Theory of Average Neutron Reaction Cross Sections in the Resonance Region*

P. A. MOLDAUER
Argonne National Laboratory, Argonne, Illinois

(Received February 2, 1961)

The scattering matrix for compound nucleus processes is studied in the R-matrix formalism, using a series expansion which is due to Thomas. It is shown that this series generally converges when (a) the average total resonance width is less than the average resonance spacing, (b) the number of important channels is not too large, and (c) the width amplitudes have random signs. The treatment also suggests strongly that the series does not converge in the continuum region. In the region of convergence the exact relationship between the channel transmission factor T_c and the ratio of partial width to level spacing is found, in the absence of direct scattering reactions, to be $T_c=2\pi\langle\Gamma_{\lambda c}\rangle/D-\pi^2\langle\Gamma_{\lambda c}\rangle^2/D^2$. The quadratic term is shown to be important in the vicinity of optical-model maxima. Correction terms to the Hauser-Feshbach relations for average reaction cross sections arising from the higher order terms of the series are obtained and are found to depend on the statistical properties of both resonance widths and resonance spacings. The effect on average neutron inelastic, compound elastic, and capture cross sections is discussed and an example of a calculation is presented.

I. INTRODUCTION

THE development of improved experimental techniques, as well as the needs of technological applications, have resulted recently is precise measurements of neutron reaction cross sections averaged over resonances for neutron energies up to several Mev.[1] Given correct theoretical interpretation, such data can yield information of interest in the study of a number of topics of nuclear structure. Thus the analysis of average cross sections for radiative capture, charged-particle emission, or multiple-neutron emission yields information on the statistical properties of highly excited nuclear states. Average elastic and inelastic scattering cross sections can be used to determine optical model parameters and to identify low-lying states of the target nucleus. The examination of average fission cross sections may be expected to yield knowledge about the fission process.

The formal theory of resonance reactions has been highly developed in several different ways. A summary of these theoretical structures has been given by Lane and Thomas.[2] In order to calculate average cross sections it is more convenient to adopt the R-matrix formalism of Wigner and Eisenbud[3] because in the otherwise simpler Kapur-Peierls formalism[4] the strong energy dependence of the resonance parameters complicates the evaluation of energy averages.[5] In the R-matrix formalism, however, one is faced with the well-known difficulty of inversion of the R matrix. This problem was solved by Thomas by means of a power series expansion of the inverse matrix.[6] In this study we follow the method of Thomas and enlarge upon his results in two ways. We study more closely the convergence of Thomas' series which had previously been inferred by examination of the not yet typical second term. Then we find the contribution to the cross sections of the higher order terms which had previously been neglected. Since the formal reaction theory does not provide values for its own parameters, we also make the connection with the optical model of Feshbach, Porter, and Weisskopf[7] by obtaining expressions for the model cross sections.

Definitions are established and basic relations are reviewed in Sec. II. Section III deals with the optical model cross sections, and Sec. IV with other reaction cross sections. The results are discussed and applications are given in Sec. V. The convergence of Thomas' series is discussed in the Appendix.

II. DEFINITIONS

We shall employ the symbol c (loosely called a "channel") to denote the collection of all quantum numbers besides energy needed to specify the internal and relative states of a scatterer and a projectile in the absence of the scattering interaction. Thus c implies the internal states of scatterer and projectile, their relative spin orientations and orbital angular momentum, and, of course, the total angular momentum and parity of the system. The rules for selecting and com-

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] Summaries of recent data may be found in *Neutron Cross Sections*, compiled by D. J. Hughes, B. A. Magurno, and M. K. Brussel, Brookhaven National Laboratory Report BNL-325 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1960), 2nd ed., Suppl. No. 1; and in "Tabulated Neutron Cross Sections," compiled by R. J. Howerton, University of California Lawrence Radiation Laboratory Report UCRL-5226, Rev. 1959 (unpublished), Part I.

[2] A. M. Lane and R. G. Thomas, Revs. Modern Phys. **30**, 257 (1958). See also G. Breit, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41/1.

[3] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[4] P. L. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938). For a discussion of the extreme low-energy s-wave neutron scattering cross section employing the Kapur-Peierls formalism, see G. E. Brown and C. T. DeDominicis, Proc. Phys. Soc. (London) **A72**, 70 (1958).

[5] Still another formulation relating resonance parameters directly to nucleon-nucleon interactions has been given by H. Feshbach, Ann. Phys. **5**, 357 (1958).

[6] R. G. Thomas, Phys. Rev. **97**, 224 (1955). For other approaches to the problem see T. Teichman, *ibid.* **77**, 506 (1950).

[7] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

bining channels to obtain observable combinations of states are well known and are not repeated here. Denoting the normalized wave functions for incoming and outgoing waves in channel c in the absence of a scattering interaction by $\mathcal{I}_c$ and $\mathcal{O}_c$, respectively, the scattering wave function for incoming waves in channel c only is asymptotically given by

$$\Psi_c^{as}=\mathcal{I}_c-\sum_{c'} U_{cc'}\mathcal{O}_{c'}. \tag{1}$$

Cross sections can be expressed in terms of the elements of the scattering matrix $U_{cc'}$ in a well-known way. Since we are interested in average cross sections we shall place a bar over all fluctuating quantities to indicate an average over an energy interval to be described more precisely later. Calling $2\pi\lambda\!\!\!^{-}_c$ the wavelength of relative motion in channel c and g_c its statistical factor,[2] we have for the average total, elastic, nonelastic, and partial reaction cross sections in units of $\pi\lambda\!\!\!^{-}_c{}^2 g_c$:

$$\bar{\sigma}_c(\text{tot})=2(1-\text{Re}\bar{U}_{cc}), \tag{2a}$$

$$\bar{\sigma}_c(\text{el})=\{|1-U_{cc}|^2\}_{\text{av}}, \quad \{A\}_{\text{av}}\equiv\bar{A}, \tag{2b}$$

$$\bar{\sigma}_c(\text{nonel})=1-\{|U_{cc}|^2\}_{\text{av}}, \tag{2c}$$

$$\bar{\sigma}_{c,c'}=\{|U_{c,c'}|^2\}_{\text{av}}, \quad c\neq c'. \tag{2d}$$

Here "elastic" means same entrance and exit channels and not merely preservation of the internal state of the scatterer. In addition the following cross sections in the same units were defined by Feshbach, Porter, and Weisskopf to facilitate the application of the optical model which is a theory of $\bar{U}_{cc}$.[7]

$$\sigma_c(\text{absorption})\equiv T_c=1-|\bar{U}_{cc}|^2, \tag{2e}$$

$$\sigma_c(\text{shape el})=\bar{\sigma}_c(\text{tot})-\sigma_c(\text{abs})=|1-\bar{U}_{cc}|^2, \tag{2f}$$

$$\sigma_c(\text{fluctuation})=\bar{\sigma}_c(\text{el})-\sigma_c(\text{shape el}) =\{|U_{cc}|^2\}_{\text{av}}-|\bar{U}_{cc}|^2. \tag{2g}$$

According to the R-matrix theory the scattering matrix is given by

$$U_{cc'}=U_{cc'}{}^0+e^{-i(\phi_c+\phi_{c'})}[\delta_{cc'}+2\mathcal{U}_{cc'}], \tag{3}$$

where, according to Wigner,[8]

$$\mathcal{U}_{cc'}=\tfrac{1}{2}i\sum_{\lambda\mu} g_{\lambda c}g_{\lambda c'}(E_{\lambda\mu}-E-\tfrac{1}{2}i\sum_{c''} g_{\lambda c''}g_{\mu c''})^{-1}. \tag{4}$$

Here $U_{cc'}{}^0$ is the part of the scattering matrix responsible for nonfluctuating or direct interactions. It will be assumed to include the contributions from all resonances lying outside the averaging interval, so that the sum over resonance indices λ in Eq. (4) will be carried over only those resonances whose resonance energies E_λ lie within the interval being averaged over. The partial and total widths of the resonance λ are given by

$$\Gamma_{\lambda c}=g_{\lambda c}{}^2, \quad \Gamma_\lambda=\sum_c \Gamma_{\lambda c}. \tag{5}$$

[8] E. P. Wigner, Phys. Rev. **70**, 606 (1946).

We shall require the averaging interval to be large compared to the spacing of resonance levels E_λ and large compared to any of the total widths Γ_λ in the interval. But we want the interval to be small enough that $\mathcal{I}_c$ and $\mathcal{O}_c$ and hence the channel phase shifts ϕ_c, the level shift factors S_c, and the penetrabilities P_c remain essentially constant in the interval.[9] Then we may assume that the channel boundary conditions of the R-matrix theory have been adjusted in each interval so that all the shift functions vanish in the interval, and hence we have for all resonances within the interval

$$E_{\lambda\mu}=\delta_{\lambda\mu}E_\mu. \tag{6}$$

According to an argument of Porter and Thomas, one expects in an interval such as that described above that the $g_{\lambda c}$ for a given c are randomly distributed in λ and that the distribution is normal with zero mean.[10] This has, first of all, the consequence that the $g_{\lambda c}$ have random signs. Secondly, the only parameter required to describe the $g_{\lambda c}$ is their variance which by Eq. (5) equals the average partial width $\langle\Gamma_{\lambda c}\rangle$. The brackets $\langle\ \rangle$ will be used throughout to indicate an average over all resonances in the averaging interval. The partial widths themselves are distributed according to the Porter-Thomas distribution law for $x_\lambda=\Gamma_{\lambda c}/\langle\Gamma_{\lambda c}\rangle$,

$$P_{\text{P.T.}}(x)dx=(2\pi x)^{-\frac{1}{2}}e^{-\frac{1}{2}x}dx. \tag{7}$$

The validity of this distribution law has been well established by analyses of neutron resonance scattering data.[11]

The distribution law for total widths depends on the number of channels making substantial contributions. As the number of channels increases, the distribution becomes narrower, and for very many channels n it may be approximated by a Gaussian with its mean at $\langle\Gamma_\lambda\rangle$ and a dispersion of $\sqrt{2}\langle\Gamma_\lambda\rangle/(n)^{\frac{1}{2}}$.

Basing his arguments on assumptions similar to those of Porter and Thomas, Wigner has suggested a distribution law for the resonance level spacings.[12] Calling D the average spacing of resonances of the same spin and

[9] The channel functions ϕ_c, S_c, P_c are defined in reference 2 and have been evaluated for neutrons by J. E. Monahan, L. C. Biedenharn, and J. P. Schiffer, Argonne National Laboratory Report ANL-5846 (1958) (unpublished). If the stated conditions for the length of the averaging interval are inconsistent, one may still expect to obtain useful results by replacing the R matrix with Wigner's statistical R matrix [E. P. Wigner, Ann. Math. **53**, 36 (1951)] and moving the difference to R^0 which contributes to U^0. The author is indebted to Dr. J. E. Lynn for reminding him of the importance of retaining U^0 in the formalism.

[10] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).

[11] Reference 10 and J. L. Rosen, J. S. Desjardins, J. Rainwater, and W. W. Havens, Phys. Rev. **118**, 687 (1960) and Bull. Am. Phys. Soc. **5**, 32 (1960) and **5**, 295 (1960).

[12] E. P. Wigner, Proceedings of the Conference on Neutron Physics by Time-of-Flight, Gatlinburg Tennessee, 1956 [Oak Ridge National Laboratory Report ORNL-2309, 1957 (unpublished)], p. 59; and Proceedings of the International Conference on Neutron Interactions with the Nucleus, Columbia University, 1957 [Columbia University Report CU-175, 1957 (unpublished)], p. 49; and Fourth Canadian Mathematical Congress Proceedings, 1957 (unpublished), p. 174.

parity, $y_\lambda=(E_{\lambda+1}-E_\lambda)D^{-1}$ is distributed according to

$$P_W(y)dy=\frac{\pi}{2}\frac{y}{D}\exp(-\tfrac{1}{4}\pi y^2)dy. \quad (8)$$

The statistical foundations of this Wigner distribution law have been further discussed by Porter and by Rosenzweig and others,[13] and Eq. (8) has been shown to be consistent with existing neutron resonance data.[14–16]

Following Thomas,[6] we shall employ the following expansion of the matrix $\mathfrak{U}$ as given in Eq. (4).

$$\mathfrak{U}=\sum_\lambda A_\lambda+\sum_{\lambda\neq\mu} A_\lambda\times A_\mu+\sum_{\lambda\neq\mu,\mu\neq\nu} A_\lambda\times A_\mu\times A_\nu +\cdots, \quad (9)$$

where the matrices A_λ are given by[17]

$$A_\lambda{}^{cc'}=\tfrac{1}{2}i(g_{\lambda c}g_{\lambda c'})/(E_\lambda-E-\tfrac{1}{2}i\Gamma_\lambda). \quad (10)$$

In the following sections, the consequences of the expansion (9) are examined in the light of the statistical laws of Eqs. (7) and (8).

III. OPTICAL-MODEL CROSS SECTIONS

The optical model determines the values of $\bar{U}_{cc}$.[7] To establish a connection with it, it is therefore necessary to average the diagonal element of Eq. (3). Since $U_{cc}{}^0$ and ϕ_c are assumed not to vary appreciably in the averaging interval, the problem is to average $\mathfrak{U}_{cc}$. We average by integrating over energy the contribution from all resonances in the interval and dividing by the length of the interval. Since the interval is assumed to be very large compared to all widths we may extend the range of integration from $-\infty$ to $+\infty$. If the expansion (9) converges uniformly in the interval we may average it term by term. Doing this, we note from Eq. (10) that each term consists of a product of analytic functions each with a single pole in the lower half plane and going to infinity as E^{-1}. Consequently the average of all terms except the first vanishes and we have

$$\{\mathfrak{U}_{cc}\}_{\text{av}}=\sum_\lambda \bar{A}_\lambda{}^{cc}=-\tfrac{1}{2}\pi\langle\Gamma_{\lambda c}\rangle/D. \quad (11)$$

The convergence requirement for the validity of this relation may be relaxed somewhat by invoking the random signs of the $g_{\lambda c}$. With this assumption all terms in Eq. (9) which are linear in any $g_{\lambda c'}$ cancel one another before integration and the condition becomes that part of Eq. (9) containing terms quadratic in the $g_{\lambda c}$ shall converge uniformly in the interval. Conditions for this convergence are found in the Appendix. It is shown there that Eq. (11) may be expected to hold, provided the average total width is less than the average level spacing for resonances of the angular momentum parity in question and provided that the number of channels contributing significantly to the decay of such resonances does not become too large.

Under these conditions we obtain then from Eqs. (3) and (11):

$$\bar{U}_{cc}=U_{cc}{}^0+e^{-2i\phi_c}[1-\tfrac{1}{2}\langle\tau_{\lambda c}\rangle], \quad (12)$$

where

$$\tau_{\lambda c}\equiv 2\pi\Gamma_{\lambda c}/D. \quad (13)$$

From this we obtain, using Eq. (2e), the absorption cross section or transmission coefficient,

$$T_c=\langle\tau_{\lambda c}\rangle-\tfrac{1}{4}\langle\tau_{\lambda c}\rangle^2 -2\,\text{Re}U_{cc}{}^0e^{2i\phi_c}[1-\tfrac{1}{2}\langle\tau_{\lambda c}\rangle]-|U_{cc}{}^0|^2. \quad (14)$$

This well-known expression[18] has been used extensively in the region where $\langle\tau_{\lambda c}\rangle$ is small and where therefore the quadratic term in $\langle\tau_{\lambda c}\rangle$ may be neglected. However, for low-angular-momentum neutron channels, and particularly at optical model maxima, T_c may approach its limiting value of unity at fairly low neutron energies. Then the quadratic term can no longer be neglected. In fact, under the assumption that $U_{cc}{}^0$ vanishes, one finds that $\langle\tau_{\lambda c}\rangle$ approaches the value two as T_c approaches unity. The effect of this quadratic term, with $U_{cc}{}^0=0$, on the neutron strength functions

$$\gamma_c{}^2/D=\langle\tau_{\lambda c}\rangle/(4\pi P_c), \quad (15)$$

at the optical model p and d wave maxima, is shown in Fig. 1. It is clear that as $\langle\tau_{\lambda c}\rangle$ exceeds the value of two it will tend to decrease T_c, eventually even in the presence of a nonvanishing $U_{cc}{}^0$. This physically unsatisfactory situation strongly suggests, as do the arguments in the Appendix, that the convergence condition

[13] S. Blumberg and C. E. Porter, Phys. Rev. **110**, 786 (1958). C. E. Porter and N. Rosenzweig, Ann. Acad. Sci. Fennicae **Ser. A. VI**, No. 44 (1960); M. L. Mehta, Nuclear Phys. **18**, 395 (1960). See also reference 15.

[14] I. I. Gurevich and M. I. Pevsner, Nuclear Phys. **2**, 575 (1957). J. A. Harvey and D. J. Hughes, Phys. Rev. **109**, 471 (1958). See also reference 11.

[15] P. A. Moldauer, Bull. Am. Phys. Soc. **4**, 319 (1959) and P. A. Moldauer (to be published).

[16] All of the mentioned statistical results are based on the assumption of the randomness of the reduced width amplitudes $\gamma_{\lambda c}=g_{\lambda c}(2P_c)^{-\frac{1}{2}}$ which are proportional to the overlap integrals of the compound nucleus and channel wave functions at the channel radius. The consequences of this assumed randomness of the compound-state wave functions have been confirmed only in the extreme resonance region where resonance levels are well separated. It should not be surprising if, in the region of overlapping resonance levels, correlations appeared due to interactions of the resonance states via the channels and if these correlations should cause a breakdown in any of the statistical laws, including the random sign assumption.

[17] The fact that the level shift factors do vary over the averaging interval introduces in each of the terms of Eq. (9) a correction factor $(1-i\Delta S_{c'}P_{c'}{}^{-1})$ for each of the channels c' arising from the implied sums over channels. By $\Delta S_{c'}$ we mean the change in shift factor from the center of the interval to the energy in question. Assuming a linear variation of $S_{c'}$ across the interval and using the tabulated values of the derivative of $S_{c'}$ given in the work cited in footnote 9, one can estimate the maximum numerical value of $\Delta S_{c'}$ to be less than the size of the averaging interval measured in Mev. But this value can always be made very small compared to the value of $P_{c'}$ for all channels making significant contributions to Eq. (9). Furthermore, these correction factors can only improve the convergence of the series at the edges of the interval as compared to the center (see Appendix).

[18] This equation has recently been written and discussed in a form in which $U_{cc}{}^0$ is combined with ϕ_c by H. Feshbach, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press, Inc., New York, 1960), Part B, p. 1041.

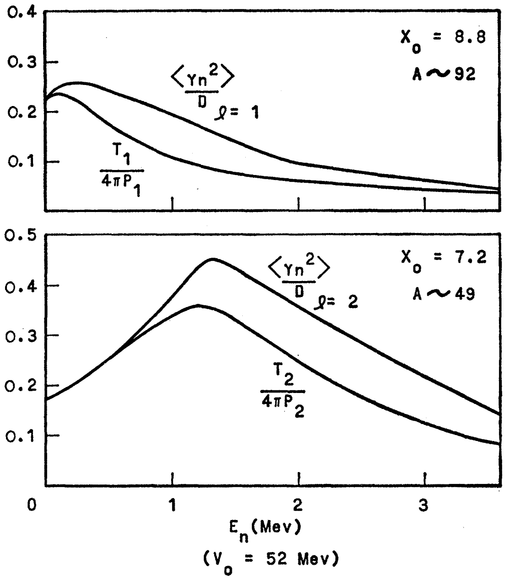

FIG. 1. A comparison of the neutron strength function $\langle\gamma_n{}^2\rangle/D$ with the ratio of neutron transmission coefficient T to the penetrability P for p-wave neutrons at the p-wave optical model maximum and for d-wave neutrons at the d-wave optical model maximum. E_n is the neutron channel energy in Mev. $X_0=K_0R$, where K_0 is the neutron wave number in the well and R the potential well radius. The curves are based on the neutron optical model phase shifts of Campbell, Feshbach, Porter, and Weisskopf (see reference 32).

derived there for the validity of Eq. (11) and hence also of Eq. (14) does in fact correspond closely to the actual limit of validity of these expressions and that these expressions are not valid in the continuum region.[19] Another cause for the breakdown of these expressions would be any correlations in the signs of the $g_{\lambda c}$ (see footnote 16). Formally, it would of course be possible to retain the validity of Eqs. (11), (12), and (14) by transferring parts of $\mathcal{U}_{cc}$ to $U_{cc}{}^0$, in particular any correlated parts, in such a way that the convergence condition remains satisfied. The practical usefulness of such a procedure will depend on whether one has a model for this new $U_{cc}{}^0$ enabling one to calculate it and to average it if it should fluctuate.

In addition to the absorption cross section we can now write down, subject to the same conditions, expressions for the average total cross section in units of $\pi\lambda\!\!\!^{-}{}_c{}^2g_c$:

$$\bar{\sigma}_c(\text{tot})=2[1-\text{Re}U_{cc}{}^0-\cos(2\phi_c)(1-\tfrac{1}{2}\langle\tau_{\lambda c}\rangle)], \quad (16)$$

and for the shape elastic cross section in units of $\pi\lambda\!\!\!^{-}{}_c{}^2g_c$:

$$\sigma_c(\text{shape el})=|1-U_{cc}{}^0|^2+2\,\text{Re}(1-U_{cc}{}^0)e^{2i\phi_c}(1-\tfrac{1}{2}\langle\tau_{\lambda c}\rangle)+(1-\tfrac{1}{2}\langle\tau_{\lambda c}\rangle)^2. \quad (17)$$

The same argument which led to Eq. (11) gives, in view of the random sign assumption,

$$\{\mathcal{U}_{cc'}\}_{\text{av}}=0, \quad c\neq c'. \quad (18)$$

IV. REACTION CROSS SECTIONS

To calculate the remaining cross sections in Eq. (2) we need expressions for the average value of $|\mathcal{U}|^2$. Employing again the expansion (9) we have

$$\{|\mathcal{U}_{cc'}|^2\}_{\text{av}}=\sum_n\{\mathcal{U}_{cc'}{}^{nn}\}_{\text{av}}+\sum_{n<m}2\,\text{Re}\{\mathcal{U}_{cc'}{}^{nm}\}_{\text{av}}, \quad (19)$$

where the summations are carried from one to infinity and

$$\mathcal{U}_{cc'}{}^{nm}=\sum_{\substack{\lambda_i\neq\lambda_{i+1}\\ \text{for } i\neq n}}(A_{\lambda_1}{}^*\times A_{\lambda_2}{}^*\times\cdots\times A_{\lambda_n}{}^*)_{cc'}\times(A_{\lambda_{n+1}}\times A_{\lambda_{n+2}}\times\cdots\times A_{\lambda_{n+m}})_{cc'}. \quad (20)$$

Assuming no degenerate levels and no correlations between widths and spacings, we perform the average by summing over residues in the upper half plane of each term in the expression (20), obtaining in this way

$$\{\mathcal{U}_{cc'}{}^{nm}\}_{\text{av}}=\Big\langle(G_{\lambda_1}\times G_{\lambda_2}\times\cdots\times G_{\lambda_n})_{cc'}(G_{\lambda_{n+1}}{}^*\times G_{\lambda_{n+2}}{}^*\times\cdots\times G_{\lambda_{n+m}}{}^*)_{cc'}\times\frac{2\pi i}{D}\sum_{i=1}^{n}\int\frac{W_n(\epsilon_{i1},\epsilon_{i2}\cdot\overset{\prime}{\cdot}\cdot\epsilon_{i,n+m})d\epsilon_{i1}d\epsilon_{i2}\cdot\overset{\prime}{\cdot}\cdot d\epsilon_{i,n+m}}{(\epsilon_{i1}+i\tau_{i1})(\epsilon_{i2}+i\tau_{i2})\cdot\overset{\prime}{\cdot}\cdot(\epsilon_{in}+i\tau_{in})(\epsilon_{i,n+1}+i\sigma_{i,n+1})\cdots(\epsilon_{i,n+m}+i\sigma_{i,n+m})}\Big\rangle, \quad (21)$$

where

$$G_\lambda{}^{cc'}=(i/2)g_{\lambda c}g_{\lambda c'},$$
$$\epsilon_{ij}=E_{\lambda i}-E_{\lambda j},$$
$$\tau_{ij}=\tfrac{1}{2}(\Gamma_{\lambda i}-\Gamma_{\lambda j}),$$
$$\sigma_{ij}=\tfrac{1}{2}(\Gamma_{\lambda i}+\Gamma_{\lambda j}),$$

and $W_n(\epsilon_{i1},\epsilon_{i2},\cdot\overset{\prime}{\cdot}\cdot\epsilon_{i,n+m})$ is the combined density function for the $n+m-1$ resonance energies $E_{\lambda j\neq i}$ relative to $E_{\lambda i}$, subject to the restrictions that $\lambda_j\neq\lambda_{j+1}$ for $j\neq n$ and $\lambda_j\neq\lambda_i$ for $j\leq n$. The primes indicate omission of the ith term. The function W_n can be expressed in terms of the symmetric pair density function $W(\epsilon)$ which is the average over λ of the number of resonances per unit energy interval at the energy $E_\lambda+\epsilon$. If the resonance energies E_λ were randomly distributed, we would have $W(\epsilon)\equiv D^{-1}$. However, the Wigner repulsion effect embodied in Eq. (8) modifies this relation. The relationship between $W(\epsilon)$ and the distribution $P(\epsilon)$ of spacings between neighboring resonances has been previously discussed by the author, who obtained the density function $W_W(\epsilon)$ corresponding to the Wigner spacing

[19] By continuum we always mean here, of course, the continuum for resonances of the same spin and parity.

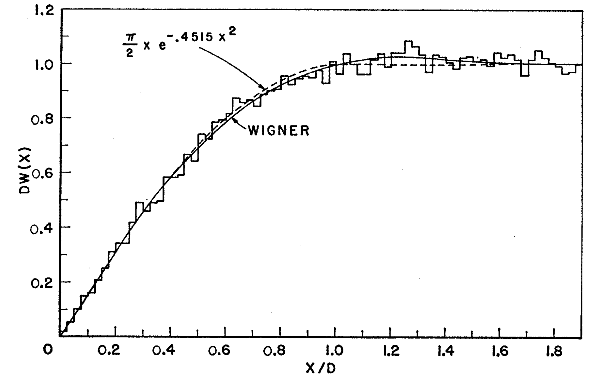

FIG. 2. The pair density function $W(x)$, giving the density of nuclear levels at an energy x from a given level. D is the average level spacing. The solid line corresponds to the density function arising from the Wigner spacing distribution. The histogram is the results of numerical diagonalization of one hundred 20×20 random matrices (see reference 15). The dashed line corresponds to the approximation of Eq. (22).

distribution law [Eq. (8)] by both analytic and numerical methods as shown in Fig. 2.[15] While no closed analytic form for $W_W(\epsilon)$ is known, the approximation

$$W(\epsilon)=\frac{\pi}{2}\frac{|\epsilon|}{D^2}\exp\left[-\left(\frac{\epsilon}{D}\right)^2\ln\frac{\pi}{2}\right],\quad |\epsilon|<D, \qquad (22)$$
$$W(\epsilon)=1/D,\quad |\epsilon|>D$$

can be seen in Fig. 2 to fit $W_W(\epsilon)$ well.[20]

With the help of $W(\epsilon)$ we now write

$$W_n(\epsilon_{i1},\epsilon_{i2}\cdots\epsilon_{in}\epsilon_{i,n+1}\cdots\epsilon_{i,n+m})$$
$$=\prod_{j=1,j\neq i}^{n}W(\epsilon_{ij})\prod_{k=n+1}^{n+m}[\delta(\lambda_i,\lambda_k)+W(\epsilon_{ik})]$$
$$\times\prod_{j=1,j\neq i}^{n-1}DW(\epsilon_{ij}-\epsilon_{i,j+1})\prod_{k=n+1}^{n+m-1}DW(\epsilon_{ik}-\epsilon_{i,k+1}). \qquad (23)$$

This expression neglects the effect of correlations between level spacings though there is both experimental[21] and theoretical[22] evidence for such correlations. Their effect would be to replace those density functions whose arguments are differences between resonance spacings, and which arise from the restrictions placed on W_n, by more complicated functions. However, it is expected that the effects of such correlations will cancel out at least partially when the summation over i is performed in Eq. (21). Furthermore, such effects can show up in Eq. (19) starting only with terms of the fourth order in the widths. In performing the integral in Eq. (21) with the help of Eqs. (22) and (23), we encounter the following integrals:

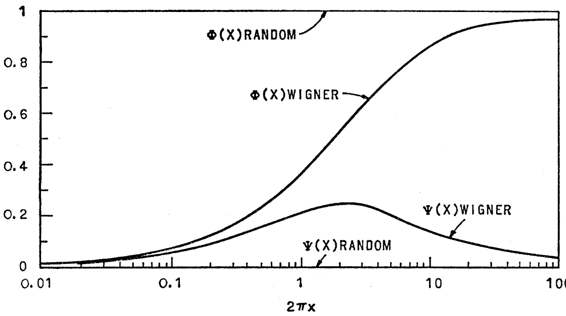

FIG. 3. The functions $\Phi(x)$ and $\Psi(x)$ plotted against $2\pi x$ [see Eqs. (24) and (25)].

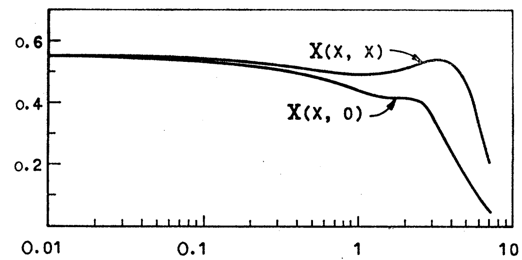

FIG. 4. The functions $X(x,x)$ and $X(x,0)$ plotted against $2\pi x$ [see Eq. (26)].

$$\Phi(\Gamma/D)=i\frac{D}{\pi}\int_{-\infty}^{+\infty}\frac{d\epsilon\, W(\epsilon)}{\epsilon+i\Gamma}=1-\frac{2}{\pi}\tan^{-1}\left(\frac{D}{\Gamma}\right)+\frac{1}{2}\frac{\Gamma}{D}\exp\left[\left(\frac{\Gamma}{D}\right)^2\ln\frac{\pi}{2}\right]\left\{\mathrm{Ei}\left[-\left(1+\frac{\Gamma^2}{D^2}\right)\ln\frac{\pi}{2}\right]-\mathrm{Ei}\left[-\frac{\Gamma^2}{D^2}\ln\frac{\pi}{2}\right]\right\}, \qquad (24)$$

$$\Psi(\Gamma/D)=\frac{D\Gamma}{\pi}\int_{-\infty}^{+\infty}\frac{d\epsilon\, W(\epsilon)}{(\epsilon+i\Gamma)^2}=\left(1+2\frac{\Gamma^2}{D^2}\ln\frac{\pi}{2}\right)\left[\Phi\left(\frac{\Gamma}{D}\right)+\frac{2}{\pi}\tan^{-1}\left(\frac{D}{\Gamma}\right)-1\right]-\left(1-\frac{2}{\pi}\right)\frac{\Gamma}{D}, \qquad (25)$$

where $\mathrm{Ei}(z)$ is the exponential integral. These two functions are plotted in Fig. 3. In addition there appears in the low-order terms of $\{|\mathfrak{U}|^2\}_{\mathrm{av}}$ the integral

$$X(\Gamma_1/D,\Gamma_2/D)=-\frac{D^2}{\pi^2}\int_{-\infty}^{+\infty}\frac{d\epsilon\, W(\epsilon)}{\epsilon+i\Gamma_1}\int_{-\infty}^{+\infty}\frac{d\eta\, W(\eta)DW(\epsilon-\eta)}{\eta+i\Gamma_2}. \qquad (26)$$

This function may be estimated by setting $DW(\epsilon-\eta)=1-[1-DW(\epsilon-\eta)]$, where the first term just integrates to $\Phi(\Gamma_1/D)\Phi(\Gamma_2/D)$ and the second term yields an integrand which vanishes except within a distance D of the

[20] Calculations show that the integral of Eq. (21) is quite insensitive to the precise form of $W(\epsilon)$. Even a density function constructed of a linear piece with slope of $\pi/2$ and a constant piece gives results differing very little from those of Eqs. (24) and (25).

[21] P. Egelstaff, J. Nuclear Energy **7**, 35 (1958).

[22] C. E. Porter (private communication).

line $\epsilon=\eta$. We approximate this part by integrating along the line $\epsilon=\eta$ and multiplying the result by D. The results have been plotted in Fig. 4 for the cases where $\Gamma_1=\Gamma_2$ and where $\Gamma_2=0$. Other functions of this type appear in the higher order terms of $\{|\mathcal{U}|^2\}_{\rm av}$.

With these definitions we calculate the expression (21) subject to the requirement, due to the random sign assumption, that the $g_{\lambda c}$ occur only quadratically. We have then for the off-diagonal elements

$$\{\mathcal{U}_{cc'}^{11}\}_{\rm av}=\frac{2\pi i}{D}\left\langle G_\lambda^{cc'}G_\lambda^{cc'*}\frac{1}{i\Gamma_\lambda}\right\rangle=\frac{\pi}{2D}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}\right\rangle, \tag{27}$$

$$\{\mathcal{U}_{cc'}^{12}\}_{\rm av}=\frac{2\pi i}{D}\left\langle G_\lambda^{cc'}[G_\lambda^{cc'*}G_\mu^{c'c'*}+G_\mu^{cc*}G_\lambda^{cc'*}]\frac{1}{i\Gamma_\lambda}\int_{-\infty}^{+\infty}\frac{d\epsilon W(\epsilon)}{\epsilon+\frac{i}{2}(\Gamma_\lambda+\Gamma_\mu)}\right\rangle_{\lambda\neq\mu}$$
$$=-\frac{\pi^2}{4D^2}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}(\Gamma_{\mu c}+\Gamma_{\mu c'})\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right\rangle_{\lambda\neq\mu}. \tag{28}$$

Assuming the number of resonances averaged over to be very large, we may disregard the restriction on the average in Eq. (28). This restriction has also been dropped in the following expressions for $n+m=4$.

$$\{\mathcal{U}_{cc'}^{13}\}_{\rm av}=\frac{\pi^2}{8D^2}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}\left\{\sum_{c''}\frac{\Gamma_{\lambda c''}\Gamma_{\mu c''}}{\Gamma_\lambda}\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)-2\frac{\Gamma_{\mu c}\Gamma_{\mu c'}}{\Gamma_\lambda+\Gamma_\mu}\Psi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)+\frac{\pi}{D}\Gamma_{\mu c}\Gamma_{\nu c'}\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\Phi\left(\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)\right.\right.$$
$$\left.\left.+\frac{\pi}{D}(\Gamma_{\mu c}\Gamma_{\nu c}+\Gamma_{\mu c'}\Gamma_{\nu c'})X\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D},\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)\right\}\right\rangle, \tag{29}$$

$$\{\mathcal{U}_{cc'}^{22}\}_{\rm av}=\frac{\pi^2}{8D^2}\left\langle\sum_{c''}\frac{\Gamma_{\lambda c}\Gamma_{\lambda c''}\Gamma_{\mu c'}\Gamma_{\mu c''}}{\Gamma_\lambda\Gamma_\mu}2\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right\rangle+\frac{\pi^3}{8D^3}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}[\Gamma_{\mu c'}\Gamma_{\nu c'}+\Gamma_{\mu c'}\Gamma_{\nu c}+\Gamma_{\mu c}\Gamma_{\nu c'}+\Gamma_{\mu c}\Gamma_{\nu c}]\right.$$
$$\left.\times\left[X\left(\frac{\Gamma_\mu+\Gamma_\lambda}{2D},\frac{\Gamma_\nu+\Gamma_\lambda}{2D}\right)-X\left(\frac{\Gamma_\mu-\Gamma_\lambda}{2D},\frac{\Gamma_\nu+\Gamma_\lambda}{2D}\right)-\Phi\left(\frac{\Gamma_\lambda-\Gamma_\mu}{2D}\right)\Phi\left(\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)\right]\right\rangle. \tag{30}$$

The corresponding expressions for the diagonal elements are

$$\{\mathcal{U}_{cc}^{11}\}_{\rm av}=\frac{\pi}{2D}\left\langle\frac{\Gamma_{\lambda c}^2}{\Gamma_\lambda}+\frac{\pi}{D}\Gamma_{\lambda c}\Gamma_{\mu c}\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right\rangle, \tag{31}$$

$$\{\mathcal{U}_{cc}^{12}\}_{\rm av}=-\frac{\pi^2}{2D^2}\left\langle\frac{\Gamma_{\lambda c}^2\Gamma_{\mu c}}{\Gamma_\lambda}\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)-\frac{\pi}{2D}\Gamma_{\lambda c}\Gamma_{\mu c}\Gamma_{\nu c}X\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D},\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)\right\rangle, \tag{32}$$

$$\{\mathcal{U}_{cc}^{13}\}_{\rm av}=\frac{\pi^2}{8D^2}\left\langle\frac{\Gamma_{\lambda c}^2}{\Gamma_\lambda}\sum_{c'}\frac{\Gamma_{\lambda c'}\Gamma_{\mu c'}}{\Gamma_\lambda}\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)-2\frac{\Gamma_{\lambda c}\Gamma_{\mu c}}{\Gamma_\lambda}\sum_{c'}\frac{\Gamma_{\lambda c'}\Gamma_{\mu c'}}{\Gamma_\lambda+\Gamma_\mu}\Psi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right.$$
$$\left.+3\frac{\pi}{D}\frac{\Gamma_{\lambda c}^2}{\Gamma_\lambda}\Gamma_{\mu c}\Gamma_{\nu c}X\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D},\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)+\text{terms of order}\left(\frac{\pi}{D}\Gamma_{\lambda c}\right)^4\right\rangle, \tag{33}$$

$$\{\mathcal{U}_{cc}^{22}\}_{\rm av}=\frac{\pi^2}{8D^2}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\mu c}}{\Gamma_\lambda}\sum_{c'}\frac{\Gamma_{\lambda c'}\Gamma_{\mu c'}}{\Gamma_\mu}2\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)+4\frac{\pi}{D}\frac{\Gamma_{\lambda c}^2}{\Gamma_\lambda}\Gamma_{\mu c}\Gamma_{\nu c}\left[X\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D},\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)-X\left(\frac{\Gamma_\mu-\Gamma_\lambda}{2D},\frac{\Gamma_\nu+\Gamma_\lambda}{2D}\right)\right.\right.$$
$$\left.\left.-\Phi\left(\frac{\Gamma_\lambda-\Gamma_\mu}{2D}\right)\Phi\left(\frac{\Gamma_\lambda+\Gamma_\nu}{2D}\right)\right]+\text{terms of order}\left(\frac{\pi}{D}\Gamma_{\lambda c}\right)^4\right\rangle. \tag{34}$$

Additional terms of the second order in $\Gamma_{\lambda c}/D$ occur in all quantities $\{\mathcal{U}^{1m}\}_{\rm av}$. Of these the only ones which make appreciable contributions are those due to $\{\mathcal{U}^{14}\}_{\rm av}$ and $\{\mathcal{U}^{15}\}_{\rm av}$. These terms are, respectively, for both diagonal and off-diagonal elements,

$$\frac{\pi^2}{8D^2}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}\sum_{c''}\frac{\Gamma_{\lambda c''}\Gamma_{\mu c''}}{\Gamma_\lambda}\frac{(\Gamma_{\mu c'}+\Gamma_{\mu c})}{\Gamma_\lambda+\Gamma_\mu}\Psi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right\rangle \tag{35}$$

and

$$\frac{\pi^2}{16D^2}\left\langle\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{\Gamma_\lambda}\sum_{c''}\frac{\Gamma_{\lambda c''}\Gamma_{\mu c''}}{\Gamma_\lambda}\sum_{c'''}\frac{\Gamma_{\lambda c'''}\Gamma_{\mu c'''}}{\Gamma_\lambda}\frac{1}{\Gamma_\lambda+\Gamma_\mu}\Psi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\right\rangle. \tag{36}$$

By substituting Eqs. (27)–(36) into Eq. (19) we obtain an expansion of the average of $|\mathfrak{U}|^2$ in ascending powers of $\langle\tau_{\lambda c}\rangle$. The convergence of this expansion is of course governed by the results of the Appendix. However, here we are not so much interested in the mere fact of convergence as in sufficiently rapid convergence so that the first few terms yield a good approximation. In general this can be established by inspection of successive terms.[23]

When the argument of the functions Φ or Ψ or X is $(\Gamma_\lambda+\Gamma_\mu)/2D$, we may take advantage of the slow variations of the functions as well as the small fluctuations of sums of widths—even for small numbers of channels—to approximate the values of these functions by their values for the average of the argument, e.g.,

$$\Phi\left(\frac{\Gamma_\lambda+\Gamma_\mu}{2D}\right)\sim\Phi\left(\frac{\langle\Gamma_\lambda\rangle}{D}\right). \tag{37}$$

In the following cross-section expressions we make use of Eq. (37). In addition we set $\Phi[(\Gamma_\mu-\Gamma_\nu)/2D]$ equal to zero and do not specify the argument $\langle\Gamma\rangle/D$ of Φ, Ψ.

We thus obtain for the reaction cross section in units of $\pi\lambda\!\!\!^{-}{}_c^2 g_c$ [Eq. (2d)], using Eqs. (3), (18), (27)–(30), and (37),

$$\begin{aligned}\bar{\sigma}_{cc'}=|U_{cc'}{}^0|^2+\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}}{\tau_\lambda}\right\rangle\Big\{1-\tfrac{1}{2}\langle\tau_{\lambda c}+\tau_{\lambda c'}\rangle\Phi-\frac{1}{4}\frac{\langle\tau_{\lambda c}\rangle\langle\tau_{\lambda c'}\rangle}{\langle\tau_\lambda\rangle}\Psi+\tfrac{1}{8}\langle\tau_{\lambda c}\rangle\langle\tau_{\lambda c'}\rangle\Phi^2\\+\tfrac{1}{8}[\langle\tau_{\lambda c}\rangle^2+\langle\tau_{\lambda c'}\rangle^2]\mathrm{X}(\langle\Gamma\rangle/D,\langle\Gamma\rangle/D)+\tfrac{1}{16}\langle\tau_{\lambda c}+\tau_{\lambda c'}\rangle^2[\mathrm{X}(\langle\Gamma\rangle/D,\langle\Gamma\rangle/D)-\mathrm{X}(0,\langle\Gamma\rangle/D)]\Big\}\\+\tfrac{1}{4}\sum_{c''}\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c''}}{\tau_\lambda}\right\rangle\left\langle\frac{\tau_{\lambda c'}\tau_{\lambda c''}}{\tau_\lambda}\right\rangle\Phi+\tfrac{1}{4}\sum_{c''}\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}\tau_{\lambda c''}}{\tau_\lambda{}^2}\right\rangle\langle\tau_{\lambda c''}\rangle\left[\Phi+\frac{1}{2}\frac{\langle\tau_{\lambda c}+\tau_{\lambda c'}\rangle}{\langle\tau_\lambda\rangle}\Psi\right]\\+\tfrac{1}{16}\sum_{c''}\sum_{c'''}\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}\tau_{\lambda c''}\tau_{\lambda c'''}}{\tau_\lambda{}^3}\right\rangle\frac{\langle\tau_{\lambda c''}\rangle\langle\tau_{\lambda c'''}\rangle}{\langle\tau_\lambda\rangle}\Psi+\cdots.\end{aligned} \tag{38}$$

For the fluctuation cross section which may be used to obtain both the elastic and nonelastic cross sections we obtain in units of $\pi\lambda\!\!\!^{-}{}_c^2 g_c$:

$$\begin{aligned}\sigma_c(\text{fluctuation})=4[\{|\mathfrak{U}_{cc}|^2\}_{\text{av}}-\{|\mathfrak{U}_{cc}|\}_{\text{av}}{}^2]=\langle\tau_{\lambda c}{}^2/\tau_\lambda\rangle\{1-\langle\tau_{\lambda c}\rangle\Phi+\langle\tau_{\lambda c}\rangle^2[\tfrac{5}{8}\mathrm{X}(\langle\Gamma\rangle/D,\langle\Gamma\rangle/D)-\tfrac{1}{4}\mathrm{X}(0,\langle\Gamma\rangle/D)]\}\\+\langle\tau_{\lambda c}\rangle^2[\tfrac{1}{2}\Phi-\tfrac{1}{4}]+\tfrac{1}{4}\sum_{c'}\left\langle\frac{\tau_{\lambda c}{}^2\tau_{\lambda c'}}{\tau_\lambda{}^2}\right\rangle\langle\tau_{\lambda c'}\rangle\Phi-\tfrac{1}{4}\sum_{c'}\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}}{\tau_\lambda}\right\rangle\frac{\langle\tau_{\lambda c'}\rangle\langle\tau_{\lambda c}\rangle}{\langle\tau_\lambda\rangle}\Psi+\tfrac{1}{4}\sum_{c'}\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}}{\tau_\lambda}\right\rangle^2\Phi\\+\tfrac{1}{4}\sum_{c'}\left\langle\frac{\tau_{\lambda c}{}^2\tau_{\lambda c'}}{\tau_\lambda{}^2}\right\rangle\frac{\langle\tau_{\lambda c'}\rangle\langle\tau_{\lambda c}\rangle}{\langle\tau_\lambda\rangle}\Psi+\tfrac{1}{16}\sum_{c'}\sum_{c''}\left\langle\frac{\tau_{\lambda c}{}^2\tau_{\lambda c'}\tau_{\lambda c''}}{\tau_\lambda{}^3}\right\rangle\frac{\langle\tau_{\lambda c'}\rangle\langle\tau_{\lambda c''}\rangle}{\langle\tau_\lambda\rangle}\Psi+\tfrac{1}{4}\langle\tau_{\lambda c}\rangle^3\mathrm{X}(\langle\Gamma\rangle/D,\langle\Gamma\rangle/D)+\cdots.\end{aligned} \tag{39}$$

V. DISCUSSION AND APPLICATIONS

The leading resonance terms in Eqs. (38) and (39) resemble in form the Hauser-Feshbach relations which are based on the notion of independence of formation and decay of the compound nucleus on the average.[24] These latter formulas are

$$\bar{\sigma}_{cc'}{}^{(\text{H.F.})}=T_cT_{c'}/\textstyle\sum_{c''}T_{c''}, \tag{40}$$

$$\bar{\sigma}_c{}^{(\text{H.F.})}(\text{compound elastic})=T_c^2/\textstyle\sum_{c'}T_{c'}. \tag{41}$$

Our results are seen to differ from these relations in three ways. First, they differ by the appearance of the $\tau_{\lambda c}$ instead of the transmission coefficients T_c. This distinction has been discussed in Sec. III. Secondly, the function of average resonance parameters in Eqs. (40)

[23] One type of term which might be thought to give difficulty occurs starting with $n=3$ and has the form

$$2\pi D^{-1}(\Gamma_\lambda-\Gamma_\mu)^{-1}\Psi(\Gamma_\lambda-\Gamma_\mu/2D),\quad \lambda\neq\mu,$$

which diverges logarithmically as $\Gamma_\lambda-\Gamma_\mu\to 0$ as a direct consequence of the fact that the spacing distribution law is linear for small spacings. When averaged over the distribution of width differences the result is, of course, finite and becomes large only for very narrow distributions. From the discussion in Sec. II we find that for large numbers n of channels (where one might fear a narrow distribution) the dispersion of the distribution of $\Gamma_\lambda-\Gamma_\mu$ is of the order of $2\langle\Gamma_\lambda\rangle n^{-\frac{1}{2}}\sim 2\langle\langle\Gamma_{\lambda c}\rangle\rangle_c n^{\frac{1}{2}}$ and is therefore expected to increase with increasing numbers of channels.

[24] W. Hauser and H. Feshbach, Phys. Rev. 87, 366 (1952).

and (41) is replaced by an average of the function in the leading terms of Eqs. (38) and (39). This "width fluctuation effect" has been discussed by a number of authors.[25–27] Thirdly, there are the correction terms due to the higher order terms in the expansion of $\mathfrak{U}$. We shall discuss the consequences of Eqs. (38) and (39) by comparison with Eqs. (40) and (41).

In order to perform the required averages of the functions of τ which occur in Eqs. (38) and (39) it is necessary to integrate these functions over each partial width, weighting the integral with the normalized Porter-Thomas distribution function [Eq. (7)]. These multiple integrals can easily be brought into the form of the following single integral.

$$\frac{\langle \tau_1^{k_1}\tau_2^{k_2}\cdots\tau_n^{k_n}/\tau^m\rangle}{\langle\tau_1\rangle^{k_1}\langle\tau_2\rangle^{k_2}\cdots\langle\tau_n\rangle^{k_n}/\langle\tau\rangle^m}=\frac{1}{(m-1)!}\int_0^\infty dt\, t^{m-1} \times\prod_{i=1}^{n}(2k_i-1)!!\,[1+2t\langle\tau_i\rangle/\langle\tau\rangle]^{-k_i-\frac{1}{2}}, \quad (42)$$

where $(2k-1)!!=1\times3\times5\times\cdots(2k-1)$ and $(-1)!!\equiv1$. This expression can also be generalized for other distribution functions of the χ^2 class discussed by Porter and Thomas.[10] For a channel with widths distributed according to a χ^2 distribution with ν degrees of freedom, one replaces the corresponding factor $(2k_i-1)!![1+2t\langle\tau_i\rangle/\langle\tau\rangle]^{-k_i-\frac{1}{2}}$ in the integrand by $\Gamma(\nu/2+k_i)[\Gamma(\nu/2)]^{-1}(\nu/2)^{-k_i}[1+2t\langle\tau_i\rangle/\nu\langle\tau\rangle]^{-\nu/2-k_i}$. In the case of radiative capture channels it is generally satisfactory to replace all capture widths by a single nonfluctuating width.[28] In that case the above factor for the capture channel γ may be replaced with $\exp[-t\langle\tau_\gamma\rangle/\langle\tau\rangle]$. Some of these integrals have been given in numerical and graphical form[25] and numerical machine codes for their evaluation exist.[26,27] Graphs of two types of averages which are of interest in inelastic neutron scattering are shown in Fig. 5. These are the two-channel case and the three-channel case, where the third channel is nonfluctuating. It is seen there that for two channels with equal average widths, one obtains a maximum fluctuation effect of

$$\frac{\langle\tau_{\lambda c}\tau_{\lambda c'}/(\tau_{\lambda c}+\tau_{\lambda c'})\rangle}{\langle\tau_{\lambda c}\rangle\langle\tau_{\lambda c'}\rangle/\langle\tau_{\lambda c}+\tau_{\lambda c'}\rangle}=\frac{1}{2}$$

and hence

$$\frac{\langle\tau_{\lambda c}^2/(\tau_{\lambda c}+\tau_{\lambda c'})\rangle}{\langle\tau_{\lambda c}\rangle^2/\langle\tau_{\lambda c}+\tau_{\lambda c'}\rangle}=\frac{3}{2}, \quad (43)$$

for $\langle\tau_{\lambda c}\rangle=\langle\tau_{\lambda c'}\rangle$. This result can be understood qualitatively by considering the fact that according to the Porter-Thomas distribution law, widths smaller than the average are much more likely to occur than widths which are larger than the average width. Hence under the assumption that the widths for channels c and c' are uncorrelated, the occurrence of a resonance λ, for which both $\tau_{\lambda c}$ and $\tau_{\lambda c'}$ are larger than their averages is unlikely. Now $\tau_{\lambda c}\tau_{\lambda c'}(\tau_{\lambda c}+\tau_{\lambda c'})^{-1}$ is always less than the smaller of $\tau_{\lambda c}$, $\tau_{\lambda c'}$ and therefore almost all resonances can be expected to contribute less than $\langle\tau_{\lambda c}\rangle$ to the average. However, $\tau_{\lambda c}^2(\tau_{\lambda c}+\tau_{\lambda c'})^{-1}$ approaches $\tau_{\lambda c}$ in value when $\tau_{\lambda c}$ is large and therefore there will be an appreciable number of resonances contributing a value greater than $\langle\tau_{\lambda c}\rangle$ to the second average.

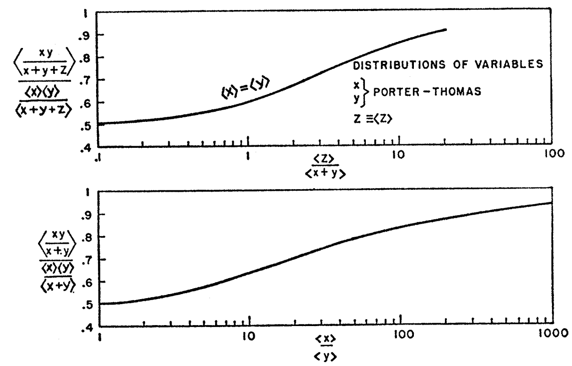

FIG. 5. Effect of width fluctuations on neutron inelastic scattering cross sections for two- and three-channel cases. Channel width-to-spacing ratios are denoted by x, y, and z.

In comparison to the Hauser-Feshbach relations the width fluctuation effect always tends to decrease the cross section for different entrance and exit channels and tends to increase the cross section for the same entrance and exit channels. The fluctuation effect for the more complicated functions occurring in the higher order terms in Eqs. (38) and (39) can, of course, go either way.

The effect of these higher order terms on $\bar{\sigma}_{cc'}$ has been calculated for the cases where c and c' are the only open channels and where there is a third channel with nonfluctuating widths and $\langle\tau_{\lambda c}\rangle=\langle\tau_{\lambda c'}\rangle$. It needs to be emphasized, however, that these results are only indicative of the order of magnitude of the correction and the convergence that may be expected. In cases differing appreciably in channel structure from those discussed here, the results may be quite different. Figure 6 shows the percent correction to the leading resonance term in Eq. (38) due to the higher order terms, for six cases falling in the above categories. The corrections are separated according to whether they arise from terms of the first order in any of the $\langle\tau_{\lambda c}\rangle$ or from terms of the second order. While the first-order contributions are essentially complete, second-order contributions are only those arising from $\{\mathfrak{U}^{13}\}_{av}$ and $\{\mathfrak{U}^{22}\}_{av}$. The contributions of second order from higher terms in the expansion may be significant for the larger values of $\langle\tau\rangle$ but are smaller than the calculated contributions. The following

[25] A. M. Lane and J. E. Lynn, Proc. Phys. Soc. (London) **A70**, 557 (1957).

[26] L. Dresner, Proceedings of the International Conference on Neutron Interactions with the Nucleus, Columbia University, 1957 [Columbia University Report CU-175, 1957 (unpublished)], p. 71.

[27] P. A. Moldauer, Bull. Am. Phys. Soc. **3**, 18 (1958) and **4**, 475 (1959), see also references 29 and 30.

[28] For exceptions, however, see R. T. Carpenter and L. M. Bollinger, Nuclear Phys. **21**, 66 (1960).

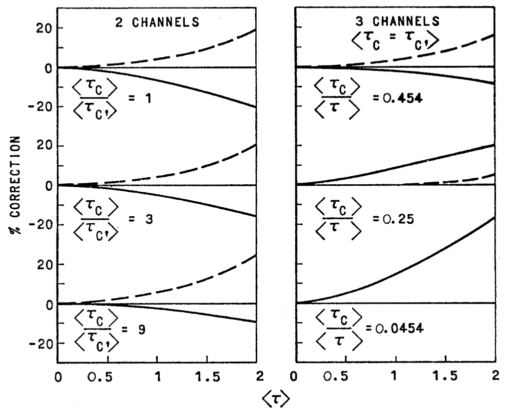

FIG. 6. Percent correction to the neutron inelastic cross section due to the higher-order terms in the scattering matrix expansion. $\langle\tau_c\rangle=2\pi\langle\Gamma_{\lambda c}\rangle/D$. $\langle\tau\rangle=\Sigma_c\langle\tau_c\rangle$.

general observations can be made from Fig. 6 and estimates of the higher contributions: (1) The expansion converges rapidly only when total and channel widths are not too large. The method can apparently be used fairly safely for $\langle\tau\rangle<1.5$ which is about half the value for which the expansion converges. (2) The correction terms are largest and positive when the widths of the participating channels are small compared to the total widths. (3) Magnitudes of the corrections in the range of rapid convergence go up to about 50%. They can therefore be comparable in importance to the fluctuation correction.

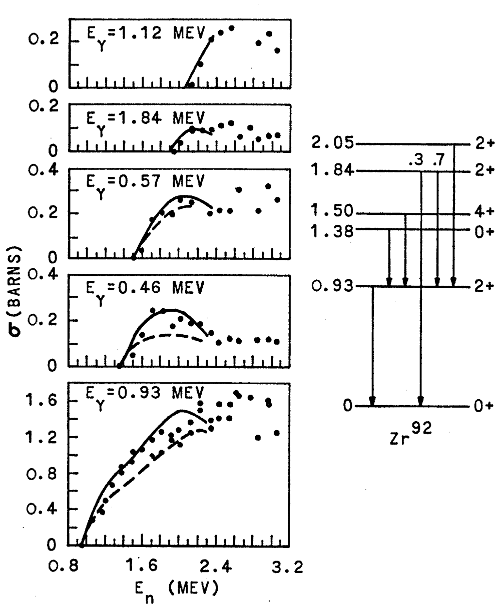

FIG. 7. Comparison of calculated with measured $Zr^{92}(n,n'\gamma)$ excitation curves. The solid curves are calculated according to Eq. (38). The dashed curves were computed using transmission coefficients. Optical model parameters of reference 32 were used throughout. The data points are by D. A. Lind and R. B. Day (see reference 33) and represent $4\pi[d\sigma(n,n'\gamma)/d\Omega](90°)$. The decay scheme was inferred from the data.

The corrections to the fluctuation cross section Eq. (39) are similar to the appropriate cases in Fig. 6 except that the second-order terms are somewhat smaller.

The above results have been used for the analysis of measured neutron inelastic scattering cross sections[29] and for the estimation of such cross sections.[30] Here we present only the results of calculations for neutron inelastic scattering from Zr^{92}, which is a particularly appropriate isotope.[31] According to the calculations of Campbell, Feshbach, Porter, and Weisskopf[32] their diffuse surface optical model gives a good account of both the total neutron cross section and the elastic scattering angular distribution for neutrons in the low Mev region scattered by Zr. The p-wave strength function according to that model is shown in Fig. 1. The s- and d-wave strength functions are very low. The f wave also has a giant resonance here but the f-wave neutron penetrability P_3 is down by at least a factor of ten from the p-wave penetrability P_1 for energies up to 2 Mev. This, together with the level structure of this even isotope (see Fig. 7) causes almost all fast neutron scattering to proceed through compound states of odd parity which decay predominantly through p-wave emission to the various levels of Zr^{92}. Furthermore, the p-wave strength function decreases with energy as more inelastic channels are opened, thus limiting the increase in total width. As a result, the equations derived above may be used with confidence up to the vicinity of 2 Mev. The results of such a calculation including neutron channels up to $l=6$ are compared in Fig. 7 with (n,n') γ-ray excitation data obtained by Lind and Day.[33] The dashed curves were obtained by using T_c instead of $\langle\tau_c\rangle$ in the formulas. The higher order correction terms give relatively small contributions in this case. Omission of the width fluctuation effect would cause the curves to rise much more steeply and give substantially higher cross sections. In these calculations the same optical model has been used to describe the excited states as was used for the ground state. This procedure generally appears to give satisfactory results.

The width fluctuation effect on average neutron

[29] A. B. Smith, Bull. Am. Phys. Soc. **5**, 19 (1960), A. B. Smith and P. A. Moldauer, *ibid.* **5**, 409 (1960). Results of these and other calculations will be published in the near future.

[30] P. A. Moldauer, Proceedings of the Conference on the Physics of Breeding, Argonne National Laboratory Report ANL-6122, 1959 (unpublished), p. 67; and S. Yiftah, D. Okrent, and P. A. Moldauer, *Fast Reactor Cross Sections* (Pergamon Press, New York, 1960).

[31] Entirely analogous results were obtained for Zr^{94}.

[32] E. J. Campbell, H. Feshbach, C. E. Porter, and V. F. Weisskopf, Massachusetts Institute of Technology Laboratory for Nuclear Science Technical Report No. 73, 1960 (unpublished).

[33] D. A. Lind and R. B. Day (to be published). The data points in Fig. 7 represent 4π times the differential cross section at 90°. The angular distribution of the gamma rays tends to be peaked in the forward-backward directions for the $2+\rightarrow0+$ transitions and at 90° for the $2+\rightarrow2+$ transition. To obtain the correct total cross section the points shown for the 0.93- and 1.84-Mev gamma rays should be increased just above threshold by up to 30% and by less at higher energies [R. B. Day (private communication)].

capture cross sections has been discussed in the references of footnotes 25, 26, 27. It reduces the cross section by at most 32% when the neutron channel width is the same as the total radiation width and by less otherwise. The magnitude of the average neutron width affects the radiative capture cross section substantially only at energies where the neutron width is of the same order of magnitude as the total radiation width or less. Since Γ_γ/D is always quite small, the distinction between $\langle\tau_{\lambda c}\rangle$ and T_c in that energy region is negligible. That distinction may, however, affect the capture cross section above an inelastic threshold through its effect on the average total width. It may therefore tend to produce a somewhat sharper decline of the capture cross section above such thresholds. Finally, the effect of the higher order terms on the capture cross section is small because of the small values of Γ_γ/D.

ACKNOWLEDGMENTS

The author is indebted to Dr. D. A. Lind and Dr. R. B. Day for making available and permitting use of their data before publication.

APPENDIX: CONVERGENCE OF THE THOMAS EXPANSION

We shall discuss here convergence conditions for that part of the expansion in Eq. (9) containing the $g_{\lambda c}$ only quadratically. This restriction means that the general term

$$\sum_{\lambda_i\neq\lambda_{i+1}} A_{\lambda_1}\times A_{\lambda_2}\times\cdots\times A_{\lambda_n} \tag{A.1}$$

can be rearranged so as to consist of a sum of products of diagonal elements of the A matrices [Eq. (10)]. The magnitude of (A.1) is overestimated if we perform the implied sum over channels by evaluating each $A_\lambda{}^{cc}$ for the channel $c(\lambda)$ with the largest partial width $\Gamma_{\lambda c(\lambda)}$ and multiply the resulting product by an average correction factor to account for the terms arising from the presence of the other channels. In this way one obtains for (A.1)

$$\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle^n \sum_{\lambda_i\neq\lambda_{i+1}} \mathcal{A}_{\lambda_1}\mathcal{A}_{\lambda_2}\cdots\mathcal{A}_{\lambda_n}, \tag{A.2}$$

where

$$\mathcal{A}_\lambda=\frac{\frac{1}{2}i\Gamma_{\lambda c(\lambda)}}{E_\lambda-E-\frac{1}{2}i\Gamma_\lambda}. \tag{A.3}$$

Here $\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle$ may be looked upon as an average effective number of channels contributing to the total width of each resonance. Because of the large fluctuations in the channel widths, this effective channel number is always considerably smaller than the true number of channels, even when the average channel widths are of comparable magnitudes. Thus, for two channels of equal average partial widths, one obtains from the Porter Thomas distribution $\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle=\frac{1}{2}+2/\pi\sim1.137$. For large numbers of equal average width channels, one may estimate the effective number of channels to be of the order of one half the actual number of channels. This number is further reduced by any differences in the average partial widths.

To perform the sum over resonances in (A.2) we initially evaluate that expression at $E=E_\mu$ (μ is any one of the λ_i) and treat all resonances other than μ statistically, disregarding for these latter the restrictions on the sum over resonances. Again we thereby increase, if anything, the magnitude of the term. Substituting this expression immediately into Eq. (9), we have

$$\mathcal{U}_{cc}(E_\mu)\sim\sum_n \langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle^n \sum_p \binom{n-p+1}{p} \times[\mathcal{A}_\mu(E_\mu)]^p[\sum_{\lambda\neq\mu}\mathcal{A}_\lambda(E_\mu)]^{n-p}. \tag{A.4}$$

The binomial coefficient $\binom{n-p+1}{p}$ occurring here is just the number of ways in which $\mathcal{A}_\mu$ can occur p times as a factor in expression (A.2) without the occurrence of any adjacent factors $\mathcal{A}_\mu$.[34] Rearranging (A.4) by summing over p and $t=n-p+1$, one obtains

$$\mathcal{U}_{cc}(E_\mu)\sim\frac{\sum_t\{\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\sum_{\lambda\neq\mu}\mathcal{A}_\lambda(E_\mu)[1+\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\mathcal{A}_\mu(E_\mu)]\}^t}{\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\sum_{\lambda\neq\mu}\mathcal{A}_\lambda(E_\mu)}, \tag{A.5}$$

Here we have, by Eq. (A.3),

$$\mathcal{A}_\mu(E_\mu)=-\Gamma_{\mu c(\mu)}/\Gamma_\mu. \tag{A.6}$$

Provided there is no unusual accumulation of resonances at E_μ, the sum over other resonances can be treated statistically

$$\sum_{\lambda\neq\mu}\mathcal{A}_\lambda(E_\mu)=\left\langle\int_{-\infty}^{+\infty}d(E_\lambda-E_\mu)W(E_\lambda-E_\mu)\mathcal{A}_\lambda(E_\mu)\right\rangle_\lambda =-\frac{\pi}{2}\langle(\Gamma_{\lambda c(\lambda)}/D)\Phi(\Gamma_\lambda/2D)\rangle_\lambda, \tag{A.7}$$

where we have made use of Eq. (24). A condition for the convergence of $\mathcal{U}_{cc}$ at E_μ is now that the magnitude of the general term of the geometric series in Eq. (A.5) be less than unity. Substituting therefore (A.6) and (A.7) into (A.5) we obtain the condition

[34] To show this, call the desired number $C_p{}^n$ and write the recursion relation $C_p{}^n=C_p{}^{n-1}+C_{p-1}{}^{n-2}$. Iterating this $k-1$ times, one obtains

$$C_p{}^n=\sum_{m=0}^{k}\binom{k}{m}C_{p-k+m}{}^{n-2k+m}.$$

By extending the definition of C to $C_0{}^n=1$ and $C_{p<0}{}^n=0$ for all n and choosing $k=n-p+1$, one obtains the above result.

$$\left|-\frac{\pi}{2}\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\langle(\Gamma_{\lambda c(\lambda)}/D)\Phi(\Gamma_\lambda/2D)\rangle \times[1-\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\Gamma_{\mu c(\mu)}/\Gamma_\mu]\right|<1. \quad (A.8)$$

To remove the restrictions placed on the resonance structure at E_μ, we must consider the following possible cases: (1) A second resonance ν has an E_ν unusually close to E_μ. (2) A resonance lying close to E_μ has an unusually large width $\Gamma_{\lambda c(\lambda)}$. (3) $\mathfrak{U}_{cc}$ is evaluated at an off-resonance energy E. For cases 1 and 2 we must add to the factor $(\pi/2)\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\langle\Gamma_{\lambda c(\lambda)}/D\Phi\rangle$ the expressions $\Gamma_{\nu c(\nu)}/\Gamma_\nu$ or $\Gamma_{\nu c(\nu)}/2D$, respectively. Case 3 is accounted for by setting $\Gamma_{\mu c(\mu)}/\Gamma_\mu=0$.

As in Sec. IV we now take Φ outside the averaging sign and we also replace $\Gamma_{\lambda c(\lambda)}/D$ by its maximum value Γ_λ/D. Then, supposing $\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle$ to be less than 2, the expression $|1-\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle\Gamma_{\mu c(\mu)}/\Gamma_\mu|$ will always be less than 1. We find then from (A.8) the convergence condition

$$2\pi(\langle\Gamma_\lambda\rangle/D)\Phi(\langle\Gamma_\lambda\rangle/2D)<\frac{4}{\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle}. \quad (A.9)$$

Consulting Fig. 3, one sees that this is satisfied if

$$\langle\Gamma_\lambda\rangle/D<\tfrac{1}{2} \quad \text{for} \quad \langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle<2, \quad (A.10)$$

which is a reasonable limit on the effective channel number up to $\langle\Gamma_\lambda\rangle/D\sim1$. If the effective channel number is 1 then $\langle\Gamma_\lambda\rangle/D$ may approach unity. Also if $\langle\Gamma_\lambda\rangle/\langle\Gamma_{\lambda c(\lambda)}\rangle$ is appreciably larger than unity, the limit on $\langle\Gamma_\lambda\rangle/D$ is increased by a corresponding factor.

Of the three alternate conditions enumerated above, the third one does not affect our conclusions provided $\langle\Gamma_\lambda/\Gamma_{\lambda c(\lambda)}\rangle<2$, while the second one affects condition (A.10) only slightly. Because of the level repulsion effect, the first alternate condition may be expected to be applicable only when $\langle\Gamma_\lambda\rangle$ approaches D. It therefore gives a convergence limit of the same order of magnitude as (A.10). However as $\langle\Gamma_\lambda\rangle/D$ increases beyond unity, the overlapping resonances at any energy give rise to rapidly increasing numbers of terms making large contributions to expression (A.1), causing the series to diverge. In fact for $\langle\Gamma_\lambda\rangle\gg D$ the expression (A.1) approaches $(\langle\Gamma_\lambda\rangle/D)^n$ in order of magnitude.

In view of the several approximations made in deriving Eq. (A.5), it is reasonable to conclude that in general the series converges and the results of Sec. III are valid when the average total width is less than the average level spacing and the effective number of channels is less than about two.

A second method proposed by Thomas,[6] employing the channel elimination method, leads to an expansion of the R matrix which may under certain conditions have a much larger range of validity than that indicated above. However, in performing the inversion of this R-matrix expansion which is still necessary to obtain an explicit scattering matrix, one is again left with an expansion whose convergence characteristics are very similar to those obtained here.

PHYSICAL REVIEW VOLUME 123, NUMBER 3 AUGUST 1, 1961

Decay of Tm^{172}†

R. G. Helmer* and S. B. Burson
Argonne National Laboratory, Argonne, Illinois
(Received March 16, 1961)

The radioactive nuclide $_{69}Tm^{172}$ was produced by successive capture of two neutrons in erbium oxide enriched in Er^{170}. The irradiations were made in the Materials Testing Reactor at Arco, Idaho. In addition to three thulium activities, these samples contained six active contaminants. Pure thulium sources were obtained by use of an ion-exchange column. Studies were conducted with a 256-channel coincidence scintillation spectrometer. These measurements indicate the presence of at least 17 gamma-ray and 5 beta-ray transitions. The beta-ray spectrum was studied with a 180° magnetic beta-ray spectrometer. This spectrum was analyzed by use of a computer program compiled by the authors in collaboration with members of the Argonne Applied Mathematics Division. The level scheme proposed for Yb^{172} has states with energies, spins, and parities of $0.0(0^+)$, $0.079(2^+)$, $0.260(4^+)$, 1.17(3), 1.46(2), 1.54(3), 1.60(1), 1.64(?), and 1.73(3) Mev. The total decay energy is found to be 1.88 Mev. The experimental data are consistent with the previously proposed interpretation that the first two excited states are members of a $K=0$ rotational band based on the ground state. The states at 1.46 and 1.54 Mev are interpreted as members of a rotational band with $K=2$. The states at 1.60 and 1.73 Mev are tentatively interpreted as members of a rotational band with $K=0$ and negative parity. It is suggested that the state at 1.17 Mev has $K=3$. From the analysis of the beta spectrum it is concluded that the ground state of thulium has $I=K=2$ and negative parity.

INTRODUCTION

Previous Studies

THE radioactive nuclide $_{69}Tm^{172}$, which decays to Yb^{172} by β^- emission, has been reported by Nethaway *et al.*[1] They obtained this isotope from the decay of Er^{172} which was produced by two successive neutron captures in stable Er^{170}. The thulium activity was identified by its genetic relationship to the Er^{172} activity. The mass assignment was verified by time-of-flight isotopic separation. The half-life of thulium was

† Work performed under the auspices of the U. S. Atomic Energy Commission.

* This material has been submitted in partial fulfillment of the requirements for the Ph.D. degree at the University of Michigan, Ann Arbor, Michigan.

[1] D. R. Nethaway, M. C. Michel, and W. E. Nervik, Phys. Rev. **103**, 147 (1956).