self-energy parts; i.e., diagrams which are not skeleton-like. This difficulty might be eliminated by reformulating the theory in terms of free-particle propagators and introducing some graph-by-graph renormalization scheme. However, the present formulation in terms of true propagators and skeleton graphs turns out to be particularly convenient for use in situations where there is present a static crystal or impurity field. In any case, the fact that quasi-particles are not stable implies that we cannot hope to find such simple relationships here as those which occur in true particle-field theories.

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Theory of Impurity Resistance in Metals. II

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The many-body theory of impurity resistance which was developed in a previous paper is extended to include all corrections resulting from electron-electron interactions. The model used is a normal Fermi fluid in the presence of a small but finite concentration of randomly scattered, fixed impurities. The resulting expression for the conductivity may be interpreted in terms of independent single-electron-like excitations, or "quasi-particles." The combined effect of the impurities and the many-body interactions causes these quasi-particles to carry current at their group velocity; but there is no effective charge correction.

I. INTRODUCTION

IN a recent paper[1] the author presented a theory of impurity-resistance in normal metals using as a model an interacting Fermi fluid and randomly placed scattering centers. The physical picture which emerged was that of single-electron-like excitations at the Fermi surface scattering from screened impurities. Interactions among these excitations, or "quasi-particles,"[2] were assumed to be negligible. It is the purpose of the present paper to complete the work of I by taking into account all of the many-body effects. The resulting expression for the resistivity will be exact to first order in the density of impurities and to all orders in the electron-electron interactions.

Although electron-electron collisions cannot contribute directly to the resistivity, there are several places where they play an important role in the theory. Perhaps the most important many-body effect is the screening of the impurities. In principle, however, this effect is included exactly when one calculates the amplitude for scattering of a single quasi-particle at an impurity according to the rules prescribed in I. This amplitude, which includes even the exchange interactions between the incident electron and the screening cloud, is merely a basic ingredient of the independent quasi-particle model. In this paper we shall be in search of more subtle effects.

One possible many-body effect has been emphasized recently in an article by Heine and Falicov.[3] These authors point out that it may be incorrect to neglect the interactions among the quasi-particles when calculating their acceleration in an external electric field. For example, the acceleration of a perfectly free electron gas in an external field is independent of the strength of the interactions between the electrons. If one uses an independent quasi-particle model to describe this gas, one destroys the Galilean invariance in a way which is quite inappropriate to this particular situation and arrives at an incorrect value for the acceleration. The introduction of fixed impurities or a lattice removes the Galilean invariance; but it is apparent that a purely kinetic formulation of transport theory in terms of quasi-particles requires careful examination from a fundamental point of view.

An even more subtle many-body effect, which apparently has not been pointed out before now, involves the current carried by a quasi-particle. In the usual Landau picture, a quasi-particle of wave vector $\mathbf{k}$ carries a current $e\mathbf{k}/m$ simply because momentum and charge are conserved in electron-electron collisions. When we make a dc measurement on an impure metal, however, we measure the current over a time much longer than any other time which appears in the problem. In particular, the period of the applied field must

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[1] J. S. Langer, Phys. Rev. **120**, 714 (1960), hereafter referred to as I.

[2] L. D. Landau, Soviet Physics—JETP **3**, 920 (1957).

[3] L. M. Falicov and V. Heine, Phil. Mag. **10**, 57S (1961).

be very much longer than the collision time of the quasi-particles. As a result, the argument based on momentum conservation no longer is valid, and we must be quite careful in calculating the quasi-particle current.

A close analog of this situation occurs in the theory of electron transport based on an independent Bloch-wave model. In this model it is obvious that the electrons are tied down to the lattice and that the correct current is given by the group velocity of the Bloch wave. In the present problem, although any particular electron will bump into an impurity only very infrequently, the Fermi fluid as a whole is "tied" to the fixed impurities via the electron-electron interactions. We shall see that this "tying down" of the Fermi fluid makes a qualitative difference in the nature of the single-particle-like excitations.

In order to solve this problem, we shall continue the frontal attack which was begun in I. That is, we first write down an exact expression for the conductivity σ in the form of an autocorrelation coefficient. This coefficient may be expanded, according to well-known procedures, as a sum of many-body Feynman diagrams. The idea is to rearrange and resum these diagrams in such a way that σ is expressed finally as a simple function of more or less physical quantities, in particular, the group velocity and decay rate for an individual quasi-particle. It is assumed throughout that the conditions for a "normal" metal[4] are satisfied; i.e., that apart from certain renormalization terms, the analytic properties of the exact propagators and vertex functions which appear are the same as those of each term in the relevant perturbation expansions. The mathematical techniques developed in the preceding paper[5] will play an important role in this work.

II. FORMAL EXPRESSION FOR THE CONDUCTIVITY

The starting point for our theory will be Eq. (I2.5) in which[5] the conductivity σ is expressed in terms of the autocorrelation coefficient for the total current.

$$\sigma=\lim_{\alpha\to 0}\left\{\frac{2}{3\alpha\Omega}\,\mathrm{Im}\int_{-\infty}^{0}e^{\alpha t}dt\langle\Psi_0|\,\mathbf{J}(0)\cdot\mathbf{J}(t)\,|\Psi_0\rangle+\frac{e^2n_e}{m\alpha}\right\}. \quad (2.1)$$

This expression was derived in I by considering the system of electrons and impurities in its ground state Ψ_0 and slowly turning on a uniform electric field. In (2.1), $\mathbf{J}$ is the current operator:

$$\mathbf{J}=(e/m)\sum_{\mathbf{k}}\mathbf{k}a_{\mathbf{k}}{}^{\dagger}a_{\mathbf{k}}, \quad (2.2)$$

$a_{\mathbf{k}}{}^{\dagger}$ and $a_{\mathbf{k}}$ being the usual creation and annihilation operators for Fermions of momentum $\mathbf{k}$;

$$\mathbf{J}(t)\equiv e^{iHt}\mathbf{J}e^{-iHt}, \quad (2.3)$$

where H is the complete Hamiltonian for the electrons interacting among themselves and with the impurities. Finally, Ω is the volume of the system, and $n_e=N/\Omega$ is the number density of electrons. In principle, Eq. (2.1) is to be averaged over random configurations of impurities as the last step in the calculation.

Let us define the function

$$\mathfrak{F}(t)\equiv\langle\Psi_0|\,T\{\mathbf{J}(0)\cdot\mathbf{J}(t)\}\,|\Psi_0\rangle, \quad (2.4)$$

where T means we are to write the operators $\mathbf{J}$ in order of increasing time from right to left. Since only negative values of t occur in (2.1), there is no difference between $\mathfrak{F}$ and the autocorrelation coefficient which appears in σ. As defined by (2.4), however, $\mathfrak{F}$ has a simple interpretation in terms of many-body diagrams. In particular, $\mathfrak{F}$ has the form of a vacuum-polarization graph in which the external photon lines are replaced by the operators $\mathbf{J}$ an the scalar product is to be taken. Due to the presence of the impurities, we may expect $\mathfrak{F}(t)$ to diminish as $|t|$ becomes large. The dominant behavior of $\mathfrak{F}$ will be that of a decaying exponential; but we cannot rule out oscillating terms which may persist to longer times. In any case, $\mathfrak{F}$ must be integrable in some generalized sense, and we shall assume that its moments exist also.

In order for Eq. (2.1) to yield a finite value for σ, the terms proportional to α^{-1} must cancel out. That is,

$$\frac{2}{3\Omega}\,\mathrm{Im}\int_{-\infty}^{0}\mathfrak{F}(t)dt=-\frac{e^2n_e}{m}. \quad (2.5)$$

This is a rather remarkable equation which, in fact, gives us our first clue concerning the last of the many-body effects mentioned in the Introduction. If there were no impurities at all in the system, $\mathfrak{F}$ would vanish because Ψ_0 would be an eigenstate of $\mathbf{J}$ with eigenvalue zero. σ then would be given correctly by $e^2n_e/m\alpha$, which simply expresses the acceleration of a perfectly free electron gas in an exponentially increasing external field. Thus Eq. (2.5) is correct only when there is a finite concentration of impurities in the sample. On the other hand, the right-hand side of Eq. (2.5) is independent of the concentration of impurities. In the work which follows we shall find a mathematical explanation for this qualitative change in the nature of $\mathfrak{F}$. A direct proof of Eq. (2.5), using some of the analytical techniques developed in the body of the paper, will be found in Appendix A. It should be mentioned here, however, that Eq. (2.5) turns out to be the f-sum rule for a metal with a finite dc conductivity.

If we assume that Eq. (2.5) is correct, we may return to Eq. (2.1) and write σ in the form

$$\sigma=\frac{2}{3\Omega}\,\mathrm{Im}\int_{-\infty}^{0}tdt\,\mathfrak{F}(t). \quad (2.6)$$

This expression may profitably be rewritten in terms

[4] J. M. Luttinger, Phys. Rev. **121**, 942 (1961). See also Sec. V in I.

[5] J. S. Langer, preceding paper [Phys. Rev. **124**, 997 (1961)], hereafter referred to as II.

of the Fourier transform of $\mathfrak{F}(t)$:

$$F(\nu)=\frac{1}{2\pi}\lim_{\eta\to 0}\int_{-\infty}^{\infty}\mathfrak{F}(t)e^{i\nu t-\eta|t|}dt. \tag{2.7}$$

$F(\nu)$ has the familiar spectral representation:

$$F(\nu)=\frac{1}{2\pi i}\int_0^{\infty}\rho(\nu')d\nu'\left\{\frac{1}{\nu'+\nu-i\eta}+\frac{1}{\nu'-\nu-i\eta}\right\}, \tag{2.8}$$

where

$$\rho(\nu)=\sum_n|\langle\Psi_0|\,\mathbf{J}\,|\Psi_n\rangle|^2\delta(E_n-E_0-\nu), \tag{2.9}$$

Ψ_n being an eigenstate of H with energy E_n. Inversion of the Fourier transform for negative times yields

$$\mathfrak{F}(t)=\int_0^{\infty}\rho(\nu)e^{i\nu t+\eta t}d\nu,\quad (t<0), \tag{2.10}$$

where the convergence factor has been retained for use in the next step. On inserting Eq. (2.10) into Eq. (2.6), exchanging the order of integration, and integrating once by parts, we find

$$\begin{aligned}\sigma&=\frac{2}{3\Omega}\int_0^{\infty}\rho(\nu)d\nu\ \mathrm{Im}\int_{-\infty}^{0}te^{i\nu t+\eta t}dt\\&=-\frac{2\pi}{3\Omega}\int_0^{\infty}\rho(\nu)\frac{d}{d\nu}\delta(\nu)d\nu=\frac{\pi}{3\Omega}\frac{d\rho}{d\nu}\bigg|_{\nu=+0},\end{aligned} \tag{2.11}$$

as long as $\rho(0)=0$.

Now ρ is just the discontinuity across the branch cut of $F(\nu)$, and may be evaluated immediately by means of the analytic techniques developed in II. We see from Eqs. (II 4-3) and (II 4-4) that only the simplest kind of reduced graph (Fig. 8(a), paper II) contributes to $d\rho/d\nu$ as $\nu\to 0$. Essentially what has happened here is that we have expressed ρ in a form which, if the quasi-particles were stable, would be a unitarity sum over intermediate states containing well-defined numbers of quasi-particle-hole pairs. Because a quasi-particle is very nearly stable when its excitation energy is small, the above verbal description becomes more and more accurate as ν goes to zero. In particular, the factor ν^{2n-1} which occurs in Eq. (II 4.4) is the same phase-space factor which would appear in a true particle-field theory. As a result of this factor, the only nonvanishing contributions to σ come from reduced graphs which contain only a single particle-hole pair.

The relevant reduced graph is drawn in Fig. 1. According to the prescription given in II, the associated contribution to σ is

$$\begin{aligned}\sigma=\frac{2\pi}{3\Omega}\sum_{i=1}^{3}\sum_{\substack{\mathbf{k}_1,\mathbf{k}_1'\\ \mathbf{k}_2,\mathbf{k}_2'}}&\Lambda_i(\mathbf{k}_1,\mathbf{k}_1',\mu)\\&\times A(\mathbf{k}_1',\mathbf{k}_2',\mu)B(\mathbf{k}_2,\mathbf{k}_1,\mu)\Lambda_i^*(\mathbf{k}_2',\mathbf{k}_2,\mu).\end{aligned} \tag{2.12}$$

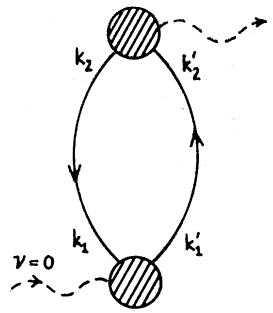

FIG. 1. The reduced graph associated with the conductivity σ.

In this expression, A and B are the spectral densities which determine the single-electron propagator in the case of particle-like and hole-like lines, respectively. These functions are discussed in detail in I and are used extensively in II. Note that we no longer are using the notation in which the chemical potential μ has been shifted to the origin [see Eqs. (II 2.1) and (II 2.2)]. The function Λ_i represents the sum of all proper vertex functions (or, alternatively, amplitudes for single pair production) and is denoted graphically by a small shaded circle in Fig. 1. The external photon line associated with Λ_i carries the interaction J_i defined by Eq. (2.2). An extra factor 2 in (2.12) accounts for a spin sum. It should be emphasized that Eq. (2.12) is still an exact expression for σ.

III. IMPURITY EFFECTS

A detailed description of the procedure for including impurity interactions in expressions like (2.12) has been given in I. We shall use exactly the same technique here; that is, we shall calculate with an impurity potential

$$V_{\mathrm{imp}}=\sum_{s=1}^{N_i}v(\mathbf{r}-\mathbf{r}_s), \tag{3.1}$$

and then average over random configurations of these impurities. In (3.1), $N_i=n_i\Omega$ is the number of impurities and $\mathbf{r}_s$ represents the position of the sth impurity in a particular configuration. As a result of the averaging process, the total momentum transfer at any impurity site must be zero; thus the averaged graph retains a sort of momentum conservation. Finally, remember that each impurity site which appears in a graph contributes a factor n_i.

In order to see where the impurity effects occur in Eq. (2.12), consider what happens to that equation in the limit $n_i\to 0$. In the absence of impurities,

$$A(\mathbf{k},\mathbf{k}',\mu)=B(\mathbf{k},\mathbf{k}',\mu)\underset{n_i\to 0}{\longrightarrow}\delta[\epsilon_{\mathbf{k}}-\mu-\Sigma'(\mathbf{k},\mu)]\delta_{\mathbf{k},\mathbf{k}'}, \tag{3.2}$$

where $\epsilon_{\mathbf{k}}=k^2/2m$ and $\Sigma'(\mathbf{k},\mu)$ is the proper self-energy function evaluated at the Fermi surface. The vertex functions Λ are perfectly well behaved in this limit:

$$\Lambda_i(\mathbf{k},\mathbf{k}',\mu)\underset{n_i\to 0}{\longrightarrow}\Lambda_i(\mathbf{k},\mu)\delta_{\mathbf{k},\mathbf{k}'}=\frac{e}{m}k_i g(k^2)\delta_{\mathbf{k},\mathbf{k}'}, \tag{3.3}$$

where $g(k^2)$ is some dimensionless function of order unity. If we were to insert Eqs. (3.2) and (3.3) into (2.12), we should find σ to be infinite by virtue of a delta function of zero argument. As discussed in I, the effect of the impurities is to replace the delta functions in (3.2) by sharply peaked distributions whose width is proportional to n_i. Then the infinity in (2.12) is replaced by a term of order n_i^{-1}, which is exactly correct for a conductivity.

It follows that, if we want to evaluate the conductivity only to lowest order in the density of impurities, we may use Eq. (3.3) for the vertex corrections and then proceed exactly as in I. Our new result is

$$\sigma=\frac{e^2k_F{}^4N_{kF}|g(k_F{}^2)|^2}{6m^2\pi^2u_F}[\Gamma(k_F)-\Gamma'(k_F)]^{-1}, \quad (3.4)$$

which differs from Eq. (I 8.15) only by a factor g^2. The symbols occurring in (3.4) were defined in I as follows:

$$\Gamma(k_F)=\frac{n_ik_F{}^2N_{kF}}{4\pi u_F}\int_0^\pi |t^+(k_F,\theta)|^2\sin\theta d\theta; \quad (3.5)$$

$$\Gamma'(k_F)=\frac{n_ik_F{}^2N_{kF}}{4\pi u_F}\int_0^\pi |t^+(k_F,\theta)|^2\cos\theta\sin\theta d\theta. \quad (3.6)$$

$$N_{\mathbf{k}}{}^{-1}=1+\left.\frac{\partial\Sigma'(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega=\omega_{\mathbf{k}}'}, \quad (3.7)$$

where $\omega_{\mathbf{k}}'$ satisfies:

$$\epsilon_{\mathbf{k}}-\omega_{\mathbf{k}}'-\Sigma'(\mathbf{k},\omega_{\mathbf{k}}')=0 \quad (3.8)$$

(The primes here indicate functions computed in the absence of impurities.) $u_F=(d\omega_k'/dk)|_{k=k_F}$ is the group velocity of a quasi-particle at the Fermi surface. Finally, t^+ is the proper scattering amplitude for an electron in interaction with a single impurity. t^+ includes the electron-electron interactions to all orders; but the two external electron lines must not contain self-energy parts.

Now we must look for physical interpretations of the various terms which occur in Eq. (3.4).

A. Relaxation Time

In I we defined the function

$$\tilde{S}(\mathbf{k},\omega)=\frac{1}{2\pi i}\frac{1}{\epsilon_{\mathbf{k}}-\omega-\Sigma(\mathbf{k},\omega)}, \quad (3.9)$$

which is the single-particle propagator averaged over configurations of impurities. $\tilde{S}$ has a branch cut along the real axis; and

$$\lim_{\epsilon\to+0}\Sigma(\mathbf{k},\omega\pm i\epsilon)=\Delta(\mathbf{k},\omega)\pm i\Gamma(\mathbf{k},\omega), \quad (3.10)$$

Δ and Γ being real. To first order in n_i, $\Gamma(k_F,\mu)=\Gamma(k_F)$ as expressed in Eq. (3.5). This Γ, however, is not quite the correct expression for the decay rate of a quasi-particle. If we analytically continue $\tilde{S}$ from, say, the upper half ω plane, through the cut, and into the lower half plane, we generally find a simple pole at $\tilde{\omega}=\tilde{E}_{\mathbf{k}}-i\tilde{\Gamma}_{\mathbf{k}}$. Strictly speaking, $\tilde{\Gamma}_{\mathbf{k}}$ is the correct decay rate.

In order to calculate $\tilde{\Gamma}_{\mathbf{k}}$, we may expand Σ about the point $\omega=\tilde{E}_{\mathbf{k}}$ and insert the resulting expression into the equation

$$\epsilon_{\mathbf{k}}-\tilde{\omega}-\Sigma(\mathbf{k},\tilde{\omega})=0. \quad (3.11)$$

We find

$$\epsilon_{\mathbf{k}}-\tilde{E}_{\mathbf{k}}+i\tilde{\Gamma}_{\mathbf{k}}-\Sigma(\mathbf{k},\tilde{E}_{\mathbf{k}})+\left.\frac{\partial\Sigma}{\partial\omega}\right|_{\omega=\tilde{E}_{\mathbf{k}}}(i\tilde{\Gamma}_{\mathbf{k}})=0 \quad (3.12)$$

Thus

$$\tilde{E}_{\mathbf{k}}\cong\epsilon_{\mathbf{k}}-\Delta(\mathbf{k},\tilde{E}_{\mathbf{k}}); \quad (3.13)$$

and

$$\tilde{\Gamma}_{\mathbf{k}}\cong\Gamma(\mathbf{k},\tilde{E}_{\mathbf{k}})\Big/\left(1+\left.\frac{\partial\Delta}{\partial\omega}\right|_{\omega=\tilde{E}_{\mathbf{k}}}\right). \quad (3.14)$$

To lowest order in n_i, these equations are

$$\tilde{E}_{\mathbf{k}}=\epsilon_{\mathbf{k}}-\Delta(\mathbf{k},\tilde{E}_{\mathbf{k}}), \quad (3.15)$$

and

$$\tilde{\Gamma}_{\mathbf{k}}=N_{\mathbf{k}}\Gamma(\mathbf{k},\tilde{E}_{\mathbf{k}}). \quad (3.16)$$

Finally, the expression relevant to the conductivity calculation is

$$\tilde{\Gamma}(k_F)=\frac{n_ik_F{}^2}{4\pi u_F}N_{kF}{}^2\int_0^\pi |t^+(k_F,\theta)|^2\sin\theta d\theta. \quad (3.17)$$

The physical relaxation time must be:

$$\tau^{-1}=2[\tilde{\Gamma}(k_F)-\tilde{\Gamma}'(k_F)]. \quad (3.18)$$

The factor N_{kF} in Eq. (3.17) represents a wave-function renormalization. That is, N_{kF} normalizes the scattering states so that they contain one quasi-particle rather than one electron per unit volume. This factor will be discussed further in the next section.

B. Vertex Function

In the complete absence of impurities, it can be shown that

$$\Lambda_i(\mathbf{k},\omega_{\mathbf{k}}')=ek_i/mN_{\mathbf{k}}, \quad n_i=0. \quad (3.19)$$

We see that, apart from the normalization factor $N_k{}^{-1}$, Λ is the expectation value of the current carried by a quasi-particle. This expression, however, is not exactly the vertex function which we must use in Eq. (3.4); therefore we shall relegate proof of (3.19) to an appendix. The difficulty is that (3.19) is valid only when $\tilde{\Gamma}$ is very small even when compared to the single-particle level spacing. This will be true only if we keep N_i fixed when we let the volume Ω tend to infinity. But this is not what is meant by the limit $n_i\to 0$ in Eq. (3.3). The correct procedure is to compute Λ for fixed n_i in the limit $\Omega\to\infty$, and then let $n_i\to 0$. We now shall prove

that, if we take these limits in their correct order, we must replace the factor k_i/m by the group velocity in Eq. (3.19).

First let us consider any single electron line which might occur in a vertex graph. The single-particle propagator is

$$S_F(\mathbf{k},\omega)=\frac{1}{2\pi i}\frac{1}{\epsilon_\mathbf{k}-\omega-i\alpha_\mathbf{k}},\ \alpha_\mathbf{k}\to\begin{cases}+0 \text{ for } \epsilon_\mathbf{k}>\epsilon_F\\ -0 \text{ for } \epsilon_\mathbf{k}\leq\epsilon_F.\end{cases} \tag{3.20}$$

If we insert impurity interactions into S_F in such a way that none of these interactions connect to other electron lines in Λ, we find the function

$$\bar{S}_1(\mathbf{k},\omega)=\frac{1}{2\pi i}\frac{1}{\epsilon_\mathbf{k}-\omega-i\alpha_\mathbf{k}-\Sigma_1(\mathbf{k},\omega)}. \tag{3.21}$$

Σ_1 is the sum of all graphs in which the impurity interactions form a proper self-energy part. For example, the second order contribution to Σ_1 is:

$$\Sigma_1^{(2)}(\mathbf{k},\omega)=\frac{n_i}{\Omega}\sum_{\mathbf{k}'}\frac{|v(\mathbf{k}-\mathbf{k}')|^2}{\epsilon_{\mathbf{k}'}-\omega-i\alpha_{\mathbf{k}'}}. \tag{3.22}$$

Note that:

$$\begin{aligned}\Gamma_1^{(2)}(\mathbf{k},\omega)&=\mathrm{Im}\Sigma_1^{(2)}(\mathbf{k},\omega)\\ &=(n_i/\Omega)\pi\textstyle\sum_{\mathbf{k}'}|v|^2\delta(\epsilon_{\mathbf{k}'}-\omega)\ \mathrm{sgn}(\epsilon_{\mathbf{k}'}-\epsilon_F)\\ &=(n_i/\Omega)\pi\ \mathrm{sgn}(\omega-\epsilon_F)\textstyle\sum_{\mathbf{k}'}|v|^2\delta(\epsilon_{\mathbf{k}'}-\omega).\end{aligned} \tag{3.23}$$

As usual, Σ_1 may be defined as a function of the complex variable ω with a branch cut along the real axis. The final form of Eq. (3.23) implies that the physical values of Σ_1 are to be found for ω just above the real axis if $\omega>\epsilon_F$, and just below if $\omega\leq\epsilon_F$, i.e., along the contour C_F.

It should be mentioned that, in order to be completely self-consistent, we should renormalize the calculation of $\bar{S}_1$ so that the real part of Σ_1 vanishes at the place where the ω contour crosses the real axis. That is, the impurities, and eventually the electron-electron interactions, will shift the chemical potential from ϵ_F to μ; and the ultimately correct ω contour, C, will cross the real axis at μ. The impurity shift, however, is of order n_i, and may be neglected for our purposes.

The point to be emphasized here is that $\Gamma_1(\mathbf{k},\omega)$ always is finite and much larger than the single-particle spacing. Thus the perturbation expansion for $\bar{S}_1$ in powers of n_i diverges inside a region of radius $\gtrsim\Gamma(\mathbf{k},\epsilon_\mathbf{k})$ about the point $\omega=\epsilon_\mathbf{k}$, even when $\epsilon_\mathbf{k}=\epsilon_F$. Because of the special nature of the interaction $\mathbf{J}$, it turns out that there are diagrams in Λ in which the entire contribution comes from just this divergent part of $\bar{S}_1$. It is in these "anomalous" diagrams that we find deviations from Eq. (3.19).

The simplest anomalous vertex is drawn in Fig. 2. In the complete absence of impurities, diagram 2(a) vanishes because $\epsilon_{\mathbf{k}-\mathbf{q}}$ cannot be both $>\epsilon_F$ and $\leq\epsilon_F$ at the same time. Diagram 2(b), of course, does not vanish. We may write the sum of all such diagrams, illustrated schematically in Fig. 2(c), in the form

$$\begin{aligned}\Lambda_i^{(1)}=&\frac{2\pi ie}{m}\sum_\mathbf{q}(k_i-q_i)\\ &\times\int_{-\infty}^{\infty}\bar{S}_1(\mathbf{k}-\mathbf{q},\omega)\bar{S}_1(\mathbf{k}-\mathbf{q},\omega)d\omega\frac{4\pi e^2}{q^2}.\end{aligned} \tag{3.24}$$

Now note that, if $\Gamma_1(\mathbf{k},\omega)$ remains finite even at $\omega=\mu$, the quantity $i\alpha_\mathbf{k}$ is meaningless in (3.21). We may drop $i\alpha_\mathbf{k}$ and perform the ω integration along the contour C. On closing this contour in the upper half-plane, we find

$$\begin{aligned}\Lambda_i^{(1)}=&-\frac{1}{2\pi i}\frac{e}{m}\sum_\mathbf{q}(k_i-q_i)\frac{4\pi e^2}{q^2}\\ &\times\int_{-\infty}^{\mu}\frac{4i[\epsilon_{\mathbf{k}-\mathbf{q}}-\omega-\Delta_1(\mathbf{k}-\mathbf{q},\omega)]\Gamma_1(\mathbf{k}-\mathbf{q},\omega)}{[(\epsilon_{\mathbf{k}-\mathbf{q}}-\omega-\Delta_1)^2+\Gamma_1^2]^2}d\omega,\end{aligned} \tag{3.25}$$

where Δ_1 is the real part of Σ_1. [Remember that Eq. (3.25) would be incorrect in the absence of impurities even if we took $\bar{S}_1$ to be the true one-particle propagator including electron-electron interactions. It is absolutely necessary that Γ_1 not vanish at $\omega=\mu$.] The integrand in (3.25) is sharply peaked near $\omega=\epsilon_{\mathbf{k}-\mathbf{q}}$. Considered as a function of the momentum variable $(\mathbf{k}-\mathbf{q})$, Γ_1 is very nearly constant across this peak for sufficiently small n_i [see Eq. (I 7.19)]. If we let Γ_1 become small, but still much larger than the single-particle level spacing, we may write

$$\begin{aligned}\lim_{n_i\to0}\Lambda_i^{(1)}&=\lim_{\Gamma_1\to0}\frac{1}{2\pi i}e\frac{\partial}{\partial k_i}\textstyle\sum_\mathbf{q}\frac{4\pi e^2}{q^2}\int_{-\infty}^{\epsilon_F}\frac{2i\Gamma_1}{(\epsilon_{\mathbf{k}-\mathbf{q}}-\omega)^2+\Gamma_1^2}d\omega\\ &=e\frac{\partial}{\partial k_i}\textstyle\sum_\mathbf{q}\frac{4\pi e^2}{q^2}\int_{-\infty}^{\epsilon_F}\delta(\epsilon_{\mathbf{k}-\mathbf{q}}-\omega)d\omega\\ &=e\frac{\partial}{\partial k_i}\sum_{\epsilon_{\mathbf{k}-\mathbf{q}}<\epsilon_F}\frac{4\pi e^2}{q^2}=\Sigma_{\mathrm{ex}}^{(1)}(k),\end{aligned} \tag{3.26}$$

where $\Sigma_{\mathrm{ex}}^{(1)}(k)$ is the first order exchange self energy.

The final form of Eq. (3.26) suggests that there exists a Ward's identity of the sort

$$\lim_{n_i\to0}\Lambda_i(\mathbf{k},\omega)=\frac{ek_i}{m}-e\frac{\partial}{\partial k_i}\Sigma'(\mathbf{k},\omega), \tag{3.27}$$

where $\mathbf{k}$ and ω here refer to the momentum and energy of the external electron lines. To prove this identity, we insert for each free electron line in Λ_i the propagator S_F, but omit $i\alpha_\mathbf{k}$ and perform ω integrations along C_F. Any internal self-energy parts which might occur should be renormalized according to the rules given in I, Sec. V. Once the nondifferentiable $\alpha_\mathbf{k}$'s have been

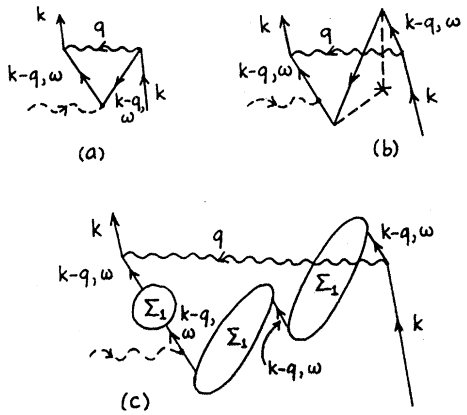

FIG. 2. Illustration of the role of impurity interactions in the development of an anomalous vertex.

discarded, we can insert the vertex J_i into any electron line by writing

$$S_F(\mathbf{k},\omega)\frac{ek_i}{m}S_F(\mathbf{k},\omega)=-\frac{e}{2\pi i}\frac{\partial}{\partial k_i}S_F(\mathbf{k},\omega). \quad (3.28)$$

Ordinarily this relation would be incorrect because it allows anomalous vertices of the kind drawn in Fig. 2(a). But these are just the graphs we want, as demonstrated by Eq. (3.26). All possible vertex graphs may be obtained by inserting J_i in all possible ways into all possible proper self-energy diagrams. We may label the lines in any self-energy graph in such a way that the external momentum $\mathbf{k}$ appears only along the open electron line which runs through the diagram. The right-hand side of (3.27) obviously inserts J_i into each segment of this open line. There may remain a number of closed electron loops in Σ' but these loops make no contribution to Λ_i. Let us label the lines in any loop so that a momentum variable $\mathbf{k}'$ runs around this loop and appears nowhere else. Then we may insert J_i into each segment of the loop by differentiating with respect to k_i'. The result is a total derivative of a smooth function (no $i\alpha_\mathbf{k}$'s) to be summed over $\mathbf{k}'$, and this sum obviously vanishes. This completes the proof of (3.27).

As in I [Eq. (I 7.15)], we now apply Eqs. (3.7) and (3.8) to find

$$\Lambda_i(\mathbf{k},\omega_\mathbf{k}')=eN_\mathbf{k}^{-1}d\omega_\mathbf{k}'/dk_i=eN_\mathbf{k}^{-1}u_i(\mathbf{k}); \quad (3.29)$$

thus

$$g(k_F^2)=mu_F/k_FN_{k_F}. \quad (3.30)$$

Finally:

$$\sigma=\frac{e^2k_F^2u_F\tau}{3\pi^2}=\frac{n_ee^2\tau}{m}\left(\frac{mu_F}{k_F}\right). \quad (3.31)$$

IV. QUASI-PARTICLES

The reader will note that, although the term "quasi-particle" has been used often throughout this paper, the final expression for σ in Eq. (3.31) really has been derived without reference to any particular quasi-particle model. In view of the fact that such models recently have generated some controversy in the literature,[3,6] we conclude this work with a discussion of those properties of a quasi-particle which may be deduced from the analysis leading to Eq. (3.31).

Let us *define* a quasi-particle to be that excitation of the Fermi fluid whose propagation is described by the single-particle propagator in the absence of impurities; i.e.,

$$S'(\mathbf{k},\omega)=\frac{1}{2\pi i}\frac{1}{\epsilon_\mathbf{k}-\omega-\Sigma'(\mathbf{k},\omega)}. \quad (4.1)$$

S' has a pole at $\omega=\omega_\mathbf{k}'$; and $\omega_\mathbf{k}'$ must be the energy of the quasi-particle of wave vector $\mathbf{k}$. Remember that $\omega_\mathbf{k}'$ has an imaginary part which vanishes only at $k=k_F$; i.e., these excitations are not stable except at the Fermi surface.

The current carried by a quasi-particle may be deduced from the vertex function Λ_i. This function contains the expectation value of the operator $\mathbf{J}=e\mathbf{P}/m$. Just as in the case of the scattering amplitude t^+, Λ must be multiplied by the normalization factor $N_\mathbf{k}$ in order for it to measure the current in a state normalized to a single quasi-particle per unit volume. Thus, from Eq. (3.19),

$$\mathbf{J}_\mathbf{k}=e\mathbf{k}/m, \quad n_i=0, \quad (4.2)$$

in the case of perfect translational symmetry. When there is a finite concentration of impurities, however, Eq. (3.29) implies

$$\mathbf{J}_\mathbf{k}=e\mathbf{u}_\mathbf{k}. \quad (4.3)$$

As mentioned in the Introduction, Eq. (4.3) is in exact analogy with the case of Bloch waves. In each case there exists a persistent perturbation arising from fixed scattering centers; and in each case the expectation value of the momentum leads to the group velocity. Experimentally we may bridge the gap between Eqs. (4.2) and (4.3) by changing the frequency ν of the external field. Equation (4.3) was derived on the assumption that $\nu\tau\ll1$. Direct evaluation of a few graphs easily shows that the anomalous vertices vanish when $\nu\tau\gg1$. Thus we may expect Eq. (4.2) to be the correct form of the vertex function in the high frequency limit.

Equation (4.3) is in direct opposition to the theories of Falicov, Heine, and Stern,[3,6] although our picture of a quasi-particle is very much the same as theirs. When there are impurities in the system, the single-particle-like excitations form a wave packet. Because of the smearing of the Fermi surface in momentum space, this wave packet must be localized within a region of size $\Delta x\sim u_\mathbf{k}/\tilde{\Gamma}$, i.e., the mean free path. The present paper differs from those mentioned previously only in that we assert that this wave packet must be normalized

[6] "The Fermi Surface," *Proceedings of the Cooperstown Conference, New York, 1960* (John Wiley & Sons, Inc., 1960). In particular, see the papers of L. A. Stern, L. M. Falicov, and J. M. Luttinger.

to unit charge. We now present two arguments in support of this assertion.

First, we compute the proper vertex function $Q(\mathbf{k},\omega)$. Q is the same as Λ_i except that the external interaction line measures the charge density rather than the current. We may insert this interaction into each line of a proper self-energy diagram by using the relation

$$S_F(\mathbf{k},\omega)eS_F(\mathbf{k},\omega)=\frac{e}{2\pi i}\frac{\partial}{\partial\omega}S_F(\mathbf{k},\omega), \tag{4.4}$$

which is valid whether or not S_F contains the term $i\alpha_k$. Then the Ward's identity appropriate for Q is

$$Q(\mathbf{k},\omega)=e(1+\partial\Sigma'/\partial\omega), \tag{4.5}$$

which may be proved by the same argument which led to Eq. (3.27). When we evaluate Q on the energy shell ($\omega=\omega_k'$) and multiply by the normalization factor N_k, we find for the effective charge of a quasi-particle

$$\tilde{e}_k=N_kQ(\mathbf{k},\omega_k')=e. \tag{4.6}$$

The second argument runs as follows. Suppose we solve our entire problem in the presence of a uniform scalar potential ϕ. Then we expect the quasi-particle energy to be $\omega_k'+\tilde{e}_k\phi$, $\tilde{e}_k$ being the effective charge. In the free-particle propagator S_F, ϵ_k is replaced by $\epsilon_k+e\phi$; and we must make this modification when computing self-energy graphs. But this modification can have no effect on a closed electron loop because ϕ can be eliminated simply by displacing the zero of the energy variable which runs around the loop. We may account for the entire change in Σ' by replacing the external energy ω by $\omega-e\phi$. The new quasi-particle energy satisfies

$$\epsilon_k-\omega+e\phi-\Sigma'(\mathbf{k},\omega-e\phi)=0. \tag{4.7}$$

We know that this equation has the solution

$$\omega-e\phi=\omega_k', \tag{4.8}$$

thus

$$\tilde{e}_k=e. \tag{4.9}$$

It should be emphasized that the intrinsic charge as discussed here must be thought of as the unscreened charge of the quasi-particle. The proper vertex Q accounts for no vacuum polarization parts which might be inserted into the external interaction line. Such polarization graphs can have no effect on the current carried by the quasi-particle or its energy in a constant external potential. Of course, collisions between quasi-particles or between quasi-particles and impurities proceed via screened interactions; that is, one must correct for the dielectric constant of the Fermi fluid. It would be a mistake, however, to try to account for this screening by modifying the effective charge.

Equation (3.31) now may be seen to follow exactly from a kinetic theory formulated in terms of independent quasi-particles. For the acceleration due to an electric field $\mathbf{E}$, we write

$$d\mathbf{k}/dt=e\mathbf{E}; \quad \text{and} \quad d\omega_k'/dt=\mathbf{J}_k\cdot\mathbf{E}=e\mathbf{u}_k\cdot\mathbf{E}. \tag{4.10}$$

We take the unperturbed distribution of quasi-particles to be the Fermi function f_0 evaluated at zero temperature. The Boltzmann equation is

$$\left.\frac{\partial f}{\partial t}\right|_{\text{field}}=-\frac{\partial f_0}{\partial\omega_k'}e\mathbf{u}_k\cdot\mathbf{E}=\delta(\omega_k'-\mu)e\mathbf{u}_k\cdot\mathbf{E}$$
$$=-\left.\frac{\partial f}{\partial t}\right|_{\text{collisions}}=\frac{f-f_0}{\tau_k}. \tag{4.11}$$

Thus

$$\sigma=\frac{2e^2}{3\Omega}\sum_{\mathbf{k}}\delta(\omega_k'-\mu)\mathbf{u}_k\cdot\mathbf{u}_k\tau_k$$
$$=\frac{2e^2}{3\Omega}\sum_{\mathbf{k}}\delta(k-k_F)u_k\tau_k=\frac{n_ee^2\tau}{m}\left(\frac{mu_F}{k_F}\right). \tag{4.12}$$

APPENDIX A. DIRECT PROOF OF EQUATION (2.5)

We offer here a direct proof of the equation

$$\operatorname{Im}\int_{-\infty}^{0}\mathfrak{F}(t)dt=-\frac{3Ne^2}{2m}, \tag{A.1}$$

for systems in which there is a finite concentration of impurities.

First we may obtain a diagrammatic prescription for evaluation of the left-hand side of (A.1) by inserting the spectral representation of $\mathfrak{F}$ as given in Eqs. (2.8) and (2.10).

$$\operatorname{Im}\int_{-\infty}^{0}\mathfrak{F}(t)dt=\lim_{\eta\to0}\operatorname{Im}\int_{-\infty}^{0}dt\int_{0}^{\infty}\rho(\nu')\exp(i\nu't+\eta t)d\nu'$$
$$=-\operatorname{Im}i\int_{0}^{\infty}\rho(\nu')\frac{d\nu'}{\nu'}=\pi\operatorname{Im}F(0). \tag{A.2}$$

Because $\rho(\nu)\sim\nu$, the real part of $F(0)$ must vanish, and we may rewrite (A.1) in the form

$$F(0)=-i(3Ne^2/2\pi m). \tag{A.3}$$

$F(\nu)$ is the same vacuum polarization part which we have been discussing throughout this paper. To evaluate F at $\nu=0$, we close the single-particle propagator upon itself to form one of the external vertices and then use Eq. (3.28) to insert the second vetex into each electron line in the graph. S is considered to be averaged over configurations of impurities; and it is to be emphasized that use of Eq. (3.28) is not justified unless the density of these impurities is finite. The resulting graph consists of a number of independent closed electron loops. According to the discussion leading to Eq. (3.27), we need consider only those graphs in which

both external vertices appear on the same loop. Thus

$$F(0)=\frac{e^2}{2\pi i m}\sum_{i=1}^{3}\sum_{\mathbf{k}}k_i\frac{\partial}{\partial k_i}\int_C d\omega\, S(\mathbf{k},\omega)\exp i\omega 0^+. \quad (A.4)$$

But

$$\int_C d\omega\, S(\mathbf{k},\omega)\exp i\omega 0^+ = -\langle\Psi_0|\, a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}}\,|\Psi_0\rangle = -n(\mathbf{k}), \quad (A.5)$$

where $n(\mathbf{k})$ is the occupation probability of the state of wave vector $\mathbf{k}$. Again we invoke the presence of the impurities so that we can say that $n(\mathbf{k})$ is a continuous function, even at $k=k_F$. Roughly speaking, the impurities tend to smear the Fermi surface in momentum space, removing the discontinuity described by Luttinger. This may be seen easily by inserting in the left-hand side of (A.5) a propagator in which the discontinuity across the cut does not vanish at $\omega=\mu$. Thus we may integrate over $\mathbf{k}$ by parts to find

$$F(0)=-\frac{e^2}{2\pi i m}\sum_{i=1}^{3}\sum_{\mathbf{k}}k_i\frac{\partial n(\mathbf{k})}{\partial k_i}$$

$$=-\frac{3ie^2}{2\pi m}\sum_{\mathbf{k}}n(\mathbf{k})=-\frac{3ie^2}{2\pi m}N. \quad (A.6)$$

Equation (A.6) is a special case of the f-sum rule:

$$\sum_n f_{0n}=Ne^2, \quad (A.7)$$

where

$$f_{0n}=\frac{2m}{3}\frac{|\langle\Psi_0|\,\mathbf{J}\,|\Psi_n\rangle|^2}{E_n-E_0}, \quad (A.8)$$

according to Eqs. (2.8) and (2.9). This sum rule usually applies to longitudinal external fields of finite wavelength. Because of the impurities, we are able to write Eq. (A.7) for the case of an applied field which is strictly uniform.

APPENDIX B. VERTEX FUNCTION, EQUATION (3.19)

When the many-body system is translationally invariant, we may obtain the vertex function Λ in the following manner. We insert the external interaction J_i into each line of the propagator S' by writing

$$\mathfrak{L}_i(\mathbf{k},t,t')=\langle\Psi_0|\, T\{a_{\mathbf{k}}(t)J_i(t')a_{\mathbf{k}}{}^\dagger(0)\}\,|\Psi_0\rangle, \quad (B.1)$$

so that:

$$L_i(\mathbf{k},\omega,\nu)$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\,\mathfrak{L}_i(\mathbf{k},t,t')\exp i\omega t\exp i\nu(t-t') \quad (B.2)$$

$$=S'(\mathbf{k},\omega+\nu)\Lambda_i(\mathbf{k},\omega,\nu)S'(\mathbf{k},\omega).$$

The translational symmetry of the system implies that

$$\begin{aligned}\mathfrak{L}_i(\mathbf{k},t,t')&=(e/m)k_iS'(\mathbf{k},t) \quad \text{for} \quad t>t'>0;\\ &=-(e/m)k_iS'(\mathbf{k},t) \quad \text{for} \quad 0>t'>t; \quad (B.3)\\ &=0 \text{ otherwise.}\end{aligned}$$

If we substitute this expression for $\mathfrak{L}$ into Eq. (B.2), we find

$$L_i(\mathbf{k},\omega,\nu)$$

$$=\frac{ek_i}{4\pi^2m}\int_0^{\infty}dt\int_0^{t}dt'\,S'(\mathbf{k},t)\exp i\omega t\exp i\nu(t-t')$$

$$-\frac{ek_i}{4\pi^2m}\int_{-\infty}^{0}dt\int_t^{0}dt'\,S'(\mathbf{k},t)\exp i\omega t\exp i\nu(t-t') \quad (B.4)$$

$$=\frac{ek_i}{m}\frac{1}{2\pi i\nu}[S'(\mathbf{k},\omega+\nu)-S'(\mathbf{k},\omega)].$$

Thus

$$\Lambda(\mathbf{k},\omega,\nu)=\frac{ek_i}{m}\frac{1}{2\pi i\nu}\left[\frac{1}{S'(\mathbf{k},\omega)}-\frac{1}{S'(\mathbf{k},\omega+\nu)}\right]$$

$$=\frac{ek_i}{m}\left[1+\frac{\Sigma'(\mathbf{k},\omega+\nu)-\Sigma'(\mathbf{k},\omega)}{\nu}\right]. \quad (B.5)$$

Equation (3.19) is obtained by letting ν go to zero and choosing $\omega=\omega_{\mathbf{k}}'$.

It is also of interest to consider the vertex function when $\mathbf{J}$ is defined to be

$$J_i=\lim_{\mathbf{q}\to 0}J_i(\mathbf{q})=\lim_{\mathbf{q}\to 0}\frac{e}{m}\sum_{\mathbf{k}}(k_i+\tfrac{1}{2}q_i)a_{\mathbf{k}+\mathbf{q}}{}^\dagger a_{\mathbf{k}}. \quad (B.6)$$

In this case we are concerned with only the localized part of the current fluctuation; and Eq. (3.29) (involving the group velocity) is the correct form of the vertex function. This result may be proved directly using techniques of the sort developed in this paper. It is relevant, for example, to the theory of anomalous skin effect.