placing $j_{l'}(k_0R_0)$ and $h_{l'}^{(1)}(k_0R_0)$ in (26) by their asymptotic forms for large k_0R_0 one finds, after some algebra, that

$$\langle L|T|0\rangle=-\tfrac{1}{2}[2(2L+1)]^{\frac{1}{2}}(-i)^{L+1}(2\lambda+1)\times\begin{pmatrix}L & \lambda & \lambda\\ 0 & 0 & 0\end{pmatrix}\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)\left(\frac{V_0}{E_0}\right)\left(\frac{V_0R_0^3}{k_0R_0}\right)\times[j_L(qR_0)-(-1)^\lambda j_L(pR_0)e^{2ik_0R_0}]. \quad (27)$$

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Approximation Methods in Nuclear Intermediate Coupling Applied to the $1p$ Shell*†

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The intermediate-coupling shell model for nuclei is considered in terms of the classification of states in a harmonic oscillator according to the irreducible representations of the unitary unimodular group in three dimensions, SU_3, introduced by Elliott. The properties of this group are used to produce the approximate spectrum of a quadrupole force, acting within an oscillator shell. When specialized to the $1p$ shell, a more general interaction, including exchange forces, is shown to be approximately diagonal in the chosen representation, and its approximate spectrum is computed. A method is developed for calculating the matrix elements of interactions not diagonal in the representation, in particular the single-particle spin-orbit potential, using the generating functions of the group, SU_3. The intermediate-coupling energy spectra of the nuclei of the $1p$ shell are then calculated to the first or second order in perturbation theory. The results are compared with experimental spectra, and with calculations of Kurath.

I. INTRODUCTION

THIS paper is an investigation of new methods of calculation of energy spectra in the nuclear shell model, with particular application to the $1p$ shell. The energy levels in this model are the eigenvalues of an interaction matrix in the space defined by restricting the particles outside the closed shells (the core) to the lowest available unfilled shell. The interactions considered usually contain a two-particle central potential and a single-particle spin-orbit potential. The competition of the two potentials produces "intermediate coupling" eigenfunctions, which are pure in neither L-S nor j-j coupling.

The calculation of the matrix elements of the interaction is central to the problem. This is usually done by factoring the many-particle basis functions into products of functions of smaller numbers of particles, so that the many-particle matrix elements required can be expressed in terms of those for fewer particles, and ultimately in terms of single- or two-particle matrix elements of the single- or two-particle potentials which are calculated directly. The fractional parentage methods of Racah[1] are useful for this reduction. However, for many particles, or for particles with high angular momentum, this may be a difficult program.

We have investigated a different method of obtaining the matrix elements, based on Elliott's group-theoretic classification of states for a harmonic oscillator shell model.[2] In Sec. II, we review some of Elliott's results, in a slightly different presentation. In Sec. III, we show that Elliott's classification scheme approximately diagonalizes a two-particle quadrupole interaction.

In Sec. IV we specialize to the $1p$ shell, where Elliott's classification scheme is related to the supermultiplet scheme of Wigner.[3] Here the group theory of the Elliott scheme provides a direct way to calculate the spectrum of a central, spin-independent potential. Even for the spin-dependent potentials used in intermediate-coupling calculations, we may use the group theory to obtain approximate spectra, with a small correction term which is not diagonal.

The remaining problem is the calculation of the spin-orbit matrix elements, and those of the nondiagonal

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[1] G. Racah, Phys. Rev. **63**, 367 (1943), and further references therein.

[2] J. P. Elliott, Proc. Roy. Soc. (London) **245A**, 128 and 562 (1958).

[3] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

central term. In Secs. V through VII we introduce a method for calculating those elements which will be useful for a perturbation treatment. We again use group theoretic methods to avoid the necessity of the fractional parentage reduction. This method is not restricted to the $1p$ shell.

In Sec. VIII we calculate the spectra of $1p$ shell nuclei in the perturbation scheme developed. The results are compared with experimental spectra, and with previous matrix calculations of Kurath.[4]

II. CLASSIFICATION OF STATES BY SU_3

Elliott[2] has introduced the classification of states of many particles in a three-dimensional harmonic oscillator potential, using the unitary unimodular group in three dimensions, SU_3. This group is defined in terms of linear transformations in three dimensions among the oscillator quanta contained in the wave function of a single particle in the potential. We define the operators which create one oscillator quantum, with angular momentum projection along the z axis of $1, 0, -1$, by u_1^*, u_0^*, and u_{-1}^*, respectively. The three operators which annihilate one quantum are then u_1, u_0, and u_{-1}. They obey the usual commutation relations:

$$[u_\mu,u_\nu^*]=\delta_{\mu\nu};\quad [u_\mu,u_\nu]=[u_\mu^*,u_\nu^*]=0. \tag{2.1}$$

The generators of the three-dimensional linear transformations of the quanta are the operators which annihilate, say, quantum μ, and replace it with quantum ν. These operators are of the form $u_\nu^* u_\mu$, where ν and μ each take any of the values $1, 0, 1$. From (2.1) we obtain new commutation relations:

$$[u_\kappa^* u_\lambda,u_\mu^* u_\nu]=u_\kappa^* u_\nu\delta_{\mu\lambda}-u_\mu^* u_\lambda\delta_{\kappa\nu}. \tag{2.2}$$

Since the commutators of these operators are simply the operators again, they are the generators of a linear continuous group of transformations of the form $\exp(i\theta u_\nu^* u_\mu)$. If we restrict the coefficients of $i\theta$ to the Hermitian combinations $u_\nu^* u_\mu+u_\mu^* u_\nu$ or $i(u_\nu^* u_\mu-u_\mu^* u_\nu)$, the group becomes the unitary group, U_3. If we further eliminate combinations of the generators which are simply multiples of the unit operator, we are restricted to the unimodular unitary group, SU_3.

We shall find it convenient to redefine the generators:

$$\begin{aligned}
&H_0=u_1^* u_1+u_{-1}^* u_{-1}+u_0^* u_0,\\
&H_1=u_1^* u_1-u_{-1}^* u_{-1},\\
&H_2=2u_0^* u_0-u_1^* u_1-u_{-1}^* u_{-1};\\
&F_1=-u_0^* u_1,\quad F_5=-u_0^* u_{-1},\quad F_4=-u_1^* u_{-1},\\
&F_{-1}=-u_1^* u_0,\quad F_{-5}=-u_{-1}^* u_0,\quad F_{-4}=-u_{-1}^* u_1.
\end{aligned} \tag{2.3}$$

From (2.2) we obtain the commutators quite simply:

$$\begin{aligned}
&[H_1,F_{\pm1}]=\mp F_{\pm1},\quad [H_2,F_{\pm1}]=\pm 3F_{\pm1},\\
&[H_1,F_{\pm5}]=\pm F_{\pm5},\quad [H_2,F_{\pm5}]=\pm 3F_{\pm5},\\
&[H_1,F_{\pm4}]=\pm 2F_{\pm4},\quad [H_2,F_{\pm4}]=0;\\
&[F_1,F_{-1}]=\tfrac{1}{2}(H_2-H_1),\quad [F_5,F_{-5}]=\tfrac{1}{2}(H_2+H_1),\quad [F_4,F_{-4}]=H_1,\\
&[F_{-5},F_4]=F_{-1},\quad [F_4,F_1]=F_5,\quad [F_5,F_{-1}]=F_4,\\
&[F_{-4},F_5]=F_1,\quad [F_{-1},F_{-4}]=F_{-5},\quad [F_1,F_{-5}]=F_{-4}.
\end{aligned} \tag{2.4}$$

We notice that H_0 commutes with all the other operators, and is thus a multiple of the unit operator. H_0 is simply a multiple of the oscillator Hamiltonian, and must commute with the other generators, since the number of quanta is conserved in these transformations. We drop H_0 to restrict to SU_3, as we mentioned.

The subscripts of the F_n have been chosen for convenience. It turns out that $[F_n,F_m]=\pm F_{n+m}$ for $n\neq m$. The commutator is zero if there is no F_{n+m} defined. If $n=-m$, $[F_n,F_{-n}]$ is a combination of H_1, H_2. Any commutator not appearing in (2.4) is zero.

These generators have been defined for single-particle wave functions. If we label the operator for the ith particle by $G(i)$, we may define a many-particle operator by $G=\sum_i G(i)$. The commutation relations (2.4) will hold for the set G, as well as for the $G(i)$, since $[G(i),G(j)]=0$ for $i\neq j$.

The many-particle space of m particles in the Nth oscillator level is spanned by functions containing mN quanta, in any combination of the three kinds $(1,0,-1)$. This space carries a reducible representation of the group SU_3. The reduction under U_3 breaks up the many-particle space into invariant subspaces, characterized by the symmetry of the wave functions under permutation of the oscillator quanta. The permutation symmetries are classified, as usual, by the partitions f_1, f_2, f_3, of the mN quanta into the three kinds, with $f_1+f_2+f_3=mN$. We may choose $f_1\geq f_2\geq f_3$. When we restrict ourselves to the unimodular group SU_3, only the differences f_1-f_2, f_2-f_3 are needed to classify the invariant subspaces. We follow Elliott's definitions, and use (λ,μ) to label the invariant subspaces, or the irreducible representations of SU_3 they carry, where $\lambda=f_1-f_2$ and $\mu=f_2-f_3$.

Elliott notices that two of the generating operators, which we have called H_1 and H_2, commute. Since the generators do not mix functions from different invariant subspaces (λ,μ), H_1 and H_2 can be simultaneously diagonalized within each subspace, yielding eigen-

[4] D. Kurath, Phys. Rev. **101**, 216 (1956).

values K and ϵ, respectively. From (2.3) we obtain by inspection the eigenvalues K, ϵ for states of one oscillator quantum:

$$\begin{aligned} H_1|m\rangle &= m|m\rangle \\ H_2|m\rangle &= 2|m\rangle \quad \text{for} \quad m=0, \\ &= -|m\rangle \quad \text{for} \quad m=\pm 1. \end{aligned} \tag{2.5}$$

For a state with many oscillator quanta, the eigenvalues will be simply the sum of single-quantum eigenvalues.

From the commutation relations (2.4) of the F_m with H_1 and H_2, we find that the F_m are raising and lowering operators, which change the values of K and ϵ. In particular, K is increased by 1 by F_{-1} and F_5, lowered by 1 by F_1 and F_{-5}, increased by 2 by F_4, and lowered by 2 by F_{-4}. ϵ is increased by 3 by F_1 and F_5, lowered by 3 by F_{-1} and F_{-5}, and is not changed by $F_{\pm 4}$.

Since these raising and lowering operators are generators of the group SU_3, they do not take us out of a given invariant subspace (λ,μ). If we now operate repeatedly with various of the F_m, on any state of the space, we shall eventually reach a state with a maximum value of ϵ for that (λ,μ). There may be several states in (λ,μ) with that value of ϵ, which we shall call ϵ_0. These states will have values of K differing by 2, and can be generated from any given one by repeated operations of F_4 or F_{-4}.

Using the results of (2.5), we can construct the maximum ϵ by filling the largest partition (f_1) with (0) quanta, and f_2 and f_3 with $(1, -1)$ quanta. One can see simply then that

$$\epsilon_0 = 2\lambda+\mu. \tag{2.6}$$

Clearly the maximum value of K possible for maximum ϵ is $K=\mu$.

The basis functions in the many-particle space are not completely labeled by λ, μ, ϵ, K. Other quantum numbers may be required to determine the functions uniquely. For example, the symmetry of the wave function under permutation of the particles can be specified, since this operation commutes with the transformations of SU_3, which are symmetric in the particles.

No use has been made of the group of three-dimensional rotations, R_3, which is a subgroup of SU_3. However, it turns out that H_2 does not commute with L^2, the angular momentum operator, so that we cannot simultaneously diagonalize both. We note from the definition (2.3) that H_1 is identical with the operator L_0, whose eigenvalues are the z projections of the angular momentum. Therefore, H_1 does commute with L^2.

In physical problems, we shall want to distinguish states with different angular momenta L. Elliott[2] has shown that states of different L and M can be generated from the basis functions $f_{\lambda\mu\epsilon K}(\mathbf{r})$ defined by λ, μ, ϵ_0, K, by rotating the function in space, and averaging over all directions (Ω) in space. The averaging integral is weighted by the functions $D^*{}_{MK}{}^L(\Omega)$, the representation functions of R_3. Then the averaged function,

$$\frac{2L+1}{8\pi^2}\int D^*{}_{MK}{}^L(\Omega) f_{\lambda\mu\epsilon K}(\Omega^{-1}\mathbf{r})\,d\Omega = \psi_{MK}{}^L(\mathbf{r}), \tag{2.7}$$

is an eigenfunction of L^2 and L_0, with eigenvalues $L(L+1)$ and M, respectively.

This averaging integration is equivalent to an operator which first projects the part of the function $f_{\lambda\mu\epsilon K}(\mathbf{r})$ with angular momentum L, and which then changes the z projection of angular momentum from K to M. We can define the first operation by the projection operator P^L, and the second by $(L_+)^{M-K}$ for the case that $M>K$. The successive operations of L_+ must be divided by the appropriate matrix element, to make the operation unitary. The combination of the projection of L, and normalized change of K to M, defines an operator $P_M{}^L$. Then (2.7) becomes simply

$$P_M{}^L f_{\lambda\mu\epsilon K}(\mathbf{r}) = \psi_{MK}{}^L(\mathbf{r}). \tag{2.8}$$

The usefulness of this procedure of obtaining the functions $|\lambda,\mu,L^2,M\rangle$ from the functions $|\lambda,\mu,\epsilon,K\rangle$ is demonstrated by a theorem proved by Elliott[2]: The space carrying an irreducible representation (λ,μ) of SU_3 can be completely spanned, by projecting all possible L and M from only the states of maximum ϵ, $|\lambda,\mu,\epsilon_0,K\rangle$ with all available K. (Alternately, the states of minimum ϵ have the same property.) In some cases the set of functions generated in this way will be overcomplete; there may be linear relations among some of the states produced by $P_M{}^L$ on functions of maximum ϵ_0, with different K.

We shall find that the states with maximum ϵ are particularly convenient for calculating matrix elements, so that the fact that all states of a given (λ,μ) can be generated in this way from ϵ_0 will prove useful.

III. THE CASIMIR OPERATOR AND THE QUADRUPOLE FORCE

We introduce the following bilinear combination of the infinitesimal generators of SU_3:

$$\begin{aligned} C = (1/36)[&3H_1{}^2+H_2{}^2+6(F_4F_{-4}+F_{-4}F_4) \\ &+6(F_1F_{-1}+F_{-1}F_1)+6(F_5F_{-5}+F_{-5}F_5)]. \end{aligned} \tag{3.1}$$

It can be seen from (2.4) that C commutes with every generator of the group. It follows that C also commutes with every finite transformation of the group. By Shur's lemma, C must be a multiple of unity within each invariant subspace. Therefore, within a given subspace, every basis function is an eigenfunction of C, with the same eigenvalue. The eigenvalue in general will be different for different invariant subspaces, and may be used to distinguish the invariant subspaces of SU_3, as the eigenvalues $L(L+1)$ of L^2 distinguish the invariant subspaces of the rotations R_3. C is the Casimir operator for the group SU_3.[5]

[5] G. Racah, mimeographed notes, Institute for Advanced Study, Princeton, New Jersey, 1951 (unpublished).

The eigenvalue of C for a given invariant subspace (λ,μ) may most easily be calculated by taking as an eigenfunction the function with maximum ϵ and maximum K for that ϵ_0, since F_1, F_5, and F_4 all yield zero, acting on this function. Then, using the commutation relations (2.4),

$$\begin{aligned} C|K_0,\epsilon_0\rangle &= (1/36)\{3H_1^2+H_2^2+6([F_4,F_{-4}]+[F_1,F_{-1}]+[F_5,F_{-5}])\}|K_0,\epsilon_0\rangle \\ &= (1/36)\{3K_0^2+\epsilon_0+6[K_0+\tfrac{1}{2}(\epsilon_0-K)+\tfrac{1}{2}(\epsilon_0+K)]\}|K_0,\epsilon_0\rangle \\ &= (1/36)[3K_0^2+\epsilon_0^2+6(K_0+\epsilon_0)]|K_0,\epsilon_0\rangle \equiv C_{\lambda\mu}|K_0,\epsilon_0\rangle. \end{aligned} \tag{3.2}$$

And from (2.6), finally,

$$C_{\lambda\mu}=[(\lambda+\mu)(\lambda+\mu+3)-\lambda\mu]/9. \tag{3.3}$$

The Casimir operator C can be put into an interesting form if we replace the infinitesimal generators (2.3) by an equivalent set defined by Elliott,[2] in terms of components of irreducible tensors. The new generators are simply linear combinations of the old:

$$\begin{aligned} &L_{\pm1}=\pm(F_{\pm5}+F_{\mp1}); \quad L_0=H_1; \\ &Q_{\pm1}=\pm\sqrt{3}(F_{\pm5}+F_{\mp1}); \\ &Q_{\pm2}=(6)^{\frac{1}{2}}F_{\pm4}; \quad Q_0=H_2. \end{aligned} \tag{3.4}$$

The three components L_μ form a rank one tensor, which is simply the usual operator for orbital angular momentum. The five Q_μ form a second rank tensor. Elliott[2] shows that the matrix elements of the single-particle operator $Q_\mu(i)$ between harmonic oscillator states in the same shell are equal to the matrix elements of the solid harmonic

$$2(4\pi/5)^{\frac{1}{2}}(r_i/b)^2Y_\mu^2(\Omega_i); \quad b^2=\hbar/M\omega. \tag{3.5}$$

Between single-particle states in different oscillator shells, the matrix elements of $Q_\mu(i)$ are clearly zero; the generators of SU_3 commute with the number of oscillator quanta.

Using (3.4), C can be written

$$C=(1/36)(\mathbf{Q}\cdot\mathbf{Q}+3\mathbf{L}\cdot\mathbf{L}), \tag{3.6}$$

where the dot represents the scalar product of the tensors. Now

$$\mathbf{Q}\cdot\mathbf{Q}=\sum_{i\neq j}^{n}\mathbf{Q}(i)\cdot\mathbf{Q}(j)+\sum_{i=1}^{n}Q^2(i).$$

As long as we operate only within the space of functions with all particles in the same oscillator shell, we may use the result of (3.5) to replace $\mathbf{Q}(1)\cdot\mathbf{Q}(2)$ by

$$4\frac{4\pi}{5}\frac{r_1^2}{b^2}\frac{r_2^2}{b^2}\sum_m Y_m^{2*}(\Omega_1)Y_m^2(\Omega_2)=4\frac{r_1^2}{b^2}\frac{r_2^2}{b^2}P_2(\cos\theta_{12}). \tag{3.7}$$

This is simply the form of a two-particle quadrupole-quadrupole interaction.[6] $\mathbf{Q}\cdot\mathbf{Q}$ acting within a single oscillator shell is then the interaction summed over pairs, plus a single-particle term:

$$\sum_{i\neq j}^{n}4\frac{r_i^2}{b^2}\frac{r_j^2}{b^2}P_2(\cos\theta_{12})+\sum_{i=1}^{n}Q^2(i). \tag{3.8}$$

Now C and $L^2=\mathbf{L}\cdot\mathbf{L}$ can be diagonalized simultaneously, and the basis functions labeled by (λ,μ), L. This is the complete reduction of the many-particle space under the groups SU_3 and R_3. Thus, the representation "almost" diagonalizes the quadrupole interaction, except for the single-particle term, $Q^2(i)$. Using (3.6) again, this last term becomes

$$Q^2(i)=36C(i)-3L^2(i).$$

$C(i)$ can have only one value for each shell, since for a single particle, only the symmetric partition $(\lambda,\mu)=(N,0)$ is possible, where N is the number of quanta.

Although the quadrupole-quadrupole force (3.7) is not the most general two-body interaction which can be considered in the shell model, it may represent the long-range part of a more general interaction. In those cases where a long-range approximation may be made, the reduction of the many-particle space under SU_3 and R_3 provides a useful representation in which (3.8) is diagonal, with eigenvalues $36C_{\lambda\mu}-3L(L+1)$. One expects that $L^2(i)$ can be treated as a perturbation, so that

$$\begin{aligned} &36C_{\lambda\mu}-3L(L+1)-36nC_{N0} \\ &\qquad +3\langle\lambda\mu,L|\sum_{i=1}^{n}L^2(i)|\lambda\mu,L\rangle \end{aligned} \tag{3.9}$$

gives an approximate energy spectrum for n particles in the Nth shell, interacting through the quadrupole force.

IV. CENTRAL INTERACTION IN THE $1p$ SHELL

The first excited harmonic oscillator level, the nuclear $1p$ shell, has several simplifying features which makes the treatment of a general central two-particle interaction quite easy. First, from consideration of angular momentum and parity, it is clear that only monopole and quadrupole forces interact between pairs of particles. That is, in the usual Slater treatment, in which radial integrals are done first, the angular part of any potential $V(r_1-r_2)$ becomes

$$F_0+F_2P_2(\cos\theta_{12})=F_0+F_2\frac{4\pi}{5}\sum_m Y_m^2(1)Y_m^{2*}(2). \tag{4.1}$$

[6] J. P. Elliott, Proceedings of the University of Pittsburgh Conference on Nuclear Structure, 1957 (University of Pittsburgh and Office of Ordnance Research, U. S. Army, 1957), p. 298 ff.

Within the $1p$ shell, $(r_i/b)^2=\frac{5}{2}$, and $L^2(i)=2$. Therefore

$$\mathbf{Q}\cdot\mathbf{Q}=25\sum_{i\neq j}^{n}P_2(\cos\theta_{12})+36\sum_{i=1}^{n}C(i)-6n. \quad (4.2)$$

From (3.3), with $(\lambda,\mu)=(1,0)$, we find $C(i)=4/9$. So, we find we can rewrite the entire interaction:

$$\begin{aligned}\tfrac{1}{2}\sum_{i\neq j}^{n}V(r_i-r_j)&=F_0\frac{n(n-1)}{2}+F_2\left(\frac{\mathbf{Q}\cdot\mathbf{Q}}{50}-\frac{n}{10}\right)\\&=F_0n(n-1)/2+F_2(1/50)(36C-3L^2-5n). \quad (4.3)\end{aligned}$$

This is diagonal in the (λ,μ), L scheme, and has eigenvalues

$$F_0n(n-1)/2+F_2[36C_{\lambda\mu}-3L(L+1)-5n]/50, \quad (4.4)$$

where $C_{\lambda\mu}$ is obtained from (3.3).

A more general central interaction, which has spin and isobaric spin dependence, may be written

$$V_{ij}=V(r_i-r_j)[W1_{ij}+MP_{ij}^{x}+BP_{ij}^{\sigma}-HP_{ij}^{\tau}], \quad (4.5)$$

where the coefficients satisfy $W+M+B+H=1$. The operators P_{ij}^{x}, P_{ij}^{σ}, and P_{ij}^{τ} exchange the space, spin, and isobaric spin coordinates, respectively, of particles i, j.

Again the $1p$ shell provides a simplification. Since in this shell there is one oscillator quantum for each particle, the permutation symmetry of the quanta is identical with that of the particles. Thus, the particle symmetry denoted by the partition (f_1,f_2,f_3) is identical with the quantum symmetry $(\lambda,\mu)=(f_1-f_2,f_2-f_3)$.

In particular, for two particles, the space-symmetric state has $(\lambda,\mu)=(2,0)$, with $C_{20}=10/9$, and the space-antisymmetric state has $(\lambda,\mu)=(0,1)$, with $C_{01}=4/9$. Since

$$\begin{aligned}(3C-7/3)(2,0)&=(2,0) \quad \text{and}\\(3C-7/3)(0,1)&=-(0,1), \quad (4.6)\end{aligned}$$

we see that $(3C-7/3)$ has the same effect as the space exchange operator P_{ij}^{x}, and can be used in its place.

The other exchange operators can be simply written $P_{ij}^{\sigma}=(S_{ij}^2-1)$ and $P_{ij}^{2}=(T_{ij}^2-1)$. We can still use

$$V(r_i-r_j)=F_0+(F_2/50)(36C_{ij}-3L_{ij}^2-10). \quad (4.7)$$

Using the above expressions for the exchange operators in (4.6) yields awkward expressions like $C_{ij}C_{ij}$ and $C_{ij}L_{ij}^2$. To reduce these, we use the fact that the two-particle space-antisymmetric state (0,1) has $C_{\lambda\mu}=4/9$ and $L(L+1)=2$. Then

$$\begin{aligned}(C-4/9)(3C-7/3)&=(C-4/9),\\(L^2-2)(3C-7/3)&=(L^2-2),\end{aligned} \quad (4.8)$$

on any two-particle state in the $1p$ shell. In particular

$$\begin{aligned}(36C_{ij}-3L_{ij}^2-10)P_{ij}^{x}&=(36C_{ij}-3L_{ij}^2-10)(3C-7/3)\\&=(36C_{ij}-3L_{ij}^2-10). \quad (4.9)\end{aligned}$$

Since $P_{ij}^{x}P_{ij}=-P_{ij}^{\sigma}$ for wave functions totally antisymmetric in x, σ, τ,

$$\begin{aligned}(36C_{ij}-3L_{ij}^2-10)P_{ij}^{\tau}&=-(36C_{ij}-3L_{ij}^2-10)P_{ij}^{x}P_{ij}^{\sigma}\\&=-(36C_{ij}-3L_{ij}^2-10)P_{ij}^{\sigma}\\&=-(36C_{ij}-3L_{ij}^2-10)(S_{ij}^2-1). \quad (4.10)\end{aligned}$$

Thus

$$\begin{aligned}V_{ij}=F_0[&W+M(3C_{ij}-7/9)+B(S^2-1)-H(T^2-1)]\\&+(F_2/50)[36C_{ij}-3L_{ij}^2-10]\\&\times[(W+M)+(B+H)(S_{ij}^2-1)]. \quad (4.11)\end{aligned}$$

We can reduce the term $C(S^2-1)$ by rewriting it:

$$\begin{aligned}(\tfrac{1}{3}P^{x}+7/9)P^{\sigma}&=\tfrac{1}{3}P^{x}P^{\sigma}+(7/9)P^{\sigma}\\&=-\tfrac{1}{3}(T^2-1)+(7/9)(S^2-1). \quad (4.12)\end{aligned}$$

We can now write the interaction (4.5), dropping additive constants. We shall not need these terms, since we shall restrict our considerations to relative energies within each nucleus.

$$\begin{aligned}V_{ij}=F_0\{&3MC_{ij}+BS_{ij}^2-HT^2\}\\&+(F_2/50)\{36(W+M)C_{ij}-3(W+M-B-H)L_{ij}^2\\&-10(B+H)S_{ij}^2-12(B+H)T_{ij}^2+28(B+H)S_{ij}^2\\&-3(B+H)L_{ij}^2S_{ij}^2\}. \quad (4.13)\end{aligned}$$

This form of the two-particle interaction is particularly convenient when we sum over pairs to obtain the total interaction $\frac{1}{2}\sum_{i\neq j}^{n}V_{ij}$. Now

$$\begin{aligned}\tfrac{1}{2}\sum_{i\neq j}^{n}L_{ij}^2&=\tfrac{1}{2}\sum_{i\neq j}^{n}[L^2(i)+L^2(j)+2\mathbf{L}(i)\cdot\mathbf{L}(j)]\\&=(n-2)\sum_{i}^{n}L^2(i)+L^2; \quad (4.14)\end{aligned}$$

similarly,

$$\begin{aligned}\tfrac{1}{2}\sum_{i\neq j}^{n}C_{ij}&=C+(n-2)\sum_{i}C(i),\\\tfrac{1}{2}\sum_{i\neq j}S_{ij}^2&=S^2+(n-2)\sum_{i}S^2(i),\\\tfrac{1}{2}\sum_{i\neq j}T_{ij}^2&=T^2+(n-2)\sum_{i}T^2(i).\end{aligned}$$

But $C(i)$, $L^2(i)$, $S^2(i)$, and $T^2(i)$ are all constants which can be dropped for considerations of energy splittings. Thus, each two-particle term in the interaction V_{ij}, with the exception of $L_{ij}^2S_{ij}^2$, can be expressed as a many-particle operator of the same kind, and with the same coefficients it had in V_{ij}.

C, L^2, S^2, and T^2 can all be diagonalized simultaneously. Thus, the energy spectrum for $\frac{1}{2}\sum V_{ij}$ without

the $\sum L_{ij}^2 S_{ij}^2$ term can be expressed exactly as a sum of terms C, $L(L+1)$, $S(S+1)$, and $T(T+1)$, with the appropriate coefficients from (4.13). If the term in $\sum L^2S^2$ can be shown to be small, its effect on the spectrum can be calculated in perturbation theory. This will be done below.

In Table I we list eigenvalues of C (or $C_{\lambda\mu}$) for the $1p$ shell. For each case we give the particle symmetry label $[f]$, and the equivalent (λ,μ). For each (λ,μ) we give the maximum value of ϵ, ϵ_0, and the possible values of L. We only include the particle numbers $n=1$ to 6, since the two-particle interaction $\frac{1}{2}\sum V_{ij}$ has the same spectrum in the $1p$ shell, for $12-n$ particles as for n particles, up to an additive constant. That is, n holes behave like n particles, under $\frac{1}{2}\sum V_{ij}$.

Finally, we shall make specific choices for the form of the interaction. For oscillator orbitals the Slater F^k's can be expressed in terms of Talmi integrals[7] I_p;

$$F^0=\tfrac{1}{12}[5(I_0+I_2)+2I_1],$$
$$F^2=(25/12)[(I_0+I_2)-2I_1], \tag{4.15}$$

where

$$I_p=\frac{4}{\sqrt{\pi}}\frac{2^p}{(2p+1)!!}\int_0^\infty V(r)x^{2p+2}\exp(-x^2)dx;$$
$$x^2=\frac{r^2}{2b^2}. \tag{4.16}$$

For a Gaussian two-particle potential,

$$V(r_1-r_2)=V(r)=V_0\exp(-r^2/r_0^2). \tag{4.17}$$

TABLE I. $C_{\lambda\mu}$ for $1p$ shell.

n	f	$(\lambda\mu)$	C	L	ϵ_0
1	[1]	(10)	4/9	1	2
2	[2]	(20)	10/9	0, 2	4
	[11]	(01)	4/9	1	1
3	[3]	(30)	2	1, 3	6
	[21]	(11)	1	1, 2	3
	[111]	(00)	0	0	0
4	[4]	(40)	28/9	0, 2, 4	8
	[31]	(21)	16/9	1, 2, 3	5
	[22]	(02)	10/9	0, 2	2
	[211]	(10)	4/9	1	2
5	[41]	(31)	25/9	1, 2, 3, 4	7
	[32]	(12)	16/9	1, 2, 3	4
	[311]	(20)	10/9	0, 2	4
	[221]	(01)	4/9	1	1
6	[42]	(22)	24/9	0, 2, 2, 3, 4	6
	[411]	(30)	2	1, 3	6
	[33]	(03)	2	1, 3	3
	[321]	(11)	1	1, 2	3
	[222]	(00)	0	0	0

[7] I. Talmi, Helv. Phys. Acta XXV, 185 (1952). R. Thieberger, Nuclear Phys. 2, 533 (1956–57).

We obtain

$$I_p=V_0\left(\frac{r_0^2}{2b^2+r_0^2}\right)^{p+\frac{3}{2}}=V_0\eta^{p+\frac{3}{2}}. \tag{4.18}$$

We choose the ratio $(\sqrt{2}b/r_0)=1.3$, which is equivalent to Kurath's choice[4] $L/K=6.8$. We also use his choice of $V_0=-45$ Mev. The choice of range is not very critical; a variation of $2b^2/r_0^2$ from 1 to 2 will change η only from 0.5 to 0.33.

Substituting these choices into (4.15) and (4.13), we obtain

$$\begin{aligned}V_{ij}=-\{&[(3.74)3M+(4.03)^{\frac{3}{2}}(W+M)]C_{ij}\\&+(-4.03)^{\frac{1}{8}}(W+M-B-H)L_{ij}^2\\&+[(3.74)B+(4.03)^{\frac{3}{4}}(B+H)]S_{ij}^2\\&+[(-3.74)H-(4.03)^{\frac{1}{2}}(B+H)]T_{ij}^2\\&+(-4.03)^{\frac{1}{8}}(B+H)L_{ij}^2S_{ij}^2\}.\end{aligned} \tag{4.19}$$

Kurath[4] uses the exchange mixture, $W=H=0$, $M=0.8$, $B=0.2$, which gives a stronger spin triplet than spin singlet strength in even-L states, as is observed in the deuteron and Li^6. The Serber mixture, $W=M=0.5$, $B=H=0$, gives nucleon-nucleon scattering symmetric about 90°, as observed experimentally. A mixture which gives both properties has been suggested by Meshkov[8]: $W=M=0.4$, $B=0.2$, $H=0$. For these three mixtures we obtain:

$$V(\text{Kurath})=-13.8C+0.30L^2-1.35S^2+0.40T^2+0.10L^2S^2; \tag{4.20a}$$

$$V(\text{Serber})=-11.6C+0.51L^2; \tag{4.20b}$$

$$V(\text{Meshkov})=-9.3C+0.30L^2-1.35S^2+0.40T^2+0.10L^2S^2. \tag{4.20c}$$

The Kurath and Meshkov mixtures differ solely in the coefficient of C. For all three mixtures, the L^2S^2 term is small.

V. THE INTRINSIC REPRESENTATION

In the previous section we have diagonalized the major part of the two-particle central force in the $1p$ shell, using a representation whose basis functions are eigenfunctions of C, L^2, S^2, and T^2. Clearly we can also diagonalize L_0, S_0, and T_3, and write the basis functions

$$|(\lambda\mu)LM_L,SM_S,TT_3\rangle.$$

Alternatively, we may obtain these approximate eigenfunctions by extending the basis $|(\lambda\mu)\epsilon,K\rangle$, defined in Sec. II, into spin and isobaric spin space. These basis functions, which diagonalize C, S^2, T^2, H_1 (or L_0), H_2 (or Q_0), S_0, and T_3, may be written

$$|(\lambda\mu)\epsilon,K,SM_S,TT_3\rangle.$$

Although these functions are not eigenfunctions of L^2, they may be used to generate such solutions by pro-

[8] S. Meshkov and C. W. Ufford, Phys. Rev. **101**, 734 (1956).

TABLE II. Intrinsic functions $|(\lambda\mu)\epsilon_0 K,M_S\rangle$ for the $1p$ shell.

$n=2$	$T=1, S=0$	$\|(20)40,0\rangle = \|0\uparrow 0\downarrow\|$
	$T=0, S=1$	$\|(20)40,1\rangle = \|0\uparrow, 0\uparrow\|$
		$\|(20)40,0\rangle = (\sqrt{\tfrac{1}{2}})\{\|0\uparrow, 0\downarrow\| + \|0\downarrow, 0\uparrow\|\}$
$n=3$	$T=\tfrac{1}{2}, S=\tfrac{1}{2}$	$\|(30)60,\tfrac{1}{2}\rangle = \|0\uparrow 0\downarrow, 0\uparrow\|$
$n=4$	$T=0, S=0$	$\|(40)80,0\rangle = \|0\uparrow 0\downarrow, 0\uparrow 0\downarrow\|$
$n=5$	$T=\tfrac{1}{2}, S=\tfrac{1}{2}$	$\|(31)71,\tfrac{1}{2}\rangle = \|1\uparrow 0\uparrow 0\downarrow, 0\uparrow 0\downarrow\|$
		$\|(31)71,-\tfrac{1}{2}\rangle = \|1\downarrow 0\uparrow 0\downarrow, 0\uparrow 0\downarrow\|$
$n=6$	$T=1, S=0$	$\|(22)62,0\rangle = \|1\uparrow 1\downarrow 0\uparrow 0\downarrow, 0\uparrow 0\downarrow\|$
		$\|(22)60,0\rangle = (\sqrt{\tfrac{1}{2}})\{\|1\uparrow -1\downarrow 0\uparrow 0\downarrow, 0\uparrow 0\downarrow\| + \|-1\uparrow 1\downarrow 0\uparrow 0\downarrow, 0\uparrow 0\downarrow\|\}$
	$T=0, S=1$	$\|(22)62,1\rangle = \|1\uparrow 0\uparrow 0\downarrow, 1\uparrow 0\uparrow 0\downarrow\|$
		$\|(22)62,0\rangle = (\sqrt{\tfrac{1}{2}})\{\|1\uparrow 0\uparrow 0\downarrow, 1\downarrow 0\uparrow 0\downarrow\| + \|1\downarrow 0\uparrow 0\downarrow, 1\uparrow 0\uparrow 0\downarrow\|\}$
		$\|(22)62,-1\rangle = \|1\downarrow 0\uparrow 0\downarrow, 1\downarrow 0\uparrow 0\downarrow\|$
		$\|(22)60,1\rangle = (\sqrt{\tfrac{1}{2}})\{\|1\uparrow 0\uparrow 0\downarrow, -1\uparrow 0\uparrow 0\downarrow\| + \|-1\uparrow 0\uparrow 0\downarrow, 1\uparrow 0\uparrow 0\downarrow\|\}$
		$\|(22)60,0\rangle = \tfrac{1}{2}\{\|1\uparrow 0\uparrow 0\downarrow, -1\downarrow 0\uparrow 0\downarrow\| + \|1\downarrow 0\uparrow 0\downarrow, -1\uparrow 0\uparrow 0\downarrow\| + \|-1\uparrow 0\uparrow 0\downarrow, 1\downarrow 0\uparrow 0\downarrow\| + \|-1\downarrow 0\uparrow 0\downarrow, 1\uparrow 0\uparrow 0\downarrow\|\}$

jection, $P_M{}^L$, as described in Sec. II. Using Elliott's theorem, we need only choose $\epsilon=\epsilon_0$ to span the whole space. We therefore define a set of functions: $|(\lambda\mu)\epsilon_0 K,M_S\rangle$, which we call the "intrinsic representation," where S, T, and T_3 are not explicitly written.

A useful feature of these intrinsic functions is that they are often easily expressed in terms of Slater determinants of the single-particle orbitals in the $1p$ shell: p_1, p_0, p_{-1}, with spin up or down, proton or neutron state. In particular, if we require the maximum possible ϵ for a given number of particles, the number of determinant wave functions consistent with a given S, T, T_3 is very small. This is particularly useful, since the maximum occurs for the highest $(\lambda\mu)$ symmetry, that is, with the greatest eigenvalue of C. Because C has a large negative coefficient in the usual central-force choices (4.20), the states of highest $C_{\lambda\mu}$ or ϵ_0 lie lowest in energy. It will be useful to have simple determinant expressions for these low states, for the calculation of perturbation terms.

In Table II we give the determinant forms of the $|(\lambda\mu)\epsilon_0 K,M_S\rangle$ for the $1p$ shell, for highest $C_{\lambda\mu}$, for 2 to 6 particles (or holes). We always choose $T_3=T$. The determinants for each are denoted by one row of orbitals, since the other rows differ only in the particle labels. The orbital functions p_0, p_1, and p_{-1} we further abbreviate by 0, 1, and -1, respectively. The arrows denote spin up or down, and proton orbitals are separated from neutron orbitals by a comma. We have shown only functions with $K>0$, which are all we shall need explicitly.

We saw that the functions generated from the intrinsic functions by L projection, $P_M{}^L$, form an LSM_LM_S representation in which the central interaction $\frac{1}{2}\sum V_{ij}$ is approximately diagonal. If the total interaction also contains noncentral terms, like a spin-orbit force, states with different total angular momentum J will no longer be degenerate. We may generate eigenfunctions of J^2 from the intrinsic states, using a J projection operator, $P_{M_J}{}^J$, which is the generalization of $P_M{}^L$ to J space. This is not simply a transformation to an $LSJM_J$ representation, since there may be more than one value of L possible for a given value of J. Then J projection, $P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$, gives a linear combination of the $LSJM_J$ functions, with different values of L. In these cases, however, there are several possible values of M_S, corresponding to the multiplicity of L for a given J. The J projection for each value of M_S will, in general, produce an independent linear combination of the $LSJM_J$ functions. That is, the functions

$$P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$$

with all values of J, M, K, M_S, span the same space as the $|(\lambda\mu)LSJM_J\rangle$. However, the $P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$ do not form an orthogonal basis in general, and may be overcomplete, in the sense that some of the functions are linearly related.

The projected intrinsic functions, $P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$, form a representation in which C, S^2, T^2 can be diagonalized, and may be taken as a zero-order representation for a perturbation treatment of the other central-force terms, L^2, $\sum L_{ij}{}^2 S_{ij}{}^2$, and any noncentral terms, like the spin-orbit interaction a $\sum_i \mathbf{l}_i\cdot\mathbf{s}_i$. Since the projected intrinsic functions with different values of K and M_S have the same eigenvalues $C_{\lambda\mu}$, $S(S+1)$, $T(T+1)$ (that is, are degenerate in zero order), the first-order energies are obtained by diagonalizing the perturbation terms in the projected intrinsic representation. We shall abbreviate the basis functions: $P_M{}^J|K,M_S\rangle$.

Diagonalizing an operator V in the nonorthogonal basis $P_M{}^J|K,M_S\rangle$ means finding linear combinations of the functions, such that

$$V\sum_{K,\mu} b_{K,\mu}P_M{}^J|K,\mu\rangle = \lambda\sum_{K,\mu} b_{K,\mu}P_M{}^J|K,\mu\rangle. \quad (5.1)$$

If

$$VP_M{}^J|K,\mu\rangle = \sum_{K',\nu} a_{K'\nu,K\mu}P_M{}^J|K',\nu\rangle, \quad (5.2)$$

then solutions of (5.1) can be found by solving

$$\sum_{K\mu} a_{K'\nu,K\mu} b_{K\mu}{}^{\lambda} = \lambda b_{K'\nu}{}^{2}. \tag{5.3}$$

Since $P_M{}^J|K,\mu\rangle$ is a nonorthogonal basis, $((a_{K'\nu,K\mu}))$ is not a Hermitian matrix. However, as long as the $P_M{}^J|K,\mu\rangle$ are linearly independent, $((a))$ is related to a Hermitian matrix by a similarity transformation, and therefore has the same eigenvalues, λ, as that Hermitian matrix.

Even if there are linear relations among some of the basis functions, (5.3) still provides valid solutions of (5.1), although these are not unique. The complete set of eigenvalues, λ, of V will still be produced, but there will also be spurious solutions, λ', resulting from the spurious enlargement of the space by introduction of nonindependent basis functions. The $b_{K\mu}{}^{\lambda'}$ which correspond to these solutions are just the coefficients of the linear relations among the basis functions,

$$\sum_{K\mu} b_{K\mu}{}^{\lambda'} P_M{}^J|K,\mu\rangle = 0.$$

Treating L^2 by degenerate perturbation theory actually involves no approximation; L^2 has no matrix elements connecting the $P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$ to states of other $(\lambda\mu)$, S, or T. The matrix elements of L^2 in the J-projected intrinsic space $P_M{}^J|K,\mu\rangle$ can be obtained:

$$\begin{aligned}
L^2 P_M{}^J|K,\mu\rangle &= P_M{}^J L^2|K,\mu\rangle = P_M{}^J(J^2+S^2-2\mathbf{J}\cdot\mathbf{S})|K,\mu\rangle \\
&= P_M{}^J\{[J(J+1)+S(S+1)-2(K+\mu)\mu]|K,\mu\rangle \\
&\quad -(2(S+\mu)(S-\mu+1))^{\frac{1}{2}}J_+|K,\mu-1\rangle-(2(S-\mu)(S+\mu+1))^{\frac{1}{2}}J_-|K,\mu+1\rangle\} \\
&= [J(J+1)+S(S+1)-2(K+\mu)\mu]P_M{}^J|K,\mu\rangle \\
&\quad -((J+K-\mu)(J+K+\mu+1)(S+\mu)(S-\mu+1))^{\frac{1}{2}}P_M{}^J|K,\mu-1\rangle \\
&\quad -((J+K+\mu)(J-K-\mu+1)(S-\mu)(S+\mu+1))^{\frac{1}{2}}P_M{}^J|K,\mu+1\rangle.
\end{aligned} \tag{5.4}$$

The matrix elements of L^2 do not connect different K.

If we denote a single-particle wave function by its total magnetic quantum number, $|m_j\rangle$, for $m_j>0$ (e.g., $|\frac{1}{2}\rangle\equiv|p_0\uparrow\rangle$), then we define $|-m_j\rangle=R(\pi)|m_j\rangle$, for $m_j>0$, where $R(\pi)$ is the operator which rotates the function about the y axis through π radians. For $m_j<0$, $|-m_j\rangle=-R(\pi)|m_j\rangle$. Then we define the many-particle intrinsic wave functions $|-K,-M_S\rangle$ to be the function obtained from $|K,M_S\rangle$ by replacing every single-particle function $|m_j\rangle$ by $|-m_j\rangle$. For a particular value of J, $R(\pi)|j,m_j\rangle=(-)^{j+m}|j,-m_j\rangle$.[9] From this it can be shown simply that the relation between $|-K,-M_S\rangle$ and $|K,M_S\rangle$ is given by

$$P_M{}^J|-K,-M_S\rangle=(-)^{\phi+J+K+M_S}P_M{}^J|K,M_S\rangle, \tag{5.5}$$

where ϕ is the number of single-particle functions in $|K,M_S\rangle$ with $m_j<0$.

VI. SPIN-ORBIT IN FIRST ORDER

We shall consider as a perturbing term the single-particle spin-orbit interaction, $\sum_{i=1}^{n}\mathbf{l}_i\cdot\mathbf{s}_i$. For the first-order calculation of the energy contribution, we shall need the matrix elements of $\sum\mathbf{l}_i\cdot\mathbf{s}_i$ between the projected states $P_M{}^J|(\lambda\mu)\epsilon_0 K,M_S\rangle$ for various values of K, M_S.

Since $\sum\mathbf{l}\cdot\mathbf{s}$ commutes with $P_M{}^J$,

$$\sum\mathbf{l}\cdot\mathbf{s}P_M{}^J|K,M_S\rangle=P_M{}^J\sum\mathbf{l}\cdot\mathbf{s}|K,M_S\rangle. \tag{6.1}$$

For the perturbation calculation, we are considering only the operations of $\sum\mathbf{l}\cdot\mathbf{s}$ within the space of functions with given $(\lambda\mu)S$, T. We shall denote by $\{\sum\mathbf{l}\cdot\mathbf{s}\}$ the restriction of the operator to that space. With that restriction,

$$\{\textstyle\sum\mathbf{l}\cdot\mathbf{s}\}|(\lambda\mu)\epsilon_0 K,M_S\rangle = \sum_{\epsilon',K',M_S'} C_{\epsilon',K',M_S'}|(\lambda\mu)\epsilon' K',M_S'\rangle. \tag{6.2}$$

That is, $\{\sum\mathbf{l}\cdot\mathbf{s}\}$ simply generates functions in the $(\lambda\mu)$ space, with different ϵ, K, M_S. We could also generate these same functions by operating on $|(\lambda\mu)\epsilon_0 K,M_S\rangle$ with combinations of the SU_3 group generators $F_{\pm1}$, $F_{\pm5}$, $F_{\pm4}$, and of the spin operators, $S_\pm$. Each state $|(\lambda\mu)\epsilon' K',M_S'\rangle$ can be generated by the appropriate combination of F_m, S_n, which change ϵ_0, K, M_S to ϵ', K', M_S'. There may be more than one independent operator for each of these transformations.

Let us expand:

$$\mathbf{l}(i)\cdot\mathbf{s}(i)=l_0(i)s_0(i)+l_+(i)s_-(i)+l_-(i)s_+(i); \tag{6.3}$$

and from (3.4)

$$\begin{aligned}
l_+(i)&=-l_1(i)=-F_5(i)-F_{-1}(i), \\
l_-(i)&=l_{-1}(i)=-F_{-5}(i)-F_1(i).
\end{aligned} \tag{6.4}$$

$F_1(i)$, $F_5(i)$ give zero, operating on a function with ϵ_0. Therefore,

$$\begin{aligned}
&\mathbf{l}(i)\cdot\mathbf{s}(i)|(\lambda\mu)\epsilon_0 K,M_S\rangle \\
&\quad=[l_0(i)s_0(i)-F_{-1}(i)s_-(i)-F_{-5}(i)s_+(i)] \\
&\qquad\times|(\lambda\mu)\epsilon_0 K M_S\rangle.
\end{aligned} \tag{6.5}$$

Clearly the first term gives states with the original $\epsilon=\epsilon_0$, K, M_S; the second term lowers ϵ_0 to ϵ_0-3, and changes

[9] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 59.

K to $K+1$, M_S-1; the third term lowers ϵ_0 to ϵ_0-3, and changes K to $K-1$, and M_S to M_S+1.

The functions we are considering have S a good quantum number, therefore the changes of M_S generated by $\{\sum \mathbf{l}\cdot\mathbf{s}\}$ can alternately be generated by the spin operator S_m. The first term in $\{\sum \mathbf{l}\cdot\mathbf{s}\}$, which changes none of the quantum numbers, is clearly proportional to the unit operator, since the state characterized by $(\lambda\mu)$, $\epsilon_0 K$, M_S, T, T_3, on which it operates, is unique.

There are two independent combinations of the many-particle operators $F_m=\sum_i F_m(i)$ which generate a state $(\epsilon_0-3, K+1)$ from (ϵ_0,K); they are F_{-1} and $F_{-5}F_4$. Although one can apparently construct other operators which do this by adding factors like F_1F_{-1} to these two operators, F_1 (or F_5) can always be commuted to the right until it annihilates the state (ϵ_0,K). The terms left after commuting will always be F_{-1} and $F_{-5}F_4$, multiplied by some combination of the diagonal operators L_0 and Q_0. This multiplying factor is just a number, so that no new independent operators can be constructed to generate $(\epsilon_0-3, K+1)$. If it happens that K has its maximum value in (ϵ_0,K), then $F_{-5}F_4$ on this state yields zero, and F_{-1} is the only independent operator leading to $(\epsilon_0-3, K+1)$. Similar considerations show that $(\epsilon_0-3, K-1)$ can be obtained by operations of F_{-5} and $F_{-1}F_{-4}$ on (ϵ_0,K). If K has its minimum value in (ϵ_0,K), F_{-5} is the only independent operator.

We find then, that $\{\sum \mathbf{l}\cdot\mathbf{s}\}$ operating on (ϵ_0,K) can generate at most five independent states, and that these are the states generated by the many-particle operators

$$1,\ F_{-1}S_-,\quad F_{-5}F_4S_-,\quad F_{-5}S_+,\quad F_{-1}F_{-4}S_+.$$

Thus we may write

$$\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|\,(\lambda\mu)\epsilon_0 K,M_S\rangle = [A1+BF_{-1}S+CF_{-5}F_4S_-+DF_{-5}S_++EF_{-1}F_{-4}S_+] \times|\,(\lambda\mu)\epsilon_0 K,M_S\rangle. \quad (6.6)$$

The coefficients A, B, C, D, and E can be determined by multiplying this equation on the left by the conjugates of the five functions on the right: $\langle(\lambda\mu)\cdots|$, $\langle\cdots|s_+F_1$, $\langle\cdots|s_+F_{-4}F_5$, $\langle\cdots|s_-F_5$, and $\langle\cdots|s_-F_4F_1$. We use $F_m{}^*=F_{-m}$ and $S_-{}^*=S_+$. This yields five simultaneous algebraic equations in the coefficients A, B, C, D, and E, which can be solved in terms of the "homogeneous" matrix elements

$$\langle(\lambda\mu)\cdots|O_a{}^*O_b|\,(\lambda\mu)\cdots\rangle$$

and the "inhomogeneous" matrix elements

$$\langle(\lambda\mu)\cdots|O_a{}^*\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|\,(\lambda\mu)\cdots\rangle,$$

where O_a is a typical operator in (6.6). The homogeneous terms are obtained by commuting $O_a{}^*$ and O_b, using (2.4), since $O_a{}^*$ contains F's which will annihilate $|\,(\lambda\mu)\epsilon_0 K,M_S\rangle$. The inhomogeneous terms are calculated by direct operation of $\sum \mathbf{l}\cdot\mathbf{s}$ on the determinant forms of the intrinsic functions (Table II). Since $\sum \mathbf{l}(i)\cdot\mathbf{s}(i)$ is a single-particle operator, acting on a determinant function $|\,a,b,c,\cdots|$,

$$\textstyle\sum \mathbf{l}\cdot\mathbf{s}\,|\,a,b,c,\cdots| = |\,(\mathbf{l}\cdot\mathbf{s}a),b,c,\cdots|+|\,a,(\mathbf{l}\cdot\mathbf{s}b),c,\cdots|+\cdots. \quad (6.7)$$

Explicit formulas for the matrix elements are given in the Appendix. Many matrix elements will be zero.

In Eq. (6.6), we may immediately operate with the $F_{\pm4}$ and $S_\pm$ on the intrinsic functions, using[10]

$$F_{\pm4}|\,(\lambda\mu)\epsilon_0 K,M_S\rangle=-\left[\left(\frac{\mu\mp K}{2}\right)\left(\frac{\mu\pm K+2}{2}\right)\right]^{\frac{1}{2}} \times|\,(\lambda\mu)\epsilon_0 K\pm2,M_S\rangle, \quad (6.8)$$

$$S_\pm|\,(\lambda\mu)\epsilon_0 K,M_S\rangle=[\tfrac{1}{2}(S\mp M_S)(S\pm M_S+1)]^{\frac{1}{2}} \times|\,(\lambda\mu)\epsilon_0 K,M_S\pm1\rangle.$$

Also, from (2.4), we may replace F_{-1} and F_{-5} by $-L_+$ and $-L_-$, respectively. Further, using $L_\pm=J_\pm-S_\pm$, (6.6) becomes

$$\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|\,K,M_S\rangle = A'|\,K,M_S\rangle+B'J_+|\,K,M_S-1\rangle+C'J_-|\,K+2,M_S-1\rangle +D'J_-|\,K,M_S+1\rangle+E'J_+|\,K-2,M_S+1\rangle, \quad (6.9)$$

where the coefficients in (6.8) have been absorbed. Operating on both sides of (6.9) with $P_{M'}{}^J$, we now obtain equations of the form (5.2), relating the different $P_{M'}{}^J|\,K,M_S\rangle$. The coefficients $a_{K\mu,K'\nu}$ are simply the primed coefficients (6.9) multiplied by the appropriate matrix element of $J_\pm$:

$$P_{M'}{}^J J_\pm|\,K,M_S\rangle =[\tfrac{1}{2}(J\mp K+M_S)(J\pm K+M_S+1)]^{\frac{1}{2}} \times P_{M'}{}^J|\,K,M_S\rangle. \quad (6.10)$$

Then diagonalizing $((a_{K\mu,K'\nu}))$ yields the first-order contributions of $\sum \mathbf{l}\cdot\mathbf{s}$ to the energy.

For the special case of wave functions totally symmetric or totally antisymmetric under space exchange, (6.6) will always have the simple form

$$\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|\,(\lambda\mu)\rangle=(1/n)\mathbf{L}\cdot\mathbf{S}\,|\,(\lambda\mu)\rangle \quad (6.11)$$

as can be shown, e.g., for symmetric functions, $|s\rangle$, $P^x(ij)\,|s\rangle=|s\rangle$, and $|s'\rangle$, $P^x(ij)\,|s'\rangle=|s'\rangle$:

$$\langle s'|\mathbf{l}(i)\cdot\mathbf{s}(i)\,|s\rangle=\langle s'|P^x(ij)\mathbf{l}(i)\cdot\mathbf{s}(i)P^x(ij)\,|s\rangle =\langle s'|\mathbf{l}(j)\cdot\mathbf{s}(i)\,|s\rangle.$$

Then

$$\sum_i^n \langle s'|\mathbf{l}(i)\cdot\mathbf{s}(i)\,|s\rangle=\frac{1}{n}\sum_{i,j}^n \langle s'|\mathbf{l}(i)\cdot\mathbf{s}(i)\,|s\rangle =-\frac{1}{n}\sum_{i,j}^n \langle s'|\mathbf{l}(j)\cdot\mathbf{s}(i)\,|s\rangle=\frac{1}{n}\langle s'|\mathbf{L}\cdot\mathbf{S}\,|s\rangle.$$

[10] $(-\sqrt{\tfrac{1}{2}})F_{+4}$ has matrix elements in (ϵ_0,K) similar to those of the angular momentum operators $J_\pm$ on $|J,M_J\rangle=|\,\tfrac{1}{2}\mu,\tfrac{1}{2}K\rangle$. See reference 2.

Space symmetric functions occur for the highest $(\lambda\mu)$ for 2, 3, and 4 particles: (2,0), (3,0), (4,0), respectively. However, for (2,0), $T=L$, and (4,0), we must have $S=0$, $\langle \mathbf{L}\cdot\mathbf{S}\rangle=0$, and there is no first-order spin-orbit contribution.

For five particles, (6.6) takes the form

$$\begin{aligned}\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|1,\tfrac{1}{2}\rangle &= [\tfrac{1}{2}-\tfrac{1}{2}L_+S_-]|1,\tfrac{1}{2}\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|1,-\tfrac{1}{2}\rangle &= [-\tfrac{1}{2}-(1/15)L_-S_+ +(4/15)L_+S_+F_{-4}]|1,-\tfrac{1}{2}\rangle.\end{aligned} \tag{6.12}$$

For six particles, $T=0$, (6.6) becomes

$$\begin{aligned}\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|2,1\rangle &= [1-\tfrac{1}{2}L_+S_-]|2,1\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|2,0\rangle &= [-\tfrac{1}{2}L_+S_- -\tfrac{1}{10}L_-S_+ +\tfrac{1}{5}L_+S_+F_{-4}]|2,0\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|2,-1\rangle &= [-1-\tfrac{1}{10}L_-S_+ +\tfrac{1}{5}L_+S_+F_{-4}]|2,-1\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|0,1\rangle &= [-\tfrac{3}{10}L_+S_- +\tfrac{1}{5}L_-S_-F_4]|0,1\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}|0,0\rangle &= [-\tfrac{3}{10}(L_+S_- +L_-S_+)+\tfrac{1}{5}(L_-S_-F_4+L_+S_+F_{-4})]|0,0\rangle.\end{aligned} \tag{6.13}$$

We have obtained these equations by using the matrix elements as calculated in the Appendix.

The calculations for $12-n$ particles may be replaced by calculations for n holes, using the functions of Table II as hole functions. However, we must remember that a single-particle interaction like $\sum \mathbf{l}\cdot\mathbf{s}$ operating on a given n-hole function is equivalent to $-\sum \mathbf{l}\cdot\mathbf{s}$ acting on the equivalent n-particle function, up to an additive constant.[11] Therefore, the calculations for n particles can be used for $12-n$ particles, with an overall change of sign.

It should be pointed out that this method of calculating matrix elements within an invariant subspace $(\lambda\mu)$ depends only on the properties of the group operators, and therefore is quite general; it can be applied to any oscillator shell.

VII. HIGHER ORDER AND CENTRAL TERMS

The second-order perturbation terms are of the form

$$\sum_{(\lambda'\mu')\neq(\lambda,\mu)} \frac{\langle(\lambda\mu)\cdots|P^J\textstyle\sum \mathbf{l}\cdot\mathbf{s}|(\lambda'\mu')\rangle\langle(\lambda'\mu')|P^J\sum \mathbf{l}\cdot\mathbf{s}|(\lambda\mu)\cdots\rangle}{E_{\lambda\mu}-E_{\lambda'\mu'}}, \tag{7.1}$$

where $E_{\lambda\mu}$ are the eigenvalues of the diagonal part of the central force, and other quantum numbers are understood. Here we are forced to consider matrix elements of $\sum \mathbf{l}\cdot\mathbf{s}$ connecting different $(\lambda\mu)$. However, since $\sum \mathbf{l}\cdot\mathbf{s}$ operating on the intrinsic ground state changes ϵ by 0 or 3, it can only connect $(\lambda\mu)$ to those $(\lambda'\mu')$ for which $\epsilon=\epsilon_0$, or ϵ_0-3. Since we have taken ϵ_0 as the highest possible for the given number of particles and T, the available $(\lambda'\mu')$ are a limited set. From Table I we see that for $n=2, 3, 4$ there is only one $(\lambda'\mu')$ for each, reached by $\sum \mathbf{l}\cdot\mathbf{s}$. In these cases, the sum (7.1) has a single nonzero term, and can be rewritten

$$\frac{\langle(\lambda\mu)|P^J(\sum \mathbf{l}\cdot\mathbf{s})^2|(\lambda\mu)\rangle-[\langle(\lambda\mu)|P^J\sum \mathbf{l}\cdot\mathbf{s}|(\lambda\mu)\rangle]^2}{\Delta E}, \tag{7.2}$$

using the closure relation for the intermediate states.

Now we do not actually have to calculate the off-diagonal terms, but only the part of $(\sum \mathbf{l}\cdot\mathbf{s})^2$ within the space $(\lambda\mu)$. We denote this by $\{(\sum \mathbf{l}\cdot\mathbf{s})^2\}$. The second term in (7.2) is the square of the first-order term we have already calculated.

In general, there may be several $(\lambda'\mu')$ connected to the highest symmetry by $\sum \mathbf{l}\cdot\mathbf{s}$. Then we can approximate the second-order term by a term of the form (7.2), where we use an average excitation energy ΔE. Since the coefficient of $\sum \mathbf{l}\cdot\mathbf{s}$ is not known for each nucleus, but is adjusted to fit the data, this approximation should not be too critical, assuming $\sum \mathbf{l}\cdot\mathbf{s}$ can be treated as a perturbation at all. The relative weighting of the separate terms in the numerator of (7.1) is changed somewhat by this procedure, but since all these terms are positive, and the denominators of the same sign, no sensitive cancellation of terms is altered.

To calculate $(\sum \mathbf{l}\cdot\mathbf{s})^2$ we find what states within the representation $(\lambda\mu)$ are reached by its operation on $|(\lambda\mu)\epsilon_0 K,M_S\rangle$. Expanding,

$$\begin{aligned}(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2=\sum_{i,j}&[l_+(i)s_-(i)+l_-(i)s_+(i)+l_0(i)s_0(i)]\\ &\times[l_+(j)s_-(j)+l_-(j)s_+(j)+l_0(j)s_0(j)].\end{aligned}$$

We find we can group the states by the change of K (or M_S):

(a) $l_+(i)l_+(j)s_-(i)s_-(j)$ and $l_-(i)l_-(j)s_+(i)s_+(j)$ reach states with $\Delta K=-\Delta M_S=\pm 2$, $\Delta\epsilon=0$ or -6. These states are generated from $|(\lambda\mu)\epsilon_0 K,M_S\rangle$ by $F_4S_-S_-$ and $F_{-4}S_+S_+$ for $\Delta\epsilon=0$, and by $F_{-1}F_{-1}SS$, $F_{-5}F_{-1}F_4S_-S_-$,

[11] D. M. Brink and G. R. Satcheler, Nuovo cimento 4, 549 (1956).

$F_{-5}F_{-5}S_+S_+$, $F_{-1}F_{-5}F_{-4}S_+S_+$, $F_{-1}F_{-1}F_{-4}F_{-4}S_+S_+$, and $F_{-5}F_{-5}F_4F_4S_-S_-$ for $\Delta\epsilon=-6$. For most cases in the $1p$ shell, many or all of these terms will be zero.

(b) $l(i)l_0(j)s_-(i)s_0(j)$ and $l_-(i)l_0(j)s(i)s_0(j)$ reach states with $\Delta K=-\Delta M_S=\pm1$, $\Delta\epsilon=-3$. These are the same states reached by $\{\sum \mathbf{l}\cdot\mathbf{s}\}$, and are generated by $F_{-1}S_-$, $F_{-5}F_4S_-$, $F_{-5}S_+$, $F_{-1}F_{-4}S_+$.

(c) $l_+(i)l_-(j)s_-(i)s_+(j)$, $l_-(i)l_+(j)s_+(i)s_-(i)$, $l_0(i)l_0(j)\times s_0(i)s_0(j)$ reach states with $\Delta\epsilon=0$ or -6, and no change of K, M_S. $\Delta\epsilon=0$ is generated by the unit operator, and $\Delta\epsilon=-6$ by $F_{-5}F_{-1}$, $F_{-5}F_{-5}F_4$, and $F_{-1}F_{-1}F_{-4}$. To separate the $\Delta\epsilon=0$ and -6 parts of the first two operators, we write

$$l_+(i)l_-(j)=[-F_{-1}(i)-F_5(i)][-F_1(j)-F_5(j)]\rightarrow F_{-5}(i)F_{-1}(j)+\tfrac{1}{2}[Q_0(i)+L_0(i)]\delta_{ij}$$

and

$$l_-(i)l_+(j)\rightarrow F_{-1}(i)F_{-5}(j)+\tfrac{1}{2}(Q_0(i)-L_0(i))\delta_{ij}$$

operating on ϵ_0.

We find this classification also allows us to calculate the diagonal matrix elements of the central-force term $\frac{1}{2}\sum L^2(ij)S^2(ij)$, which we previously ignored. We write

$$L^2(ij)S^2(ij)=[l^2(i)+l^2(j)+2\mathbf{l}(i)\cdot\mathbf{l}(j)]S^2(ij). \quad (7.3)$$

In the $1p$ shell, $l^2(i)=2$, so that

$$[l^2(i)+l^2(j)]S^2(ij)=4S^2(ij)=4S^2+\text{const} \quad (7.4)$$

for a given number of particles.

The remaining term in (7.3) operating on $|(\lambda\mu)\epsilon_0K, M_S\rangle$, reaches the same states as part (c) of $\{(\sum \mathbf{l}\cdot\mathbf{s})^2\}$, treated above, and can be handled in the same way.

As in the calculation of the first order terms, we write the operation of $\{(\sum \mathbf{l}\cdot\mathbf{s})^2\}$ as a sum of operators from (a), (b), and (c), above:

$$\{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}|K,M_S\rangle=\sum_a C_aO_a|K,M_S\rangle. \quad (7.5)$$

This equation is multiplied on the left by the orthogonal set $\langle K,M_S|O_a{}^*$. The resulting matrix elements can be calculated, and the equations solved for the C_a. The operators O_a must be broken up into operators $F_{\pm4}$, $S_\pm$ which change K, M_S, and functions of operator $\mathbf{J}$. Then projection P_{M^J} yields equations of the form, for $\{(\sum \mathbf{l}\cdot\mathbf{s})^2\}$,

$$\{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}P_{M^J}|K,M_S\rangle=\sum_{K'\nu} u_{K'\nu,K\mu}P_{M^J}|K',\nu\rangle. \quad (7.6)$$

For a spin-orbit interaction $\alpha\sum \mathbf{l}\cdot\mathbf{s}$, with average excitation energy ΔE, then diagonalizing the matrix

$$\alpha a_{K'\nu,K\mu}+(\alpha^2/\Delta E)u_{K'\nu,K\mu} \quad (7.7)$$

yields the combined first and second contributions to the energy.

We list the second-order spin-orbit terms for the cases $S=0$ only, for which there is no first-order term:

$$\begin{aligned}
&n=2,\quad T=1,\quad \{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}|(20)\rangle=[2-\tfrac{1}{4}L^2]|(20)\rangle\\
&n=4,\qquad\quad \{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}|(40)\rangle=[(8/3)-\tfrac{1}{12}L^2]|(40)\rangle\\
&n=6,\quad T=1,\quad \{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}|(60)62,0\rangle=[\tfrac{3}{2}-(3/20)(L^2-10)+\tfrac{1}{10}(L_+L_+F_{-4}-2)]|(60)62,0\rangle\\
&\qquad\qquad\quad \{(\textstyle\sum \mathbf{l}\cdot\mathbf{s})^2\}|(60)60,0\rangle=[\tfrac{5}{2}-\tfrac{1}{5}(L^2-6)+(1/20)(L_+L_+F_{-4}+L_-L_-F_4]|(60)60,0\rangle.
\end{aligned} \quad (7.8)$$

It turns out that for $n=2$, 3, 4, or 5,

$$\tfrac{1}{2}\{\sum_{i\neq j} L^2(ij)S^2(ij)\}|(\lambda\mu)\rangle=[L^2+\text{const}\rangle]|(\lambda\mu)\rangle \quad (7.9)$$

so that this term can be treated along with L^2.

For $n=6$, up to an additive constant,

$$\begin{aligned}
T=1:\ &\tfrac{1}{2}\{\textstyle\sum L^2S^2\}|2,0\rangle &&=[\tfrac{3}{5}(L^2-10)+\tfrac{3}{5}(L_+L_+F_{-4}-2)]|2,0\rangle\\
&\tfrac{1}{2}\{\textstyle\sum L^2S^2\}|0,0\rangle &&=[\tfrac{3}{10}(L^2-6)+\tfrac{3}{10}(L_+L_+F_{-4}+L_-L_-F_4-4]|2,0\rangle\\
T=0:\ &\tfrac{1}{2}\{\textstyle\sum L^2S^2\}|2,M_S\rangle &&=[8+(7/5)(L^2-10)-\tfrac{3}{5}(L_+L_+F_{-4}-2)]|2,M_S\rangle\\
&\tfrac{1}{2}\{\textstyle\sum L^2S^2\}|0,M_S\rangle &&=[(17/10)(L^2-6)-\tfrac{3}{10}(L_+L_+F_{-4}+L_-L_-F_4-4)]|0,M_S\rangle.
\end{aligned} \quad (7.10)$$

VIII. COMPARISON WITH EXPERIMENT

We can now calculate the energy spectra of the $1p$ shell nuclei, using the zero-order wave functions $P_{M^J}|(\lambda\mu)\epsilon_0K,M_S\rangle$ in which C, S^2, and T^2 are diagonal, and L^2, $\frac{1}{2}\sum L^2S^2$, and $\sum \mathbf{l}\cdot\mathbf{s}$ are perturbations. By comparison with the experimental energy levels[12] we shall try to find a set of exchange parameters for the central interaction (4.19). Since the nuclei of the p shell are not of radically different size, we would expect one choice of these parameters to suffice for the entire shell. The strength parameter of the spin-orbit potential will be allowed to vary with the number of particles, to provide the best fit to the spectra.

Intermediate-coupling calculations have already been done in the $1p$ shell,[4] in which complete matrices for the

[12] F. Azjenberg-Selove and T. Lauritsen, Nuclear Phys. **11**, 1 (1959).

interaction in the shell have been constructed, and diagonalized by digital computer. The perturbation method used in this paper is an approximation to the exact diagonalization of the matrix when we choose the Kurath exchange mixture (4.20a). For several cases of interest, we have compared the approximate and exact predictions of the spectra, as a function of the spin-orbit strength parameter, a (Figs. 4, 6, and 8), to see how good the approximation is.

The approximation method has the advantage that only a few matrix elements are required, and these can be obtained quite simply, without use of fractional parentage. The matrices are small, and can be diagonalized by hand. Also, the central force parameters can be changed without requiring recalculation of the matrices, so that various exchange mixtures can be tried. Only the coefficients of L^2 and $\frac{1}{2}\sum L^2S^2$ affect the energy in first order.

The approximation is somewhat better than first-order perturbation in LS coupling as used in Inglis' paper,[13] since the matrix elements of $\sum \mathbf{l}\cdot\mathbf{s}$ between different L in the same $(\lambda\mu)$ are taken into account. Thus we obtain first-order splittings for six particles, $T=0$, while in LS coupling, first order gives zero.[13] The ability to calculate second-order contributions also improves the approximation. We have not exhibited second-order results except when the first order gives zero.

The number of $1p$ particles for a given A is $n=A-4$.

$n=2, 3, 4$

The highest $(\lambda\mu)$ states of two, three, and four particles are space symmetric, so that the first-order spin-orbit energies are given by (6.11):

$$\langle a\sum \mathbf{l}\cdot\mathbf{s}\rangle=\frac{a}{n}\langle \mathbf{L}\cdot\mathbf{S}\rangle$$

$$=\frac{a}{2n}[J(J+1)-L(L+1)-S(S+1)]. \quad (8.1)$$

We find that the spectra for these nuclei can be fit fairly well with $a\simeq-1.5$ Mev; in particular, this gives the correct splitting of the $J=2$, $J=3$ levels for $n=2$, $T=0$. Since these two states are unique for two $1p$ particles, the first-order splitting is "exact."

If we use the range and strength assumed above for the central force, we can try to find the best exchange mixture (4.20) by comparing the calculated with the experimental spectra (Figs. 1, 2, and 3). For $n=2$, $T=0$, the "center of mass" of the $L=2$ triplet ($J=1, 2, 3$) is predicted by all three mixtures to be 3 Mev above $L=0$, $J=1$, since $0.3L^2+0.1L^2S^2=0.5L^2=3$ for $S=1$. The experimental result is 3.2 Mev (Fig. 1).

For $n=2$, $T=1$, we have $S=0$ and the Kurath and Meshkov mixtures both give a splitting of 1.8 Mev for

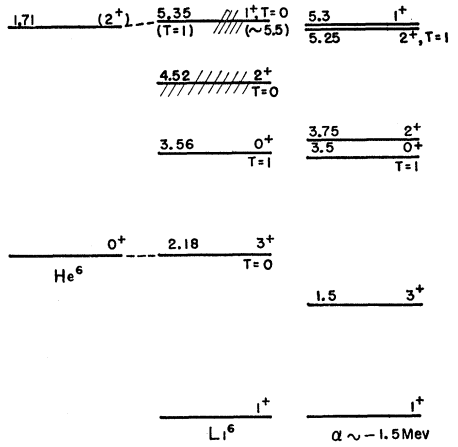

FIG. 1. Comparison of calculated and experimental spectra for $A=6$, using the Kurath or Meshkov exchange mixtures, $a=-1.5$ Mev.

$J=0$, 2, while the Serber mixtures predicts 3 Mev. The data give 1.71 Mev. Also, the Kurath and Meshkov mixtures predict the splitting of the $J=0$, $T=1$ above the $J=1$, $T=0$ level to be 3.5 Mev, compared to 3.56 experimentally. The Serber force gives no splitting. This seems to eliminate the Serber choice.

All three mixtures predict the $(\lambda\mu)=(0,1)$ states to be above 8 Mev, where no spins have been identified. For this excitation, and $a\simeq-1.5$ Mev, the second-order perturbation effects are small.

For $n=3$, (8.1) gives energy splittings $E(J=\frac{1}{2})-E(J=\frac{3}{2})=-\frac{1}{2}a$, and $E(J=\frac{5}{2})-E(J=\frac{7}{2})=-(7/6)a$, so that the latter splitting is 7/3 the former. The experimental spectrum (Fig. 2) seems to show a much

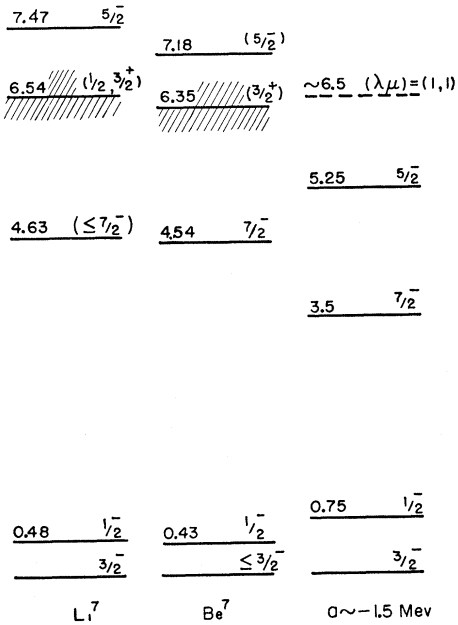

FIG. 2. Comparison of calculated and experimental spectra for $A=7$, using the Meshkov exchange mixture, $a\simeq-1.5$ Mev.

[13] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

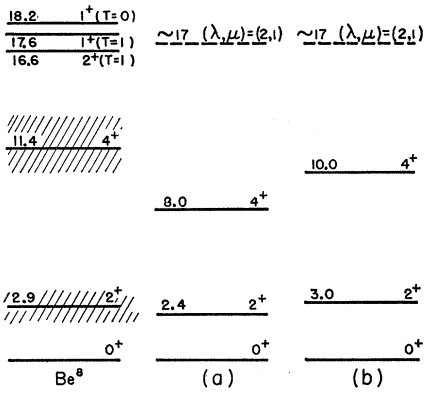

FIG. 3. Comparison of calculated and experimental spectra for $A=8$, using (a) the Kurath (b) the Serber mixture, $a\simeq-1.5$ Mev.

larger ratio of splittings, but there is evidence that the $\frac{5}{2}^-$ level seen at 7.47 Mev in Li^7 comes from a lower $(\lambda\mu)$ symmetry (1.1) than the ground state. There is also evidence of the $\frac{5}{2}^-$, (3,0) state in the region of the 6.54-Mev level.[14] This would make the splittings somewhat closer to the prediction, with $a\simeq-1.5$ Mev.

The Meshkov and Kurath mixtures put the unperturbed splitting of $L=1, 3$ at 4 Mev; the Serber mixture predicts 5 Mev, which is closer to experiment. However, second order tends to raise the $\frac{7}{2}$ energy relative to $\frac{3}{2}$. The Kurath and Serber forces put the lowest unperturbed (1,1) $S=\frac{3}{2}$ level above 10 Mev, while the Meshkov force puts it at about 5.5 Mev, quite close to the $\frac{5}{2}^-$ (30) level, as the experimental evidence suggests (Fig. 2).

For $n=4$, $S=0$ and (8.1) is zero. The second-order term is $(a^2/\Delta E)[(8/3)-\frac{1}{12}L^2]$ from (7.8). All mixtures give the zero-order excitation of (2,1) levels above (4,0), $L=0$, greater than 10 Mev. Thus the coefficient L^2 in the second-order term is less than 0.1 for $|a|\leq 3$, or 0.2 for $|a|\leq 5$. The central force has a coefficient of L^2 of 0.4 for Kurath and Meshkov, and of 0.5 for Serber mixtures. Thus to fit the $J=2, 4$ levels of Be^8 for the range and strength of the force we have chosen requires a large spin-orbit force with the first two mixtures, or a smaller $|a|$ with the Serber mixture. From the neighboring $n=2, 3$ nuclei, we might expect $|a|$ to be $\simeq 1$ to 2 Mev. Both the Kurath and Serber mixtures put the (2,1), $T=1$ levels at about 17 Mev, while the Meshkov mixture puts them at about 10 Mev, for which there is no experimental evidence (Fig. 3).

Thus we find that the low-lying energy levels of the nuclei with $A=6, 7, 8$ can be reproduced with fair accuracy using a two-body central force and a single-particle spin-orbit term which can be treated as a perturbation. However, we get conflicting evidence for the proper exchange mixtures: The Kurath choice conflicts with the close $\frac{5}{2}^-$ levels in $A=7$, the Meshkov choice conflicts with $n=4$, $T=1$ data, and the Serber choice does not agree with the $n=2$ or 3. The fits in Figs. 1, 2, and 3 could be improved somewhat by increasing the coefficient of L^2 in the central force [(4.20a) and (4.20c)], which could be accomplished by a change of strength, range, or even the radial shape of $V(r_1-r_2)$ [(4.15) to (4.19)].

$n=5$

The first-order terms for five particles in the projected intrinsic space are obtained from (6.12):

$$\begin{aligned}\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}P_M{}^J|1,\tfrac{1}{2}\rangle&=\tfrac{2}{3}P_M{}^J|1,\tfrac{1}{2}\rangle\\&\quad-\tfrac{1}{6}[(J-\tfrac{1}{2})(J+\tfrac{3}{2})]^{\frac{1}{2}}P_M{}^J|1,-\tfrac{1}{2}\rangle,\\\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}P_M{}^J|1,-\tfrac{1}{2}\rangle&\\&=[-(7/15)-(2/15)(-)^{J+\frac{1}{2}}(J+\tfrac{1}{2})]P_M{}^J\\&\quad-(1/30)[(J-\tfrac{1}{2})(J+\tfrac{3}{2})]^{\frac{1}{2}}P_M{}^J|1,\tfrac{1}{2}\rangle,\end{aligned}\tag{8.2}$$

where we have used (6.8) and (5.5):

$$\begin{aligned}P_M{}^J|-1,\tfrac{1}{2}\rangle&=(-)^{J+\frac{1}{2}}P_M{}^J|1,-\tfrac{1}{2}\rangle\\&=[-(-)^{J+\frac{1}{2}}/2]P_M{}^JS_+F_{-4}|1,\tfrac{1}{2}\rangle.\end{aligned}\tag{8.3}$$

The total perturbation is $[cL^2+a\sum \mathbf{l}\cdot\mathbf{s}]$, where c is the coefficient of L^2 in the central force, including the effect of $\sum L^2S^2$ in first order. The perturbation takes the matrix form in the space $P_M{}^J|1,\frac{1}{2}\rangle$, $P_M{}^J|1,-\frac{1}{2}\rangle$, using (8.2) and (5.4),

$$\begin{array}{c|cc}&1,\tfrac{1}{2}&1,-\tfrac{1}{2}\\\hline 1,\tfrac{1}{2}&(-\tfrac{3}{2}c+\tfrac{2}{3}a)&(-c-(1/30)a)[(J-\tfrac{1}{2})(J+\tfrac{3}{2})]^{\frac{1}{2}}\\1,-\tfrac{1}{2}&(-c-\tfrac{1}{6}a)[(J-\tfrac{1}{2})(J+\tfrac{3}{2})]^{\frac{1}{2}}&\{\tfrac{1}{2}c-a[(7/15)+(2/15)(-)^{J+\frac{1}{2}}(J+\tfrac{1}{2})]\},\end{array}\tag{8.4}$$

where we have omitted the additive constant $c(J^2+S^2)$.

The solutions of the perturbation matrix are exhibited in Fig. 4 (solid curves) as a function of a/c, with the lowest root of $J=\frac{3}{2}$ set to zero. For $J=\frac{1}{2}$, $P_M{}^{\frac{1}{2}}|1,\frac{1}{2}\rangle=0$, and the matrix becomes one-dimensional. For $J=9/2$ there is a spurious solution, since there is only one state of that spin in (3,1), that coming from $L=4$. The spurious solution can be found by diagonalizing L^2 in the degenerate space. The roots are 20 and 30, the former corresponding to $L(L+1)$ for $L=4$; the second root is spurious, and its eigenfunction must have zero norm.

It is interesting to compare the perturbation results with the results obtained by Kurath, in intermediate coupling, shown as dashed curves in Fig. 4. We have used $c=0.4$ Mev to correspond to Kurath's choice of parameters. The perturbation curves follow Kurath's fairly well for the range shown, -6 Mev$<a<0$, with

[14] C. Levinson and M. K. Banerjee, Ann. Phys. **2**, 489 (1957).

the exception of the second $\frac{5}{2}$ level. It is surprising that the higher order perturbation terms do not make themselves more strongly felt by $a\simeq-4$ Mev, since then $a^2/\Delta E\simeq-1$ for the Kurath force.

The experimental information on Be^9 is still rather incomplete, but a tentative fit of the spectrum can be attempted for $a\simeq-3.0$ Mev (Fig. 5). We assume the 3.04 level is $\frac{1}{2}^-$ and the 1.75 level is $\frac{1}{2}^+$, which does not appear in the p-shell configuration. The broad level at 4.74 could be a combination of $\frac{7}{2}$, $\frac{5}{2}$, and $\frac{3}{2}$ as shown.

The predictions for the Meshkov mixture are the same in first order. For the Serber mixture, $c=0.5$ Mev and the entire energy scale is changed by 5/4, which would improve the fit to the 2.43 and 3.04 levels. No information on $T=\frac{3}{2}$ levels is available.

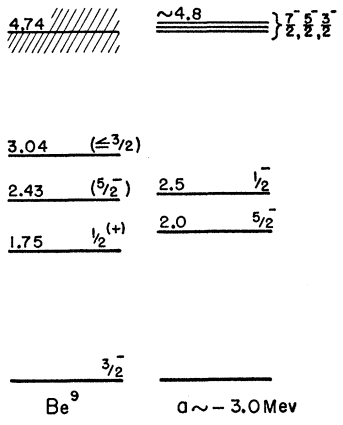

FIG. 5. Comparison of calculated and experimental spectra for $A=9$, using the Kurath or Meshkov mixtures, $a\simeq-3.0$ Mev.

$n=7$

The first-order perturbation results for $n=7$ can be obtained immediately from those for $n=5$, simply by using $|1,\frac{1}{2}\rangle$ and $|1,-\frac{1}{2}\rangle$ as functions for 5 holes. This simply changes the sign of $\sum \mathbf{l}\cdot\mathbf{s}$ in the perturbation, but not the two-particle term L^2. Thus we need the solutions of (8.4) for positive (a/c), which are shown in Fig. 6. Again we have compared the perturbation curves (solid) with $c=0.4$ Mev to Kurath's (dashed) which were given only for $a>3$ Mev. We notice that the Kurath curves rise more steeply than the first order curves, indicating the contribution of higher orders. Still the agreement is not too poor except for $J=\frac{7}{2}$, even at $a=-6$ Mev.

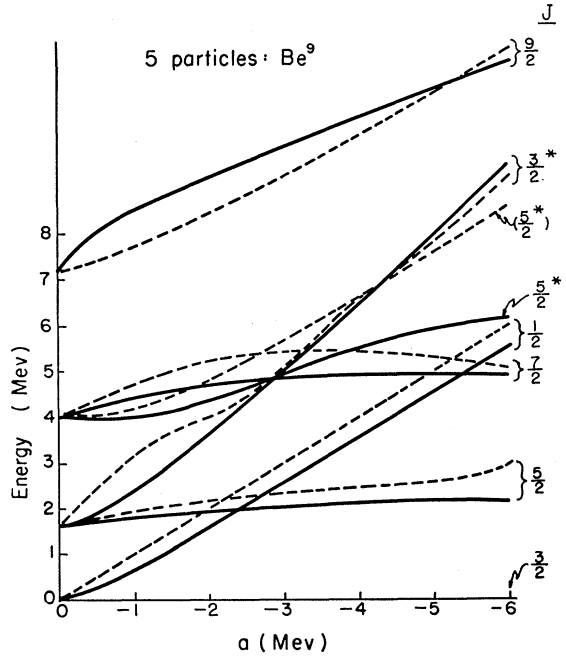

FIG. 4. Energy spacing as a function of a, relative to $J=\frac{3}{2}$, for $n=5$. Solid line: perturbation calculation. Broken line: Kurath calculation, with $K=-1$ Mev.

The predicted spectra for B^{11} are given for $a=-4$ and -5 Mev in Fig. 7 for the Kurath or Meshkov mixtures. Again the Serber force simply changes the energy scale by 5/4.

$n=6$

For six particles, $T=0$, the terms in the total perturbation

$$cL^2+a\sum \mathbf{l}\cdot\mathbf{s}+\tfrac{1}{2}d\sum L^2S^2 \qquad (8.5)$$

can be put in the first-order from (5.2), using the results (5.4), (6.13), and (7.10). The last term, in the absence of the spin-orbit force, provides a small contribution to the energies, and in particular, separates the two $L=2$ states in $(\lambda\mu)=(2,2)$, which are mixed in each intrinsic state: $K=0, 2$. However, the terms which mix K in (7.10) are small compared to the equivalent K-mixing terms in (6.13), for a reasonable strength of the spin-orbit force. We can then approximate (7.10) by dropping the $F_{\pm4}$ terms, and using an average of the coefficients of L^2 in the $K=0, 2$ expressions:

$$\sum L^2S^2\simeq 4\delta_{K,2}+1.5L^2+\text{const.} \qquad (8.6)$$

Then for the Kurath or Meshkov exchange mixtures, $c=0.3$ Mev, $d=0.1$ Mev, (8.5) becomes

$$0.45L^2+0.4\delta_{K,2}+a\sum \mathbf{l}\cdot\mathbf{s}. \qquad (8.7)$$

Using (5.4) and (6.13) we can put this into the form of a matrix in the projected intrinsic space, using the five intrinsic states K, M_S from Table II. The matrix can be diagonalized for each J, and each given value of a. There will be spurious solutions for $J=3, 4, 5$, which can be eliminated, as for five particles, by diagonalizing L^2 alone. The resulting first-order energies for several low states are shown in Fig. 8, as a function of a (solid lines). The dashed lines show the Kurath solutions for those energies, $|a|>3$.

For $T=1$ there is no first-order contribution from $\sum \mathbf{l}\cdot\mathbf{s}$, and $L=J$. We use the second-order expressions (7.8) and the $\sum L^2S^2$ terms (7.10), and diagonalize the matrix in the space $P_{M^J}|2,0\rangle$, $P_{M^J}|0,0\rangle$, of the operator

$$0.1\sum L^2S^2+(a^2/\Delta E)(\sum \mathbf{l}\cdot\mathbf{s})^2. \qquad (8.8)$$

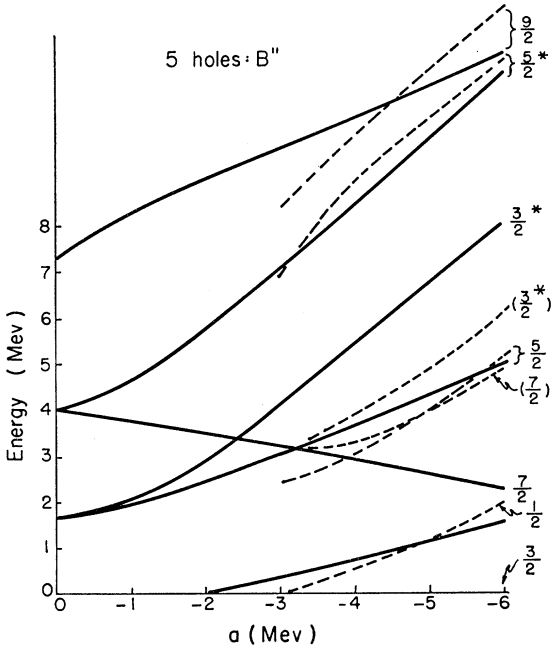

FIG. 6. Energy spacing as a function of a, relative to $J=\frac{3}{2}$ for $n=7$. Solid line: perturbation calculation. Broken line: Kurath calculation, $K=-1$ Mev.

The $J=0$ solution is shown in Fig. 8; it clearly does not follow the Kurath predictions, and drops below the experimental ground state of B^{10} ($J=3$) for $(-a)>3$. This probably means that the second-order perturbation calculation is not consistent for these values of a. The other $T=1$ levels will also be too low, and have not been included in Fig. 8.

An attempt to fit the spectra of Be^{10}, B^{10} is shown in Fig. 9. We have tried to fit the $T=0$ and $T=1$ spectra separately. The fact that a smaller value of $|a|$ is indicated for $T=1$ seem to point up the difficulty in the second-order calculation, which probably overpredicts the energy shifts compared to first order. We might expect second order in $T=0$ to make a large difference, but the Kurath curves agree well with first-order curves for $J=1$ and 3, and less well for $J=2$, so that higher orders than second would probably have to be included for consistency.

$n=8, 9, 10$

These cases can be treated as four, three, or two holes in the $1p$ shell. However, the spectra observed are very different from those for two, three, and four particles, even including the effective change of sign of $\sum \mathbf{l}\cdot\mathbf{s}$. The observed energy splittings require, in first and second order, values of $|a|$ so large that the perturbation series is not reliable.

In such cases, it seems to be necessary to include at least the second highest $(\lambda\mu)$ symmetry in a more exact way. This is not difficult to do: New intrinsic functions are defined for the new $(\lambda\mu)$. For example, for $n=2$, $T=0$, we add the intrinsic function for (0,1) symmetry, $|(0,1)11,0\rangle$, to the two functions for (2,0) from Table II, $|(2,0)40,1\rangle$ and $|(2,0)40,0\rangle$. For $n=4$, we acquire threy new intrinsic functions: $|(2,1)51,M_S\rangle$, $M_S=1, 0, -1$.

The matrices in the enlarged space are obtained be generalizing (6.6) to include the new functions:

$$\begin{aligned}\{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|(\lambda\mu)\epsilon_0 K,M_S\rangle&=\textstyle\sum_a O_a|(\lambda\mu)\epsilon_0 K,M_S\rangle\\&\quad+\textstyle\sum_b O_b|(\lambda'\mu')\epsilon_0' K',M_S'\rangle,\\ \{\textstyle\sum \mathbf{l}\cdot\mathbf{s}\}\,|(\lambda'\mu')\epsilon_0' K',M_S'\rangle&=\textstyle\sum_c O_c|(\lambda'\mu')\epsilon_0' K',M_S'\rangle\\&\quad+\textstyle\sum_d O_d|(\lambda\mu)\epsilon_0 K,M_S\rangle.\end{aligned}\tag{8.9}$$

The operators O_a and O_c are the same five appearing in (6.6).

Generally, (see Table II) $\epsilon_0'=\epsilon_0-3$. Since $\sum \mathbf{l}\cdot\mathbf{s}$ operating on ϵ_0 can at most reach ϵ_0-3, O_b must be unity or some combination of $F_{\pm 4}$, $S_\pm$, which will not change ϵ_0'. The operators O_d must be the same as those considered in Sec. VII, for the second-order spin-orbit terms. We can then project J by P_{M^J} in (8.9) to obtain matrices for $\sum \mathbf{l}\cdot\mathbf{s}$, as before.

For $n=2$, this second approximation is exact, since including (0,1) spans the whole space for p^2. The spacing of the three lowest levels ($J=1, 0, 1$) of N^{14} can be reproduced for $a\simeq -4$ or -5 Mev.

For $n=4$, even the inclusion of (2.1) does not repro-

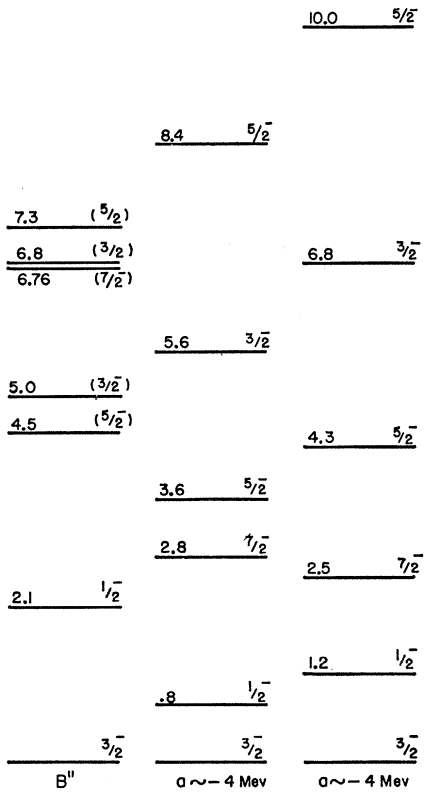

FIG. 7. Comparison of calculated and experimental spectra for $A=11$, using the Kurath or Meshkov mixtures, for $a\simeq -4.0$ and -5.0 Mev.

duce the 4.43-Mev excitation of the $J=2$ level above ground. For $n=3$, the problems is complicated by the presence of two possible spins, $S=0$ and 2, for (1,1), and was not pursued, because of the scarcity of experimental data with which to compare for C^{13} and N^{13}.

IX. CONCLUSIONS

The perturbation calculations based on the intrinsic representations seem to provide a fair picture of the lower spectra of many of the $1p$ shell nuclei. This probably results from the dominance of C, the Casimir operator of the group SU_3, in the central two-particle potential in the $1p$ shell, which causes a large separation of states of different $(\lambda\mu)$. The perturbation matrix elements were easily obtained in this scheme, because of the convenient properties of the generators of SU_3, operating on states of maximum ϵ_0.

The methods presented for calculating matrix elements can be used in other shells of the harmonic oscillator. Whether perturbation calculations will produce a reasonable picture of the energy spectra depends in part on the importance of the Casimir term in the two-particle interaction in that shell. In general, the central force problem is much more complicated. An investigation of these calculations for the $2s-1d$ shell has been carried out by Banerjee and Levinson.[15]

Kurath and Pičman[16] have developed an approach to the $1p$ shell using Nilsson's distorted wave functions to generate the intermediate-coupling wave functions. In the limit of no spin-orbit force, the Nilsson functions become the functions of our intrinsic representation.

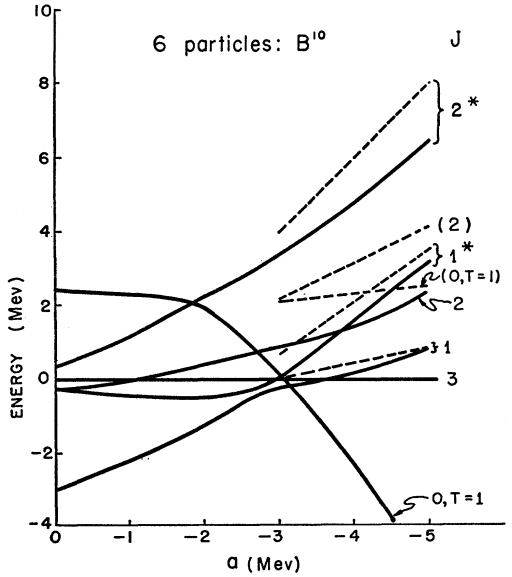

FIG. 8. Energy spacing as a function of a, relative to $J=3$, for $n=6$. Solid line: perturbation calculation. Broken line: Kurath calculation, $K=-1$ Mev.

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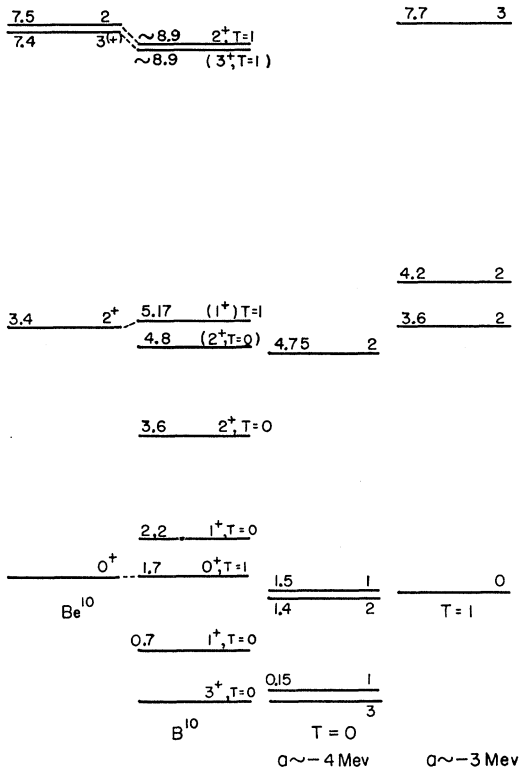

FIG. 9. Comparison of calculated and experimental spectra for $A=10$, using Kurath mixture, $a\simeq-4.0$ Mev for $T=0$, $a\simeq-3.0$ Mev for $T=1$.

The connection of the Kurath-Pičman treatment with our perturbation method is discussed in the dissertation from which this paper is extracted,[17] and will be presented in a future paper.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the continual help and interest of Professor Carl Levinson and Professor Manoj Banerjee. I would also like to thank Professor Sydney Meshkov and Dr. Ben Bayman for many helpful suggestions.

APPENDIX

Equations (6.6) become a set of simultaneous equations in A, B, C, D, and E when both sides are multiplied on the left by $\langle(\lambda\mu)\epsilon_0K,M_S|O_a{}^*$, where O_a is each of the five operators on the right side of (6.6). The homogeneous terms, which are obtained from the right side of (6.6), are calculated by using (2.4) and the fact that F_5 or F_1 acting on (ϵ_0K) yields zero. We find

$$\langle(\lambda\mu)\epsilon_0K,M_S|1\cdot1|(\lambda\mu)\epsilon_0K,M_S\rangle=1, \quad \text{(A1)}$$

$$\langle S_+F_1F_{-1}S_-\rangle=\tfrac{1}{2}\langle S_+S_-(H_2-H_1)\rangle$$
$$=\tfrac{1}{4}(S+M_S)(S-M_S+1)(\epsilon_0-K), \quad \text{(A2)}$$

[17] See asterisk reference.

where we have used $\langle S,M_S|S_+S_-|S,M_S\rangle=\frac{1}{2}(S+M_S)\times(S-M_S+1)$. Also

$$\begin{aligned}\langle S_+F_{-4}F_5F_{-5}F_4S_-\rangle&=\tfrac{1}{2}\langle S_+S_-\rangle\langle F_{-4}(H_2+H_1)F_4\rangle\\&=\tfrac{1}{2}\langle S_+S_-\rangle\langle(H_2+H_1+2)F_{-4}F_4\rangle\\&=\tfrac{1}{4}(S+M_S)(S-M_S+1)(\epsilon_0+K+2)\\&\quad\times(\tfrac{1}{2}\mu-\tfrac{1}{2}K)(\tfrac{1}{2}\mu+\tfrac{1}{2}K+1).\end{aligned}\tag{A3}$$

In the last line we have used the fact that $-(\sqrt{\frac{1}{2}})F_{\pm4}$ acts like $J_\pm$ in a space with "angular momentum" $J=\frac{1}{2}\mu$, and $M_J=\frac{1}{2}K$.[10]

Similarly,

$$\langle S_-F_5F_{-5}S_+\rangle=\tfrac{1}{4}(S-M_S)(S+M_S+1)(\epsilon_0+K),\tag{A4}$$

$$\begin{aligned}\langle S_-F_4F_1F_{-1}F_{-4}S_+\rangle&=\tfrac{1}{4}(S-M_S)(S+M_S-1)(\epsilon_0\epsilon K+2)\\&\quad\times(\tfrac{1}{2}\mu+\tfrac{1}{2}K)(\tfrac{1}{2}\mu-\tfrac{1}{2}K+1).\end{aligned}\tag{A5}$$

There are two off-diagonal terms:

$$\begin{aligned}\langle S_+F_1F_{-5}F_4S_-\rangle&=\langle S_+S_-\rangle\langle F_{-4}F_{+4}\rangle\\&=\tfrac{1}{2}(S+M_S)(S-M_S+1)\\&\quad\times(\tfrac{1}{2}\mu-\tfrac{1}{2}K)(\tfrac{1}{2}\mu+\tfrac{1}{2}K+1),\end{aligned}\tag{A6}$$

$$\langle S_-F_5F_{-1}F_{-4}S_+\rangle=\tfrac{1}{2}(S-M_S)(S+M_S+1)\times(\tfrac{1}{2}\mu+\tfrac{1}{2}K)(\tfrac{1}{2}\mu-\tfrac{1}{2}K+1).\tag{A7}$$

The inhomogeneous terms, which come from the left side of (6.6), must be calculated separately for each case, although the commutors may be used to reduce the operators:

$$\begin{aligned}\langle S_+F_1\textstyle\sum_i F_{-1}(i)S_-(i)\rangle&=\tfrac{1}{2}\langle S_+\textstyle\sum_i[H_2(i)-H_1(i)]S_-(i)\rangle,\\\langle S_-F_5\textstyle\sum_i F_{-5}(i)S_+(i)\rangle&=\tfrac{1}{2}\langle S_-\textstyle\sum_i[H_2(i)+H_1(i)]S_+(i)\rangle,\\\langle S_+F_{-4}F_5\textstyle\sum_i F_{-1}(i)S_-(i)\rangle&=\langle S_+F_{-4}\textstyle\sum_i F_4(i)S_-(i)\rangle,\\\langle S_-F_4F_1\textstyle\sum_i F_{-5}(i)S_+(i)\rangle&=\langle S_-F_4\textstyle\sum_i F_{-4}(i)S_+(i)\rangle.\end{aligned}\tag{A8}$$

The right sides of these equations are calculated by using the determinant forms (Table II) and (6.7).

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Quantum Mechanical Calculation of Mössbauer Transmission*†

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A quantum mechanical calculation of the time-dependent Mössbauer transmission has been performed neglecting solid-state effects. The source considered consists of nuclei which decay via a two-photon cascade, the second of which is emitted without recoil and is subject to resonant absorption by a foil whose resonance may be shifted due to a small relative velocity between source and absorber. The transmission is obtained when the transmitted recoiless photon is measured in coincidence with the first photon of the cascade. The result is in agreement with that obtained by considering the absorber as a classical dielectric slab capable of absorption and dispersion. The initial condition has been investigated in detail by considering the full cascade. In this manner, one sees that the usual simple assumption that the nucleus is in the first excited state immediately after the emission of the first photon, gives the correct boundary condition.

INTRODUCTION

THE most common Mössbauer experiment is performed by measuring the transmission of recoiless radiation through a thin resonant absorber which may be in motion relative to the source. In this manner, the hyperfine structure of the isotope employed may be investigated.[1] An interesting variation of this simple experiment has been performed by several groups.[2–5] They make use of the most popular Mössbauer isotope, Fe^{57}. The source contains Co^{57} which decays by electron capture to Fe^{57m} which decays in turn by a 122-kev photon followed by a

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