Excitations of Two-Phonon Surface Vibrations in Nuclei*

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The features of inelastic scattering leading to two-phonon nuclear surface vibrations have been examined in Born approximation. Comparison is made with recent experimental results of inelastic alpha-particle scattering experiments on spherical nuclei. Good agreement is found between parameters derived from these experiments and Coulomb excitation. This is not surprising since the comparisons have been carried out in a relatively model-independent way.

I. INTRODUCTION

THE characteristic behavior of direct reactions has in recent years made them an extremely useful technique for determining the spin and parity as well as detailed information on the wave functions of a large number of nuclear states. This is particularly so in the case of the deuteron-induced reactions[1]; and more recently Blair,[2] generalizing on the work of Drozdov[3] and Inopin,[4] has shown that inelastic alpha-particle scattering can give detailed information on states which are collective in nature. Prompted by the success of the Blair analysis of inelastic alpha scattering and the appearance of the octupole vibration state in nuclei in the Fe-Ni region, the group at Saclay have extended the studies of inelastic alpha scattering with better energy resolution[5] than had previously been used.[6–8] Some of their recent results are shown in Figs. 1 and 2, the group marked B being of particular interest because of their apparent contradiction of the so-called "phase rule." This rule relates to the position of the maxima and minima in the angular distribution of α's scattered from a spin-zero nucleus. According to it these maxima (minima) for the inelastic scattering to even-J states with no parity change occur at the minima (maxima) for the inelastic scattering to odd-J states with change of parity. Furthermore, *elastic* scattering behaves, for that matter, like inelastic scattering to *odd-J* states. A somewhat different phase rule has been put forth by Austern very recently but no detailed comparison with experiment has as yet been made.[9] Blair[2] and others have noted that the phase rule does not depend critically upon the model used for the interaction but is obtained in either a diffraction analysis or a first-order Born[10] approximation of inelastic scattering. In either case the "phase rule" comes about because of the appearance of periodic or near periodic functions in the expression for the differential cross section, with a phase determined by the angular momentum L transferred in the excitation.

In the case of the two levels marked by B in Figs. 1 and 2, the point of particular interest is that both of these levels are known to have spin and parity 4^+ [11] and should accordingly yield angular distributions which are out of phase with the elastic scattering angular distribution. Instead they are clearly in phase, without appreciable filling in of the minima or a distortion of the angular distribution, with the 3^- state as well as the elastic scattering except at small angles. It should also be noted, particularly for the Fe^{56} case, that the slope of the envelope of the maxima is considerably less negative than that of the other odd-parity states, or the elastic angular distribution. Furthermore, the differential cross section for the scattering to this state is, on the average, considerably below that of the other states observed in the Saclay experiments. This latter point has led the Saclay group to propose that these states are being excited by a two-phonon process. The clear but anomalous behavior of these angular distributions, in particular, has motivated our study of inelastic alpha-particle scattering.

For simplicity we prefer to study the scattering process by a simple Born approximation method rather than use the diffraction analysis of Blair[12] and Drozdov.[13] As usual we assume that the interaction

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[1] M. H. Macfarlane and J. B. French, Revs. Modern Phys. **32**, 567 (1960).

[2] J. S. Blair, Phys. Rev. **115**, 928 (1959).

[3] S. I. Drozdov, J. Exptl. Theoret. Phys. (USSR) **28**, 734 (1955), **28**, 736 (1955) [translation, Soviet Phys.—JETP **1**, 588, 591 (1955)].

[4] E. V. Inopin, J. Exptl. Theoret. Phys. (USSR) **31**, 901 (1959) [translation, Soviet Phys.—JETP **4**, 764 (1957)].

[5] R. Beurtey *et al.* (to be published); M. Crut, J. Thirion, *et al.* (to be published). Similar preliminary results at 43 Mev on Ni^{60} have recently become available from J. Yntema and B. Zeidman at Argonne National Laboratories, and W. Tippie and N. S. Wall, Massachusetts Institute of Technology, at 30 Mev on Ni^{58}.

[6] D. K. McDaniels, J. S. Blair, S. W. Chen, and G. W. Farwell, Nuclear Phys. **17**, 614 (1960).

[7] M. Crut, D. R. Sweetman, and N. S. Wall, Nuclear Phys. **17**, 655 (1960).

[8] J. Yntema, B. Zeidman and B. Raz, Phys. Rev. **117**, 801 (1960).

[9] L. Seidlitz, E. Bleuler, and D. J. Tendam, Phys. Rev. **110**, 682 (1958) show angular distributions which suggest the phase rule put forth by N. Austern and E. Rost (to be published).

[10] S. Hayakawa and S. Yoshida, Progr. Theoret. Phys. **14**, 1 (1955).

[11] *Nuclear Data Cards*, (National Research Council, Washington, D. C.).

[12] J. S. Blair (private communication to J. Thirion) has discussed second order effects in a diffraction analysis.

[13] S. I. Drozdov, J. Exptl. Theoret. Phys. (USSR) **38**, 499 (1960), [translation, Soviet Phys.—JETP **11**, 362 (1960)].

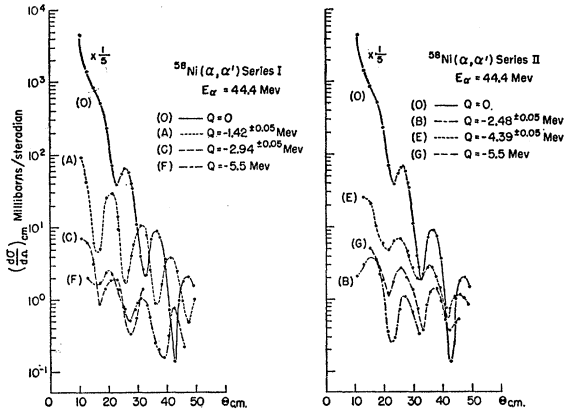

FIG. 1. The differential cross section for 44.4-Mev alpha particles scattered from the various states in Ni^{58} designated in the drawing.

between the scattered alpha particle and the nucleus is confined to within an interaction radius R. The inelastic scattering involves an excitation of the nuclear degrees of freedom also. Assuming that the states which are preferentially excited are collective states, we describe this excitation in terms of a distortion of the nuclear surface and write

$$\begin{aligned} V&=V(\mathbf{r},\alpha), \quad r<R=R_0[1+\sum_{l,m}\alpha_{lm}Y_{lm}] \\ &=0, \qquad\qquad r>R \end{aligned} \tag{1}$$

for the interaction potential. Here the α_{lm} are collective coordinates for the nuclear surface[14] and play the role of destruction and creation operators for "phonons" of angular momentum l; Y_{lm} is a spherical harmonic of order l, and R_0 some average interaction radius. An expansion of (1) in powers of α is appropriate since nuclear distortions are usually small. One finds, to second order,

$$\begin{aligned} V(\mathbf{r},\alpha)=V(r)+v_1(r)\sum_{l,m}\alpha_{lm}Y_{lm} \\ +v_2(r)\sum_{lml'm'}\alpha_{lm}\alpha_{l'm'}Y_{lm}Y_{l'm'}, \\ v_1(r)=r\frac{\partial V}{\partial r}, \quad v_2(r)=\tfrac{1}{2}r^2\frac{\partial^2 V}{\partial r^2}. \end{aligned} \tag{2}$$

The first term in Eq. (2) produces the elastic scattering; for a square well of depth V_0 the angular distribution is well known:

$$\frac{d\sigma^{(\mathrm{el})}}{d\Omega}=\left(\frac{V_0}{E_0}\right)^2 R_0^2\left[\frac{j_1(KR_0)}{KR_0}\right]^2, \quad E_0=\frac{\hbar^2}{2M_0R_0^2}. \tag{3}$$

$\mathbf{k}_0$, $\mathbf{k}_f$ are the initial and final momenta in the c.m. system, $K=|\mathbf{k}_0-\mathbf{k}_f|$ is the momentum transfer, M_0 the reduced mass, and $j_1(kR_0)$ denotes the usual spherical Bessel function of order 1. The second term in (2) linear in α_{lm} is responsible for inelastic scattering involving the excitation of a one-phonon state in the nucleus. The differential cross section has been computed by Hayakawa and Yoshida,[10] and may be written

$$\frac{d\sigma^{(\mathrm{1ph})}}{d\Omega}=\frac{2L+1}{4\pi}\left(\frac{\hbar\omega_L}{2C_L}\right)\left(\frac{V_0}{E_0}\right)^2 R_0^2\frac{k_f}{k_0}j_L^2(qR_0), \quad q=|\mathbf{k}_0-\mathbf{k}_f|, \tag{4}$$

for a square-well shape for $V(r)$ in (2); L is the angular momentum of the excited state, $\hbar\omega_L$ the excitation energy and C_L a parameter relating to the nuclear surface tension.[14] The momentum transfer q in Eq. (4) is very nearly equal to the K in Eq. (3), differing by not more than a few percent for alpha particles on medium-weight nuclei if the excitation energy is small compared to the incident alpha-particle energy.

In the next section we outline the calculation of the differential cross section in a first-order Born approximation for the last term in the expansion of the potential, Eq. (2). This term, being quadratic in α, can of course give rise to two-phonon states. However, the linear term in the potential can also give rise to a two-phonon state through a second-order correction to the first-order scattering. We have also carried out this second-order calculation and present it as well, in the next section. Details of both calculations will be found in the Appendix. The two second-order terms may further be described by saying that second-order scattering term involves an excitation which goes through an intermediate state and must be summed over all intermediate states, whereas the first-order direct two-phonon transition arises from the term in the potential quadratic in the deformation and does not involve any intermediate states in first-order. In Sec. III we compare the results of this analysis with experiment and discuss some of the approximations in the analysis.

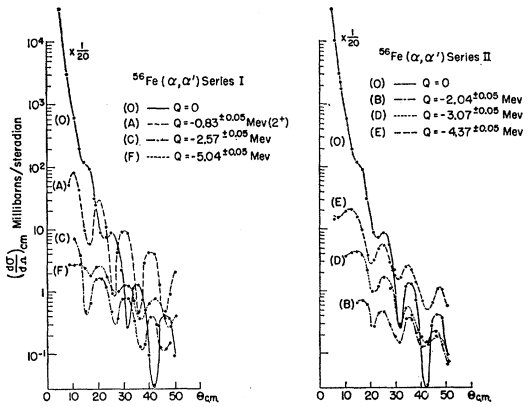

FIG. 2. The differential cross section for 44.4-Mev alpha particles scattered from the various states in Fe^{56} designated in the drawing.

[14] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14, (1952).

II. TWO-PHONON EXCITATIONS

It is clear that the higher-order terms in an expansion like (2) will lead to interactions that allow for more than one phonon to be excited in the nucleus. We consider the qualitative features of such processes in this section. Thus the third term in (2) will excite a two-phonon nuclear state. Assuming that the nuclear transition is from a zero-spin initial state $|i\rangle=|L=0\rangle$ to a final state of angular momentum L, $|f\rangle=|L\rangle$, only the $j_L(qr)$ term is picked up from the wave function of the α particle in Born approximation by the two-phonon interaction. Calling the transition operator T, one gets for the transition amplitude (see Appendix 1)

$$\langle L|T|0\rangle=(-i)^L\sum_{l,l'}\sqrt{2}e_le_{l'}\times[(2L+1)(2l+1)(2l'+1)]^{\frac{1}{2}}\times\begin{pmatrix}L & l & l'\\ 0 & 0 & 0\end{pmatrix}\int dr\, r^2v_2(r)j_L(qr), \quad (5)$$

$$e_l=\left(\frac{\hbar\omega_l}{2C_l}\right)^{\frac{1}{2}}.$$

In deriving (5) we have taken the quantization axis of L along the momentum transfer direction q in which case only the projection $M=0$ contributes. The constants $\hbar\omega_l$ and C_l still refer to a single-phonon state of spin l, and we have used the Wigner[15] round bracket notation for the vector addition coefficient.

If $V(r)$ is again taken as a square well, then $v_2(r)$ in the radial integral becomes proportional to the derivative $\delta'(r-R_0)$ of a delta function, so that

$$\begin{aligned}&\int dr\, r^2v_2(r)j_L(qr)\\ &=\tfrac{1}{2}V_0\frac{\partial}{\partial R_0}[R_0^4j_L(qR_0)]\\ &=\tfrac{1}{2}V_0R_0^3[(qR_0)j_{L-1}(qR_0)-(L-3)j_L(qR_0)],\quad L>0\\ &=-\tfrac{1}{2}V_0R_0^3[(qR_0)j_1(qR_0)-4j_0(qR_0)],\quad L=0. \quad (6)\end{aligned}$$

The differential cross section for the two-phonon state excited by a particular multipole λ in the interaction then becomes, putting $\lambda=l=l'$ in (5),

$$\frac{d\sigma^{(2\mathrm{ph})}}{d\Omega}=\frac{1}{2}\left(\frac{1}{4\pi}\right)^2(2L+1)(2\lambda+1)^2\times\begin{pmatrix}\lambda & \lambda & L\\ 0 & 0 & 0\end{pmatrix}^2\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)^2\left(\frac{V_0}{E_0}\right)^2R_0^2\frac{k_f}{k_0}\times\begin{cases}[(qR_0)j_{L-1}(qR_0)-(L-3)j_L(qR_0)]^2 & L>0\\ [(qR_0)j_1(qR_0)-4j_0(qR_0)]^2 & L=0,\end{cases} \quad (7)$$

and L can only be even, $L\leq 2\lambda$. Since for the scattering of intermediate-energy alpha particles, qR_0 is usually a fairly large number, the first term in the square bracket is much enhanced over the second, so that the characteristic oscillations in the cross section (7) with angle are determined almost completely by $(qR_0)^2j_{L-1}^2(qR_0)$. Thus we get oscillations *out* of phase with the single-phonon angular distribution which involves the same final angular momentum L and always in phase with the elastic cross section. Also the additional factor $(1/4\pi)(\hbar\omega_\lambda/2C_\lambda)$ in (7) will reduce this cross section by more than an order of magnitude. Hence these two-phonon excitations may be identified by their in-phase oscillations with the elastic cross section (3) and small cross section. The slower drop-off in magnitude of the envelope of (7) with increasing angle compared to the single-phonon cross section is another important difference. Inserting the asymptotic form for the first Bessel function in Eq. (7), one finds a $\cos^2qR_0$ behavior for the differential cross section which has a constant envelope. The reason for the slower drop-off for the two-phonon cross section in this model is the better localization of the interaction $v_2(r)$ at the surface.

Distortions of order $\lambda=2$ appear to be the most important excitations of the nuclear surface.[16] Then the first group of two-phonon states is a triplet[16] with $L=0$, 2, 4 and we find the corresponding cross sections from (7):

$$\frac{d\sigma^{(2\mathrm{ph})}}{d\Omega}=\left(\frac{1}{4\pi}\right)^2\left(\frac{\hbar\omega_2}{2C_2}\right)^2\left(\frac{V_0}{E_0}\right)^2R_0^2\frac{k_f}{k_0}F, \quad (8)$$

where

$$F=(5/2)[(qR_0)j_1(qR_0)-4j_0(qR_0)]^2,\quad \mathrm{L}=0, \quad (8a)$$

$$F=(25/7)[qR_0j_1(qR_0)+j_2(qR_0)]^2,\quad \mathrm{L}=2, \quad (8b)$$

$$F=(45/7)[qR_0j_3(qR_0)-j_4(qR_0)]^2,\quad \mathrm{L}=4, \quad (8c)$$

all of which have exactly the *same* angular distribution, namely $\cos^2qR_0$ when $qR_0\gg 1$. In this limit the relative intensities are 1:1.43:2.57 for $L=0$, 2 and 4.

A two-phonon final state such as we have just discussed can also be reached via a second-order process, going through intermediate one-phonon states. We compute the cross section for this process in second-order Born approximation, going through one multipole, λ, in $v_1\sum_{lm}\alpha_{lm}Y_{lm}$ only. The transition amplitude can be obtained in closed form for this case if one assumes in addition that v_1 has a delta-function space dependence, i.e., $v_1=V_0R_0\delta(r-R_0)$. Then for a transition to a final state $|f\rangle=|L\rangle$ one has the approximate result

$$\langle L|T|0\rangle=-\tfrac{1}{2}[2(2L+1)]^{\frac{1}{2}}(-i)^{L+1}(2\lambda+1)\times\begin{pmatrix}L & \lambda & \lambda\\ 0 & 0 & 0\end{pmatrix}\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)\left(\frac{V_0}{E_0}\right)\left(\frac{V_0R_0^3}{k_0R_0}\right)\times[j_L(qR_0)-(-1)^\lambda j_L(pR_0)e^{2ik_0R_0}],\quad k_0R_0\gg 1, \quad (9a)$$

[15] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[16] K. Alder *et al.*, Revs. Modern Phys. **28**, 432 (1960).

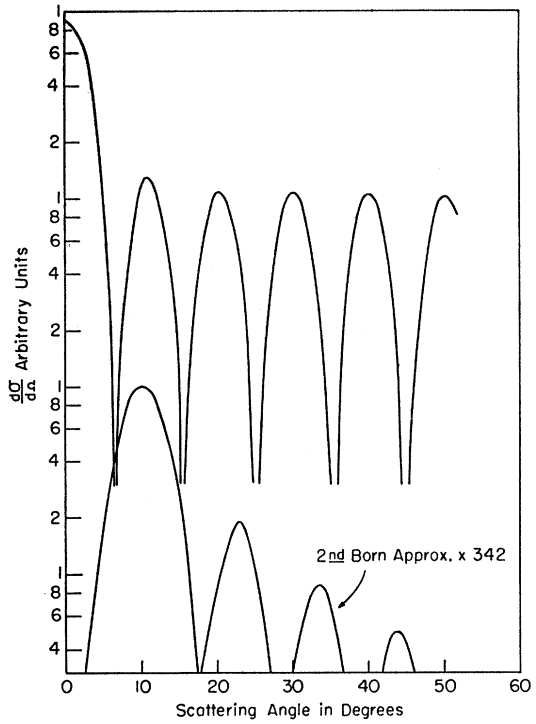

FIG. 3. A comparison of the second-order Born approximation calculation with the direct two-phonon excitation for a state with angular momentum 0. The normalization of the relative cross sections is described in the text.

if $L>0$, and

$$\langle 0|T|0\rangle=\tfrac{1}{2}[2(2\lambda+1)]^{\frac{1}{2}}i^{\lambda+1}\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)\left(\frac{V_0}{E_0}\right)\left(\frac{V_0R_0^3}{k_0R_0}\right)$$
$$\times[j_\lambda(qR_0)-j_\lambda(pR_0)e^{2ik_0R_0}]\quad k_0R_0\gg1, \quad (9b)$$

when $L=0$. The momentum transfer q has been defined previously, $p=|\mathbf{k}_0+\mathbf{k}_f|$, and k_0R_0 is assumed to be large in both cases. The derivations of (9a) and (9b) are given in Appendix 2.

To illustrate the differences between the direct two-phonon and second-order single-phonon processes, let us compare Eqs. (9) with the amplitudes for the direct excitation of two phonons, also by a single multipole λ in the interaction. The expression for this amplitude follows immediately from (5) and (6):

$$\langle L|T|0\rangle=\tfrac{1}{2}[2(2L+1)]^{\frac{1}{2}}(-i)^L(2\lambda+1)$$
$$\times\begin{pmatrix}L & \lambda & \lambda\\ 0 & 0 & 0\end{pmatrix}\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)V_0\frac{\partial}{\partial R_0}[R_0^4j_L(qR_0)]. \quad (5')$$

Now since pR_0 is a fairly large number for the range of angles of interest here, the $j(pR_0)$ in (9a) and (9b) will be damped out strongly and the angular distribution for the second-order process will behave like $j_L^2(qR_0)$, or $j_\lambda^2(qR_0)$, respectively. These angular distributions are thus always *in* phase with the single-phonon transitions to the same final state, and therefore, *out* of phase with the oscillations in the two-phonon amplitude (5′). There is also a difference in magnitude of a factor like $(V_0/E_0)^2(k_0R_0)^{-4}$ in the cross sections given by (9) and (5′). Apart from this factor the scattering cross section involving the intermediate single-phonon state is seen to drop off in magnitude with increasing angle as rapidly as the direct single-phonon cross section given by Eq. (4).

The angular distributions determined by (9b) and (5′) with $L=0$ and $\lambda=2$ are compared in Fig. 3, taking $V_0/E_0=1$. The values of this ratio in an actual case turn out to be quite large and are indicated later. However, even then the $(k_0R_0)^{-4}$ factor keeps the contribution of the second-order Born approximation small generally.

III. DISCUSSION AND COMPARISON WITH EXPERIMENT

It is well-known that the positions of the maxima and minima in the angular distributions for inelastic α-particle scattering are very well reproduced by spherical Bessel functions of argument qR_0 as in Eq. (4). The period of oscillation in the angular distribution then serves to determine the appropriate interaction radius R_0. That the expression (8c) can follow the oscillations of the angular distribution to the 4^+ level as shown in Fig. 4 is therefore not too surprising. However, we will attempt to analyze the Fe^{56} and Ni^{58} data also, with respect to the magnitudes and ratios of the various cross sections, in addition to finding the appropriate values of the interaction radii for these two cases. From Eq. (3) one sees that the magnitude of the elastic differential cross section depends only on V_0 and R_0. The cross section to the first excited 2^+ state by Eq. (4) involves the additional parameter $\hbar\omega_2/2C_2$. This quantity also occurs squared in the 4^+ cross section. Figure 4 shows the computed angular distribution from Eq. (8c). The experimental points come from Fig. 2. The absolute magnitude of the calculated curve may be determined by the parameters derived from the first excited 2^+ and elastic angular distributions. The value

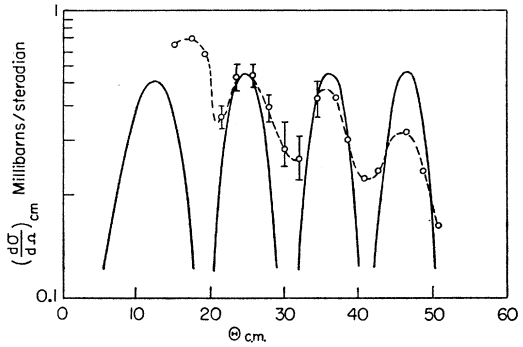

FIG. 4. A comparison of the differential cross section given by Eq. (8c) of the text, with the 2.04-Mev, 4^+, Fe^{56} excited state differential cross-section given by curve *B* of Fig. 2. The normalization is explained in the text.

for V_0 was taken from the 35° and 45° maxima in elastic cross section, and $\hbar\omega_2/2C_2$ comes from the 22° maximum in the 2^+. This value is compared with the $\hbar\omega_2/2C_2$ taken from the 4^+ angular distribution at the 28° maximum, determined from (8c) by normalizing the experimental curve to the 28° maximum. Table I shows the values of R_0, V_0, and $\hbar\omega_2/2C_2$ determined in this way, and compares the last parameter with its Coulomb excitation value.[16] Similar results have recently been obtained at 30 Mev.[5]

The values of the interaction radii are consistent with other values determined from elastic α-particle scattering.[17] The quantity V_0 has significance only as a parameter determining the strength of the interaction and must not be interpreted as the depth of an appropriate optical potential for α-particle scattering. It is rather surprising that the same V_0 seems to be appropriate for the square-well depth giving rise to the elastic scattering and for the strength of the surface interactions producing the inelastic scattering.[18] From Eqs. (4) and (8) one sees that the value of V_0 has no bearing on the ratio of the magnitudes of the scattering cross sections to the 2^+ and 4^+ states. The values for $\hbar\omega_2/2C_2$ quoted in Table I were determined at forward angles because the Born approximation for the cross section is expected to be least in error here, especially for a well shape having a sharp cutoff. Going to larger angles decreases the extracted value of this ratio, as expected.

The approximations that we have used in this analysis are clearly extreme, and one should properly also consider the effects of surface diffuseness and absorption effects in the interaction potential. A rather careful analysis of such effects has been made recently by Rost and Austern[19] and also by Pursey and McCarthy.[20] However, in the Born approximation, certain relations between the single-phonon and double-phonon angular distributions are relatively independent of any detailed assumptions about the interaction. To see this we transform the radial integral in Eq. (5) for the two-phonon amplitude by integrating by parts and then transferring the derivative from one on r to one on q,

TABLE I. Parameter values obtained by comparison with experiment.

Radius taken from elastic cross section	V_0 taken from elastic cross section	$\hbar\omega_2/2C_2\times10^3$ taken from 2^+	4^+	Coul excit.
Fe^{56} 6.84f	3.6 Mev	10	6.3	12
Ni^{58} 7.00f	4.4 Mev	5.8	1.8	9.4

[17] D. D. Kerlee, J. S. Blair and G. W. Farwell, Phys. Rev. **107**, 1343 (1957).

[18] A similar value is found from an analysis of the 30-Mev scattering data of reference 7.

[19] E. Rost and N. Austern, Phys. Rev. **120**, 1375 (1960).

[20] I. E. McCarthy and D. L. Pursey, Phys. Rev. **122**, 578 (1961).

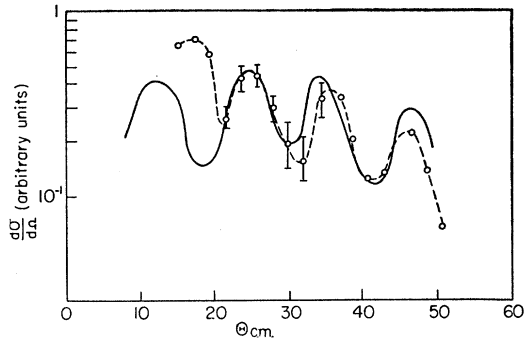

FIG. 5. The differential cross-section of the 3^- state of Fe^{56}, E of Fig. 2, multiplied by $(qR_0)^2$ and compared with the 4^+ state, B, with a normalization at the 25° maximum.

to get

$$\int dr\, r^2 v_2(r) j_L(qr) = -\int dr\, V' \frac{\partial}{\partial r}[r^4 j_L(qr)]$$
$$= -\frac{1}{q^3}\frac{\partial}{\partial q}\left[q^4 \int dr\, r^2 v_1(r) j_L(qr)\right] \quad (10)$$

recalling the definitions of v_1 and v_2. This means that the two-phonon and one-phonon scattering amplitudes are related, apart from constants, by

$$T_{(0\to L)}{}^{(2\mathrm{ph})} = -q^{-3}\frac{\partial}{\partial q}[q^4 T_{(0\to L)}{}^{(1\mathrm{ph})}]$$
$$\approx -q\frac{\partial}{\partial q} T_{(0\to L)}{}^{(1\mathrm{ph})}, \quad (11)$$

quite independent of the exact shape of $V(r)$ in Eq. (2). The last step in (11), which is a good approximation when the momentum transfer is large, clearly shows the out-of-phase relationship that holds between the one- and two-phonon transitions in this limit. Furthermore, since, in the Born approximation at any rate, the q dependence of the scattering amplitude only enters through a $j_L(qr)$, the last step in (11) can also be written approximately as

$$-q\frac{\partial}{\partial q} T_{(0\to L)}{}^{(1\mathrm{ph})} \approx -q\int dr\, r^3 v_1(r) j_{L-1}(qr)$$
$$\approx -qR_0 \int dr\, r^2 v_1(r) j_{L-1}(qr), \quad (12)$$

if we assume that v_1 is peaked around some radius $r=R_0$, and that qR_0 is a large number. Then Eq. (12) relates the two-phonon angular distributions to the one-phonon angular distributions of odd parity by

$$\frac{d\sigma^{(2\mathrm{ph})}}{d\Omega}{}_{(\mathrm{even})} \sim (qR_0)^2 \frac{d\sigma^{(1\mathrm{ph})}}{d\Omega}{}_{(\mathrm{odd})} \quad (13)$$

in a relatively model-independent way. This relation seems to be approximately true for the 4^+ and 3^- states in Fe^{56}, as Fig. 5 shows.

As pointed out in the Introduction, the 4^+ state can naturally also be reached by a single-phonon interaction with $\lambda=4$. A lower limit for the C_4 parameter which will appear in such an interaction can be estimated for Fe^{56} by equating the cross section at the 32° minimum with the cross section given by Eq. (4) for $\lambda=L=4$. The result is $C_4\geq 2\times 10^4$ Mev.

A further approximation made in the nuclear part of the wave function in calculating the two-phonon transition amplitudes has been to work entirely in the harmonic oscillator approximation. This neglects any mixing between different states of the same spin. The inclusion of such mixing removes the high degeneracy present in the harmonic oscillator model for vibrational states.[21] For instance a component of the single-phonon 2^+ state in the second 2^+ state arising from a removal of the degeneracy of the 0, 2, 4 two-phonon states introduces a part of the single-phonon transition amplitude of the opposite phase into the transition amplitude to the second 2^+, which then becomes

$$\langle 2|T|0\rangle\sim\left(\frac{\hbar\omega_2}{2C_2}\right)^{\frac{1}{2}}(qR_0)j_1(qR_0)+\eta j_2(qR_0),$$
$$\eta=\text{mixing amplitude.} \quad (14)$$

Thus the admixed one-phonon amplitude will tend to fill in the minima of the two-phonon amplitude, and pull the scattering cross section out of phase again with the elastic angular distribution especially at small angles. Clearly the extent of such an effect is dictated by the degree of admixture of the one-phonon level; at moderate momentum transfers a rather large value of η would be necessary to compensate for the rapid drop-off of the one-phonon amplitude in comparison with the two-phonon component.

APPENDIX

1. Two-Phonon Cross Section

The inelastic scattering cross section to a two-phonon state of angular momentum L is determined by the matrix element of the last term in Eq. (2) between initial and final nuclear states $|i\rangle=|00,0\rangle$ and $|f\rangle=|LM,2\rangle$ labeled by quantum numbers giving the angular momentum, its z projection, and the number of phonons (0 and 2 in this case) present in the initial and final states.

The two-phonon part of the interaction (2) may be written as

$$v_2(r)\sum_{lm,l'm',l''m''}\alpha_{lm}\alpha_{l'm'}\left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{\frac{1}{2}}\times\begin{pmatrix}l & l' & l''\\ m & m' & m''\end{pmatrix}Y_{l''m''}{}^*\begin{pmatrix}l & l' & l''\\ 0 & 0 & 0\end{pmatrix}, \quad (15)$$

[21] C. S. Shakin and A. Kerman (to be published). The thesis of the first of these authors [Harvard University, Cambridge, Massachusetts, 1960 (unpublished)] lists many references to other authors who have studied this problem.

using the well-known expansion for a product of spherical harmonics.[15] The matrix element of Eq. (15) with respect to the initial and final plane waves $\mathbf{k}_0$ and $\mathbf{k}_f$ only, is

$$T=\sum_{lml'l''}\alpha_{l,m}\alpha_{l',-m}(i)^{l''}(2l''+1)[(2l+1)(2l'+1)]^{\frac{1}{2}}\times\begin{pmatrix}l & l' & l''\\ m & -m & 0\end{pmatrix}\begin{pmatrix}l & l' & l''\\ 0 & 0 & 0\end{pmatrix}\times\int dr\, r^2v_2(r)j_{l''}(qr), \quad (16)$$

taking the momentum transfer direction $\mathbf{q}=\mathbf{k}_i-\mathbf{k}_0$ as z axis; then only the $m''=0$ term in the sum (15) is picked up and so $m'=-m$. This means that the operator T in (16) cannot change the z projection, M, of the angular momentum of the nucleus along the momentum transfer direction. Thus we only require the nuclear matrix element:

$$\langle L0;2|\alpha_{lm}\alpha_{l'-m}|00;0\rangle=e_le_{l'}(-1)^L[2(2L+1)]^{\frac{1}{2}}\times\begin{pmatrix}l & l' & L\\ m & -m & 0\end{pmatrix}, \quad (17)$$

which can be easily obtained from the matrices of α_{lm}.[14] After forming the expression $\langle f|T|i\rangle$ from (16) and inserting (17), the sum over m just expresses the orthogonality condition for the vector addition coefficients, and vanishes unless $l''=L$. Hence,

$$\langle L|T|0\rangle=(-i)^L\sum_{l,l'}\sqrt{2}e_le_{l'}[(2L+1)(2l+1)(2l'+1)]^{\frac{1}{2}}\times\begin{pmatrix}l & l' & L\\ 0 & 0 & 0\end{pmatrix}\int dr\, r^2v_2(r)j_L(qr), \quad (18)$$

and only one Bessel function $j_L(qr)$ appears in the angular distribution, as for the one-phonon excitations. The sums on l and l' are over all values consistent with the conditions that l and l' can be combined vectorially to give L, and that $L+l+l'=$even. The two-phonon cross section given in Eq. (7) in the text follows from (18) with $l=l'=\lambda$ and a square-well shape for $V(r)$, after the appropriate phase space factors are included.

2. Second Born Approximation

The contribution of the single-phonon part of the interaction given in Eq. (2) of the text to the two-phonon cross section can be evaluated in a second Born approximation calculation. We only give the most important details since the computation is rather long. The matrix element we are interested in has the

approximate form

$$\langle f|T|i\rangle=\frac{2M_0}{\hbar^2}\sum_{\mathbf{k}}\sum_{L'M'}\times\frac{\langle\mathbf{k}_f,LM:2|V_1|\mathbf{k},L'M';1\rangle\langle\mathbf{k},L'M';1|V_1|\mathbf{k}_0,00,0\rangle}{k^2-k_0^2}$$

$$V_1=v_1(r)\sum_{\lambda m}\alpha_{\lambda m}Y_{\lambda m}, \tag{19}$$

if one can neglect the excitation energy of the nucleus, as can usually be done for medium bombarding energies. Hence the sum goes over all intermediate one-phonon nuclear states $|L'M';1\rangle$ and alpha-particle momentum states $\hbar\mathbf{k}$. Let us assume that the final state has zero spin also and that only the λth multipole in V_1 is important. Then just $L'=\lambda$ contributes and the sum in (19) is over M' and k only. Putting in plane waves for the alpha particle wave function (19) becomes

$$\langle 0|T|0\rangle=\frac{2M_0}{\hbar^2}\sum_{M'}\langle 00;2|\alpha_{\lambda M'}|\lambda M';1\rangle\times\langle\lambda M';1|\alpha_{\lambda -M'}|00;0\rangle\times\frac{1}{4\pi}\int\int e^{-i\mathbf{k}_f\cdot\mathbf{r}}v_1(r)Y_{\lambda M'}(\mathbf{r})d\mathbf{r}\times\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}v_1(r')Y_{\lambda -M'}(\mathbf{r}')e^{i\mathbf{k}_0\cdot\mathbf{r}'}d\mathbf{r}', \tag{20}$$

since the sum over $\mathbf{k}$ is then just the Green's function $|\mathbf{r}-\mathbf{r}'|^{-1}e^{ik_0|\mathbf{r}-\mathbf{r}'|}$ for a free particle, with an outgoing-wave boundary condition.[22] Now the two radial integrals can be obtained by expanding the Green's function and the initial and final plane-wave states in spherical harmonics. Taking $v_1=V_0R_0\delta(r-R_0)$, the result of this integration is

$$(-1)^{M'}(V_0R_0^3)^2(2\lambda+1)ik_0\sum_{ll'l''}(2l+1)i^l j_l(k_0R_0)\times\begin{pmatrix}l&\lambda&l'\\0&0&0\end{pmatrix}\begin{pmatrix}l&\lambda&l'\\0&M'&-M'\end{pmatrix}\times h_{l'}^{(1)}(k_0R_0)(2l'+1)j_{l'}(k_0R_0)\times\begin{pmatrix}l''&\lambda&l'\\0&0&0\end{pmatrix}\begin{pmatrix}l''&\lambda&l'\\0&M'&-M'\end{pmatrix}\times(2l''+1)(-i)^{l''}j_{l''}(k_fR_0)P_{l''}(\cos\theta), \tag{21}$$

where θ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_f$, i.e., the angle of scattering, and $h_l^{(1)}$ is a spherical Hankel function of the first kind.[22] Inserting the matrix elements

$$\langle\lambda M';1|\alpha_{\lambda -M'}|00;0\rangle=(-1)^{M'}e_\lambda,$$
$$\langle 00;2|\alpha_{\lambda M'}|\lambda M';1\rangle=(-1)^\lambda e_\lambda[2/(2\lambda+1)]^{\frac{1}{2}}, \tag{22}$$

and the expression (21) in (20), the sum over M' again becomes the orthogonality relation for the vector addition coefficients, so that $l=l''$ and one finds;

$$\langle 0|T|0\rangle=ik_0R_0(V_0/E_0)V_0R_0^3(-1)^\lambda e_\lambda^2[2(2\lambda+1)]^{\frac{1}{2}}\times\sum_{l,l'}(2l+1)j_l(k_0R_0)j_l(k_fR_0)P_l(\cos\theta)\times\begin{pmatrix}l&\lambda&l'\\0&0&0\end{pmatrix}^2(2l'+1)h_{l'}^{(1)}(k_0R_0)j_{l'}(k_0R_0). \tag{23}$$

Actually this sum can be evaluated when $k_0R_0\gg 1$ using an easily proven addition theorem for spherical Bessel functions:

$$i^\lambda j_\lambda(qR_0)=\sum_{ll'}j_l(k_0R_0)(2l+1)i^l\begin{pmatrix}l&\lambda&l'\\0&0&0\end{pmatrix}^2(-i)^{l'}\times(2l'+1)j_{l'}(k_fR_0)P_{l'}(\cos\theta), \tag{24}$$

where $q=|\mathbf{k}_0-\mathbf{k}_f|$ and θ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_f$.

The asymptotic expressions for $j_l(k_0R_0)$ and $h_{l'}^{(1)}(k_0R_0)$ when k_0R_0 is large provide the necessary phase factors in (23) for it to be broken up into two sums having the form of Eq. (24) and Eq. (24) with θ replaced by $\pi-\theta$. The final result is

$$\langle 0|T|0\rangle=\tfrac{1}{2}[2(2\lambda+1)]^{\frac{1}{2}}i^{\lambda+1}e_\lambda^2V_0R_0^3(V_0/E_0)(1/k_0R_0)\times[j_\lambda(qR_0)-e^{2ik_0R_0}j_\lambda(pR_0)], \tag{25}$$

with

$$q=|\mathbf{k}_0-\mathbf{k}_f|,\quad p=|\mathbf{k}_0+\mathbf{k}_f|,\quad k_0R_0\gg 1,$$

as given in Eq. (9b) of the text. The expression analogous to (23) for a final state $|f\rangle=|L\rangle$, $L\neq 0$ is

$$\langle L|T|0\rangle=ik_0R_0(V_0/E_0)V_0R_0^3e_\lambda^2[2(2L+1)]^{\frac{1}{2}}(2\lambda+1)\times\sum_{ll'l''}(2l+1)j_l(k_0R_0)j_{l''}(k_fR_0)P_{l''}(\cos\theta)\times i^{(l-l'')}(2l''+1)\begin{pmatrix}L&l&l''\\0&0&0\end{pmatrix}\begin{Bmatrix}L&l&l''\\l'&\lambda&\lambda\end{Bmatrix}\times\begin{pmatrix}l&\lambda&l'\\0&0&0\end{pmatrix}\begin{pmatrix}l'&\lambda&l''\\0&0&0\end{pmatrix}(-1)^{l'}(2l'+1)\times h_{l'}^{(1)}(k_0R_0)j_{l'}(k_0R_0), \tag{26}$$

where as before, λ is the multipole in the interaction causing the transition. The curly bracket denotes a Racah coefficient in Wigner's notation.[15] Again, re-

[22] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1959).

placing $j_{l'}(k_0R_0)$ and $h_{l'}^{(1)}(k_0R_0)$ in (26) by their asymptotic forms for large k_0R_0 one finds, after some algebra, that

$$\langle L|T|0\rangle=-\tfrac{1}{2}[2(2L+1)]^{\frac{1}{2}}(-i)^{L+1}(2\lambda+1)\times\begin{pmatrix}L & \lambda & \lambda\\ 0 & 0 & 0\end{pmatrix}\left(\frac{\hbar\omega_\lambda}{2C_\lambda}\right)\left(\frac{V_0}{E_0}\right)\left(\frac{V_0R_0^3}{k_0R_0}\right)\times[j_L(qR_0)-(-1)^\lambda j_L(pR_0)e^{2ik_0R_0}]. \quad (27)$$

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Approximation Methods in Nuclear Intermediate Coupling Applied to the $1p$ Shell*†

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The intermediate-coupling shell model for nuclei is considered in terms of the classification of states in a harmonic oscillator according to the irreducible representations of the unitary unimodular group in three dimensions, SU_3, introduced by Elliott. The properties of this group are used to produce the approximate spectrum of a quadrupole force, acting within an oscillator shell. When specialized to the $1p$ shell, a more general interaction, including exchange forces, is shown to be approximately diagonal in the chosen representation, and its approximate spectrum is computed. A method is developed for calculating the matrix elements of interactions not diagonal in the representation, in particular the single-particle spin-orbit potential, using the generating functions of the group, SU_3. The intermediate-coupling energy spectra of the nuclei of the $1p$ shell are then calculated to the first or second order in perturbation theory. The results are compared with experimental spectra, and with calculations of Kurath.

I. INTRODUCTION

THIS paper is an investigation of new methods of calculation of energy spectra in the nuclear shell model, with particular application to the $1p$ shell. The energy levels in this model are the eigenvalues of an interaction matrix in the space defined by restricting the particles outside the closed shells (the core) to the lowest available unfilled shell. The interactions considered usually contain a two-particle central potential and a single-particle spin-orbit potential. The competition of the two potentials produces "intermediate coupling" eigenfunctions, which are pure in neither L-S nor j-j coupling.

The calculation of the matrix elements of the interaction is central to the problem. This is usually done by factoring the many-particle basis functions into products of functions of smaller numbers of particles, so that the many-particle matrix elements required can be expressed in terms of those for fewer particles, and ultimately in terms of single- or two-particle matrix elements of the single- or two-particle potentials which are calculated directly. The fractional parentage methods of Racah[1] are useful for this reduction. However, for many particles, or for particles with high angular momentum, this may be a difficult program.

We have investigated a different method of obtaining the matrix elements, based on Elliott's group-theoretic classification of states for a harmonic oscillator shell model.[2] In Sec. II, we review some of Elliott's results, in a slightly different presentation. In Sec. III, we show that Elliott's classification scheme approximately diagonalizes a two-particle quadrupole interaction.

In Sec. IV we specialize to the $1p$ shell, where Elliott's classification scheme is related to the supermultiplet scheme of Wigner.[3] Here the group theory of the Elliott scheme provides a direct way to calculate the spectrum of a central, spin-independent potential. Even for the spin-dependent potentials used in intermediate-coupling calculations, we may use the group theory to obtain approximate spectra, with a small correction term which is not diagonal.

The remaining problem is the calculation of the spin-orbit matrix elements, and those of the nondiagonal

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[1] G. Racah, Phys. Rev. **63**, 367 (1943), and further references therein.

[2] J. P. Elliott, Proc. Roy. Soc. (London) **245A**, 128 and 562 (1958).

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