=s(t,x)$ provided $|x|\leq 1$. Since only even powers of pq are present in I_J, no kinematical singularities associated with p, q appear. We are thus instructed to find solutions of the equation:

$$\begin{aligned}\frac{1}{x}t^{\frac{1}{2}}(t-c_0)&=(t-c_1)^{\frac{1}{2}}(t-c_2)^{\frac{1}{2}}(t-c_3)^{\frac{1}{2}},\\ c_0&=m_\Lambda^2+m_\Sigma^2+2m_\pi^2-2s',\\ c_1&=4M^2,\\ c_2&=4m_\pi^2,\\ c_3&=\Delta^2.\end{aligned}\tag{III.3}$$

By studying the position of the singularities as a function of x and s', we find that the singularities trace out the curve in the complex t space illustrated in Fig. 2 for the Born term singularity $s'=m_\Sigma^2$; $c_0=2m_\pi^2-2M\Delta$. For $s'\geqq(m_\Lambda+m_\pi)^2$, further singularities are introduced but none of these approach so close to the right-hand branch cut as do those of the singularities associated with the pole. If we take the rescattering corrections as going dominantly through the Y^* quasi particle then in fact all of the associated left-hand singularities lie to the left of $t=0$. Now let us write down the dispersion relation for $\Gamma_i(t)$ which follows from a knowledge of the (t plane) right-hand and left-hand singularities:

$$\Gamma_i(t)=\Gamma_i^L(t)+\frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty}dt'\frac{\text{Im}\Gamma_i(t')}{t'-t},\tag{III.5}$$

where $\Gamma_i^L(t)$ is the contribution from the left-hand singularities: the analytic structure of $\Gamma_i^L(t)$ will be discussed in some detail in the Appendix. Again following Frazer and Fulco, we observe that $\Gamma_i(t)/M(t)$ must be analytic and real for $4m_\pi^2\leq t\leq 16m_\pi^2$,

$$\text{Im}[\Gamma_i(t)/M(t)]=0,\quad 4m_\pi^2\leq t\leq 16m_\pi^2\tag{III.6}$$

because the phase of $\Gamma_i(t)$ must simply be that of $M(t)$. In fact both of them are given by the phase shift of pion-pion scattering in the $T=J=1$ state according to the theorem of Fubini, Nambu, and Wataghin.[13] This guarantees that the imaginary parts of the form factors $F_i(t)$ are real.

Applying Cauchy's theorem to $\Gamma_i(t)/M(t)$ and approximating $\Gamma_i^L(t)$ by that arising from the pole term $\Gamma_i^B(t)$, we obtain

$$\Gamma_i(t)\cong\frac{M(t)}{\pi}\int_C dt'\frac{\text{Abs}[\Gamma_i^B(t')/M(t')]}{(t'-t)},\tag{III.7}$$

where the contribution from the right-hand cut for $t'\geq 16m_\pi^2$ has been neglected and where C is the appropriate contour for the singularities arising from the pole term, and "Abs" means "the absorptive part of."

IV. LIFETIME OF THE Σ^0

Collecting together the relevant quantities Eq. (II.6) and Eq. (III.7), the form factors are given by

$$\begin{aligned}\text{Im}F_i(t)&=-\left(\frac{eM^*}{t^{\frac{1}{2}}}q^3\right)\Gamma_i(t),\\ \Gamma_i(t)&=M(t)\frac{1}{\pi}\int_C dt'\frac{\text{Abs}[\Gamma_i(t')/M(t')]}{t'-t}.\end{aligned}\tag{IV.1}$$

The $M(t')$ in the integrand of Eq. (IV.1) will be replaced by 1 since for the singularities closest to the t values of interest $M(t')\approx 1$. Our final result amounts to the following: $\text{Im}F_i(t)$ is given by the Born approximation $[\text{Im}F_i(t)]_0$ multiplied by $|M(t)|^2$, the square of the absolute value of the pion form factor:

$$\text{Im}F_i(t)=|M(t)|^2[\text{Im}F_i(t)]_0.\tag{IV.2}$$

In the absence of a realistic calculation of the rescattering corrections, we believe that the above approxima-

[13] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).

tion is, at present, the best that can be made in view of other uncertainties.

A straightforward calculation yields the following expression for the $[\mathrm{Im}F_i(t)]_0$:

$$[\mathrm{Im}F_i(t)]_0=\frac{eg_\Lambda g_\Sigma}{4\pi}\left(\frac{M}{2}\right)\frac{q_\pi^2}{2t^{\frac{1}{2}}}\frac{1}{q_\Lambda^3}\times\left\{\left[\tan^{-1}f-\frac{3}{f}\left(1-\frac{\tan^{-1}f}{f}\right)\right]\left(1-\frac{\Delta^2}{t}\right)+\frac{\Delta}{M}\frac{q_\Lambda}{q_\pi}\left(1-\frac{1}{f}\tan^{-1}f\right)\right\},$$

$$[\mathrm{Im}F_2(t)]_0=\frac{eg_\Lambda g_\Sigma}{4\pi}\frac{1}{2t^{\frac{1}{2}}}\frac{q_\pi^2}{q_\Lambda}\times\left\{\left(1-\frac{\Delta M}{2}\frac{f}{q_\Lambda q_\pi}\right)\frac{2}{f}\left(1-\frac{1}{f}\tan^{-1}f\right)-\left(M^2\right)1-\frac{\Delta^2}{t}\left(-q_\Lambda^2\right)\frac{1}{q_\Lambda^2}\times\left[\tan^{-1}f-\frac{3}{f}\left(1-\frac{1}{f}\tan^{-1}f\right)\right]\right\},$$

$$q_\pi=(\tfrac{1}{4}t-m_\pi^2)^{\frac{1}{2}},$$
$$q_\Lambda=-ip=(-\tfrac{1}{4}t+M^2)^{\frac{1}{2}}(1-\Delta^2/t)^{\frac{1}{2}},\qquad\text{(IV.3)}$$
$$f=2q_\Lambda q_\pi/(\tfrac{1}{2}t-m_\pi^2+M\Delta).$$

We recover the well-known results for the isovector nucleon form factors $[F_2^V(t)]_0$ and $[F_1^V(t)]_0$ by setting $\Delta=0$, $M=m_N$ and $g_\Lambda g_\Sigma=g^2$ with g equal to the pion-nucleon coupling constant.

The lifetime of the Σ^0 is found from $F_1(t)$. To calculate this quantity we assume that the pion-pion resonance is very sharp so that the $|M(t)|^2$ effectively acts as a delta function multiplied by some constant. By comparing Eq. (I.4) with the analogous relation for the nucleon isovector form factor $F_2^V(t)$,

$$F_2^V(t)=\frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty}dt'\frac{\mathrm{Im}F_2^V(t)}{t'-t},\qquad\text{(IV.4)}$$

the transition magnetic moment $F_1(0)$ for $\Sigma\rightarrow\Lambda+\gamma$ can be directly expressed in terms of the nucleon isovector anomalous magnetic moment:

$$F_1(t)=\frac{[\mathrm{Im}F_1(t_r)]_0}{[\mathrm{Im}F_2^V(t_r)]_0}F_2^V(t)\cong 0.64F_2^V(t)\frac{g_\Lambda g_\Sigma}{g^2},\quad\text{for } t \text{ small.}\qquad\text{(IV.5)}$$

Also, we note that the transition magnetic moment radius is the same as the magnetic moment radius in the present approximation.

If the calculation had been carried through with $\Delta=0$ and $M=(m_\Lambda+m_\Sigma)/2$ then the result would have been about 30% smaller. Not only is the result sensitive to the mass difference Δ but also to the value of M. The global symmetry model which neglects the mass difference $M-m_N$ predicts $F_1(0)=F_2^V(0)$.

In terms of $F_1(0)/F_2^V(0)$, the lifetime of the Σ^0 is[4]

$$\tau=|F_2^V(0)/F_1(0)|^2\times 4.5\times 10^{-20}\text{ sec.}\qquad\text{(IV.6)}$$

With $g_\Lambda g_\Sigma=g^2$,

$$\tau=1.1\times 10^{-19}\text{ sec,}\qquad\text{(IV.7)}$$

which may be measurable by one of the suggested experimental methods.[4]

CONCLUSION

With procedures and approximations analogous to those used in the study of the nucleon electromagnetic form factors, we have obtained an expression for the decay rate of the Σ^0. Essentially no arbitrary parameters occur in the final expression Eq. (IV.5) since g_Λ, g_Σ and g^2 are measurable by independent experimental methods. The result depends heavily on the existence of a two-pion resonant state so that if the experimentally observed lifetime is indeed that predicted above, then strong evidence will accrue for both a resonating two-pion state and the validity of the two-pion theory of the nucleon isovector magnetic moment form factor. On the other hand, if the two-pion theory is accepted as true, then the lifetime provides a measure of the strength of the coupling constant $g_\Lambda g_\Sigma$.

Finally, if the relative (Λ,Σ) parity is odd, Eq. (I.1) must be written as[4]

$$(E_\Lambda/m_\Lambda)^{\frac{1}{2}}(E_\Sigma/m_\Sigma)^{\frac{1}{2}}\langle\Lambda|j_\mu(0)|\Sigma\rangle=G_1(-\xi^2)i\bar{u}_\Lambda(\gamma_5)\sigma_{\mu\nu}\xi_\nu u_\Sigma+G_2(-\xi^2)\bar{u}_\Lambda(\gamma_5)i\gamma_\mu u_\Sigma+G_3(-\xi^2)\xi_\mu\bar{u}_\Lambda(\gamma_5)u_\Sigma.$$

In the approximation of representing Γ_i above by Γ_i^B, the Born approximation, $G_1(t)$ and $G_2(t)$ have the same expressions as $F_1(t)$ and $F_2(t)$, respectively, with the pseudoscalar coupling constant g_Λ replaced by the scalar coupling constant, and the lifetime of the Σ^0 is given by the same expression as Eq. (IV.7).

APPENDIX

Here we wish to examine the analytic properties of $\Gamma_i^L(t)$, or equivalently, of $(pq)^{-J}A_J^L$. The left-hand singularities arise from the vanishing of the denominators in the integral of $I_J(s,t)$ defined in the text:

$$I_J(s',t)=\frac{1}{(pq)^J}\int_{-1}^{+1}dx\,P_J(x)\left[\frac{1}{s'-p^2-q^2-2pqx}-\frac{1}{s'-p^2-q^2+2pqx}\right],\qquad\text{(A1)}$$

$$\frac{1}{(pq)^J}A_J(t)=\frac{1}{\pi}\int_0^\infty ds'\,a(s',t)I_J(s',t).\qquad\text{(A2)}$$

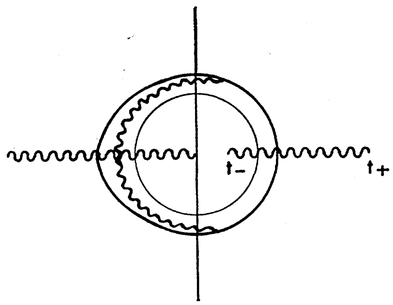

FIG. 3. Singularities of $(pq)^{-J}A_J{}^L(t)$. The heavy line in the complex region is the locus of the branch points.

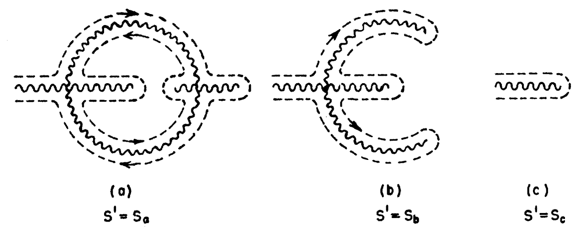

FIG. 4. Left-hand singularities of $f^L(s',t)$. The dashed lines are the contours of integration for the integral representation of $f^L(s',t)$ for fixed s'. $s_a<s_b<s_c$.

More precisely, the left-hand branch points are end-point singularities[14] and are given by the roots of $s'-p^2-q^2\pm 2pq=0$, or

$$t^{\frac{1}{2}}(t-m_\Lambda{}^2-m_\Sigma{}^2-2m_\pi{}^2-2s') = (t-c_1)^{\frac{1}{2}}(t-c_2)^{\frac{1}{2}}(t-c_3)^{\frac{1}{2}}. \quad \text{(III.3)}'$$

There are in general two roots to this equation. For $s'=m_\Sigma{}^2$ those two roots correspond to $t_\pm$ of Fig. 2. For some values of s', the branch points lie in the complex plane of t.

If we further insist that the contour of integration in Eq. (A.1) is along the real axis, $-1\leq x\leq 1$, then the branch cuts are given by Eq. (III.3) of the text and are illustrated in Fig. 2 for $s'=m_\Sigma{}^2$. In general, branch cuts corresponding to different values of s' do not overlap except in $-\infty<t\leq 0$, and $t_-\leq t\leq t_+$, and form a strip of singularities in the complex t plane (Fig. 3).

We may, of course, deform each branch line in the complex plane corresponding to different s' so that each branch line coincides with the locus of the complex branch points off the real axis.[15] The shape of singularities then would become simpler, but the discontinuity across the complex branch line would be untractably complicated, since deforming the contour in the t plane amounts to deforming the contour of integration of Eq. (A1) from the original path along the real axis. We therefore prefer to define the contour of integration of Eq. (A1) as along the real axis.

Let us consider $a(s',t)I_J(s',t)\equiv f(s',t)$ for fixed s'.

$$\frac{1}{(pq)^J}A_J{}^L(t)=\frac{1}{\pi}\int_0^\infty ds'\, f^L(s',t), \quad \text{(A3)}$$

where $f^L(s',t)$ is the contribution to $f(s,t)$ from the left-hand singularities. The left-hand branch lines of $f(s',t)$ varies for different values of s', and a few examples are shown in Fig. 4. Applying Cauchy's theorem along the contours exhibited in Fig. 4, we obtain

$$f^L(s',t)=\frac{1}{\pi}\int_{C(s')} dt'\frac{\text{Abs}f^L(s',t')}{t'-t},$$
$$\text{Abs}f^L(s',t)=\pm\frac{1}{(pq)^{J+1}}a(s',t)P_J\left(\frac{s'-p^2-q^2}{2pq}\right), \quad \text{(A4)}$$

where the signature $\pm$ is to be chosen appropriately. Note that the contour of integration $C(s')$ depends on s'.

Combining Eq. (A4) and Eq. (A2) we obtain

$$\frac{1}{(pq)^J}A_J{}^L(t)=\frac{1}{\pi^2}\int_0^\infty ds'\int_{C(s')} dt'\frac{\text{Abs}f^L(s',t')}{t'-t}. \quad \text{(A5)}$$

[14] R. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952); Phys. Rev. **119**, 1766 (1960).

[15] M. Nauenberg, thesis, Cornell University, 1959 (unpublished); the authors have enjoyed an interesting discussion with Dr. Nauenberg on this point.