When the amplitude A' is formed, there occurs a factor of $\exp[\Delta^*]$, which must be rewritten as $\exp[\Delta]S^{-1}(W^2)$. Now we further restrict our attention to the pole term in S^{-1}, which occurs at M_d^2. The other singularities need not be discussed further. Then the pole contribution to the absorptive part can be written as

$$A''(s)=\int_{2M}^{s^{\frac{1}{2}}-\mu} dW(W^2-4M^2)^{\frac{1}{2}}[(W^2-s-\mu^2)^2-4s\mu^2]^{\frac{1}{2}}$$
$$\times(W^2-M_d^2)^{-1}\int\frac{d\Omega}{2s^{\frac{1}{2}}}J^*(s,W,\Omega_f,\Omega)$$
$$\times J(s,W,\Omega,\Omega_i)\exp[2\Delta(W^2)]. \quad (5.12)$$

We now need to discuss the analyticity of A'' as a function of s. The singularity which is of interest to us is one of the endpoint singularities due to the pole at M_d^2. These occur at $(s^{\frac{1}{2}}-\mu)^2=M_d^2$. The branch point closest to the physical cut is $s^{\frac{1}{2}}=\mu+M_d$. If the coupling is now increased, this branch point moves in a path shown as the solid line in Fig. 7 (see also Fig. 6). Then the line integral over A'' from $(2M+\mu)^2$ to infinity must be deformed to the dotted line in Fig. 7 to avoid this oncoming branch cut. When this deformed integral is collapsed to the real axis, it can be rewritten as a line integral from $(M_d+\mu)^2$ to infinity. Thus the correct two particle cut has been generated in the same manner as an anomalous threshold.

Similar statements hold for all the higher inelastic states. Thus, if our assumptions about the structure of production amplitudes are true, a new particle of mass M_d has been added to the mass spectrum.

This argument concerning bound states can also be used to clarify the problem of unstable particles. First, assume that S_l has a complex zero at M^{*2}, which is near the physical cut and produces a scattering resonance. We have seen that if such a pole exists on the second sheet across the elastic cut, then there is a branch cut starting at $s=(M^*+\mu)^2$ on the unphysical sheet across the three-particle branch line. This cut can be drawn parallel to the real axis toward plus infinity, if we like. One possible interpretation which is consistent with the identification of M^{*2} as a pole due to a one (unstable) particle state is that this latter branch line singularity represents the rescattering of a pion with the unstable particle in the intermediate state. If there is a resonance in the three-particle system $(n+p+\pi)$, then it should show up in the s dependence of the function J. One would like J^* to have a simple pole in order to be consistent with the interpretation in the two-particle case. It still is not clear that this is a consistent and/or unique interpretation of these types of singularities.

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Excitation Spectrum in Many-Boson Systems*

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When only two-body correlations are fully taken into account, there appears an energy gap in the excitation spectrum for many-boson systems as shown by Girardeau and Arnowitt and confirmed by Wentzel. This energy gap is shown to disappear and the spectrum to become phononlike again and proportional to the momentum for small momentum, if we construct the eigenmodes of excitations (collective excitations), taking into consideration appropriate higher-order terms.

1. INTRODUCTION

MANY authors have studied the many-boson system and showed the presence of the phonon-roton type excitation spectrum as in the actual system of liquid helium. Especially, Bogoliubov[1] noticed the fact that the occupation number of the zero-momentum state was macroscopically large, and treated the quantum amplitude for this state $a_0^\dagger$, a_0 as a classical number $N_0^{\frac{1}{2}}$. He could get, then, the phononlike spectrum by diagonalizing the quadratic terms of $a_{\mathbf{k}}^\dagger$, $a_{\mathbf{k}}(|\mathbf{k}|\neq 0)$ in the Hamiltonian. Brueckner and Sawada[2] used essentially the same method and confirmed this result.

Recently, Girardeau and Arnowitt[3] showed that there appeared an energy gap in the excitation spectrum if one used the best trial function which fully took into consideration two-body correlations. This result was confirmed by Wentzel[4] who used a slightly different method.

Though they obtained a better ground-state energy

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[1] N. N. Bogoliubov, J. Phys. (U.S.S.R.) **9**, 23 (1947).
[2] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117, 1128 (1957).
[3] M. Girardeau and R. Arnowitt, Phys. Rev. **113**, 755 (1959).
[4] G. Wentzel, Phys. Rev. **120**, 1572 (1960).

than the previous authors, the existence of an energy gap in the excitation spectrum does not agree with the phonon spectrum in liquid helium, and moreover, is in contradiction to the theorem by Hugenholtz and Pines,[5] which states that the excitation spectrum of many-boson systems with repulsive interactions should vanish at the origin ($k=0$). Therefore, as Girardeau[6] suggested, this energy gap should vanish in a higher approximation in which more than two-body correlations are taken into consideration.

The purpose of this paper is to show that this energy gap does in fact vanish in a higher approximation. In Sec. 2, we show that the results of Girardeau and Arnowitt[2] can be obtained in a simple generalization of Bogoliubov's method. In Sec. 3, we include higher approximations in that we construct the eigenmodes for collective excitations. We solve the secular equation in the vicinity of zero momentum and show that the energy spectrum is linear in momentum (phonon spectrum).

2. ENERGY GAP

The Hamiltonian for a many-boson system is of the form

$$H=\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+\tfrac{1}{2}\sum_{\mathbf{q}}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} V_{\mathbf{q}} a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'-\mathbf{q}}^\dagger a_{\mathbf{k}'} a_{\mathbf{k}}, \quad (1)$$

where $\epsilon_{\mathbf{k}}$ is the kinetic energy,

$$\epsilon_{\mathbf{k}}=\hbar^2k^2/2m, \quad (2)$$

and $V_{\mathbf{q}}$ is the Fourier transform of the interaction potential,

$$V_{\mathbf{q}}=(1/\Omega)\int V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^3r. \quad (3)$$

The symbols $a_{\mathbf{k}}^\dagger$, $a_{\mathbf{k}}$ denote the usual creation and annihilation operators

$$[a_{\mathbf{k}},a_{\mathbf{k}}^\dagger]=\delta_{\mathbf{k},\mathbf{k}'},\quad [a_{\mathbf{k}},a_{\mathbf{k}'}]=[a_{\mathbf{k}}^\dagger,a_{\mathbf{k}'}^\dagger]=0.$$

We can expect that the occupation number of the zero-momentum state $N_0=a_0^\dagger a_0$ is very large, and so the commutator $[a_0,a_0^\dagger]$ can be neglected compared with $a_0^\dagger a_0$ itself. Noting that we are dealing with the system with constant number of particles, we can replace $a_0^\dagger$ and a_0 by an operator $(N-\sum_{\mathbf{k}\neq 0}a_{\mathbf{k}}^\dagger a_{\mathbf{k}})^{\frac{1}{2}}$, as was done by Brueckner and Sawada[2] and Wentzel.[4] The verification of this procedure will be given in the Appendix.

Then, the Hamiltonian (1) can be written as

$$H=U_0+H_2+H_3+H_4, \quad (4)$$

where

$$\begin{aligned}
U_0&=\tfrac{1}{2}V_0N^2,\\
H_2&=\textstyle\sum_{\mathbf{k}}\{[\epsilon_{\mathbf{k}}+NV_{\mathbf{k}}]a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+\tfrac{1}{2}NV_{\mathbf{k}}[a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger+a_{-\mathbf{k}}a_{\mathbf{k}}]\},\\
H_3&=\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\sum_{\mathbf{k}'}(N-\sum a_{\mathbf{k}}^\dagger a_{\mathbf{k}})^{\frac{1}{2}}(V_{\mathbf{k}}+V_{\mathbf{k}'})\\
&\quad\times[a_{\mathbf{k}+\mathbf{k}'}^\dagger a_{\mathbf{k}} a_{\mathbf{k}'}+a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}+\mathbf{k}'}], \quad (4a)\\
H_4&=\tfrac{1}{2}\textstyle\sum_{\mathbf{q}}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} V_{\mathbf{q}} a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'-\mathbf{q}}^\dagger a_{\mathbf{k}'} a_{\mathbf{k}}\\
&\quad-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\{(V_0+V_{\mathbf{k}}+V_{\mathbf{k}'})a_{\mathbf{k}}^\dagger a_{\mathbf{k}} a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'}\\
&\quad+V_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger+a_{-\mathbf{k}}a_{\mathbf{k}})a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'}\}.
\end{aligned}$$

The diagonalization of H_2 can be easily achieved by the Bogoliubov transformation,

$$\begin{aligned}
a_{\mathbf{k}}&=u_{\mathbf{k}}\alpha_{\mathbf{k}}-v_{\mathbf{k}}\alpha_{-\mathbf{k}}^\dagger,\\
a_{-\mathbf{k}}&=u_{\mathbf{k}}\alpha_{-\mathbf{k}}-v_{k}\alpha_{\mathbf{k}}^\dagger, \quad (5)\\
u_{\mathbf{k}}^2&-v_{\mathbf{k}}^2=1,
\end{aligned}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are given by

$$\begin{aligned}
u_{\mathbf{k}}^2&=\tfrac{1}{2}[(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}}/E_{\mathbf{k}})+1],\\
v_{\mathbf{k}}^2&=\tfrac{1}{2}[(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}}/E_{\mathbf{k}})-1], \quad (6)
\end{aligned}$$

and

$$E_{\mathbf{k}}=[\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}})]^{\frac{1}{2}}. \quad (7)$$

The part of the Hamiltonian labeled H_2 becomes

$$H_2=\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\{E_{\mathbf{k}}-(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}})\}+\sum_{\mathbf{k}}E_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}. \quad (8)$$

Thus, $E_{\mathbf{k}}$, the excitation energy of the quasi-particles, gives the excitation spectrum of the whole system in this approximation. As is seen from (7), for small $|\mathbf{k}|$, $E_{\mathbf{k}}\propto|\mathbf{k}|$, and for large $|\mathbf{k}|$, $E_{\mathbf{k}}\propto\epsilon_{\mathbf{k}}$, similar behavior as for the phonon-roton spectrum in liquid helium. This is the result given by Bogoliubov[1] and Brueckner and Sawada.[2]

The transformation (5), however, affects the term H_3, H_4 in the Hamiltonian (4); especially, if we rearrange every term into normal products with respect to $\alpha^\dagger$ and α, there appear extra terms from H_4 independent of and quadratic in $\alpha^\dagger$ and α. Thus, a better approximation may be obtained by taking into consideration these extra terms in the determination of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$.

After carrying out the transformation (5) and rearranging each term, the Hamiltonian (4) becomes

$$H=U_0'+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4, \quad (9)$$

where

$$\begin{aligned}
U_0'&=\tfrac{1}{2}V_0N^2+\textstyle\sum_{\mathbf{k}}[v_{\mathbf{k}}^2(\epsilon_{\mathbf{k}}+2NV_{\mathbf{k}})-2NV_{\mathbf{k}}(uv)_{\mathbf{k}}]+\sum_{\mathbf{k}}\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}-\mathbf{k}'})v_{\mathbf{k}}^2v_{\mathbf{k}'}^2\\
&\quad-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}}+V_{\mathbf{k}'})v_{\mathbf{k}}^2v_{\mathbf{k}'}^2-\sum_{\mathbf{k}}\sum_{\mathbf{k}'}V_{\mathbf{k}-\mathbf{k}'}(uv)_{\mathbf{k}}(uv)_{\mathbf{k}'}+\sum_{\mathbf{k}}\sum_{\mathbf{k}'}V_{\mathbf{k}}(uv)_{\mathbf{k}}v_{\mathbf{k}'}^2, \quad (9a)\\
\mathcal{H}_2&=\textstyle\sum_{\mathbf{k}}\{(u_{\mathbf{k}}^2+v_{\mathbf{k}}^2)[(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}})+\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}-\mathbf{k}'})v_{\mathbf{k}'}^2-\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}}+V_{\mathbf{k}'})v_{\mathbf{k}'}^2+\sum_{\mathbf{k}'}V_{\mathbf{k}'}(uv)_{\mathbf{k}'}]\\
&\quad-2(uv)_{\mathbf{k}}[NV_{\mathbf{k}}-\textstyle\sum_{\mathbf{k}'}V_{\mathbf{k}-\mathbf{k}'}(uv)_{\mathbf{k}'}-V_{\mathbf{k}}\sum_{\mathbf{k}'}v_{\mathbf{k}'}^2]\}\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+\sum_{\mathbf{k}}\{-(uv)_{\mathbf{k}}[(\epsilon_{\mathbf{k}}+NV_{\mathbf{k}})+\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}-\mathbf{k}'})v_{\mathbf{k}'}^2\\
&\quad-\textstyle\sum_{\mathbf{k}'}(V_0+V_{\mathbf{k}}+V_{\mathbf{k}'})v_{\mathbf{k}'}^2+\sum_{\mathbf{k}'}V_{\mathbf{k}'}(uv)_{\mathbf{k}'}]+\tfrac{1}{2}(u_{\mathbf{k}}^2+v_{\mathbf{k}}^2)[NV_{\mathbf{k}}-\sum_{\mathbf{k}'}V_{\mathbf{k}-\mathbf{k}'}(uv)_{\mathbf{k}'}-V_{\mathbf{k}}\sum_{\mathbf{k}'}v_{\mathbf{k}'}^2]\}\\
&\quad\times[\alpha_{\mathbf{k}}^\dagger\alpha_{-\mathbf{k}}^\dagger+\alpha_{-\mathbf{k}}\alpha_{\mathbf{k}}]. \quad (9b)
\end{aligned}$$

[5] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).
[6] M. Girardeau, Phys. Rev. **115**, 1090 (1959).

The explicit expressions for $\mathcal{H}_3$ and $\mathcal{H}_4$ are not necessary here, and will be shown when necessary.

We determine the parameters $u_\mathbf{k}$ and $v_\mathbf{k}$ so as to eliminate the nondiagonal term $\alpha_\mathbf{k}^\dagger\alpha_{-\mathbf{k}}^\dagger+\alpha_{-\mathbf{k}}\alpha_\mathbf{k}$ in $\mathcal{H}_2$. It is to be noted that this procedure is nothing but "the principle of elimination of the dangerous terms" proposed by Bogoliubov[7] in his theory of superconductivity.

In order to simplify the expression, we introduce the following quantities:

$$\begin{aligned} K&\equiv\sum_{\mathbf{k}'} v_{\mathbf{k}'}^2,\\ I_\mathbf{k}&\equiv\sum_{\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}v_{\mathbf{k}'}^2,\\ J_\mathbf{k}&\equiv\sum_{\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}(uv)_{\mathbf{k}'},\end{aligned} \tag{10}$$

and

$$\begin{aligned} f_\mathbf{k}&\equiv \epsilon_\mathbf{k}+(N-K)V_\mathbf{k}+(I_\mathbf{k}-I_0)+J_0,\\ g_\mathbf{k}&\equiv(N-K)V_\mathbf{k}-J_\mathbf{k}.\end{aligned} \tag{11}$$

Then, the equation determining $u_\mathbf{k}$ and $v_\mathbf{k}$ can be written as

$$\begin{aligned} u_\mathbf{k}^2&=\tfrac{1}{2}[(f_\mathbf{k}/E_\mathbf{k}')+1],\\ v_\mathbf{k}^2&=\tfrac{1}{2}[(f_\mathbf{k}/E_\mathbf{k}')-1],\\ (uv)_\mathbf{k}&=g_\mathbf{k}/2E_\mathbf{k}',\end{aligned} \tag{12}$$

and

$$E_\mathbf{k}'=[f_\mathbf{k}^2-g_\mathbf{k}^2]^{\frac{1}{2}}. \tag{13}$$

Since $f_\mathbf{k}$ and $g_\mathbf{k}$ involve the summation of $v_\mathbf{k}^2$ and $(uv)_\mathbf{k}$, Eq. (12) is a complicated integral equation for $u_\mathbf{k}$ and $v_\mathbf{k}$.

When $u_\mathbf{k}$ and $v_\mathbf{k}$ are thus determined, $\mathcal{H}_2$ becomes

$$\mathcal{H}_2=\sum_\mathbf{k} E_\mathbf{k}'\alpha_\mathbf{k}^\dagger\alpha_\mathbf{k}. \tag{14}$$

Therefore, $E_\mathbf{k}'$ expresses the energy of the quasi-particle in this approximation, but this does not vanish even if $|\mathbf{k}|\to 0$. That is, if $|\mathbf{k}|\to 0$,

$$\begin{aligned} f_\mathbf{k}&\to V_0(N-K)+J_0,\\ g_\mathbf{k}&\to V_0(N-K)-J_0,\end{aligned}$$

and

$$E_\mathbf{k}'\to[4V_0(N-K)J_0]^{\frac{1}{2}}. \tag{15}$$

This gives the magnitude of the energy gap.

It is to be noted that the condition for the minimum ground-state energy gives the same equations for $u_\mathbf{k}$ and $v_\mathbf{k}$ as (12). Thus, the ground-state energy in this case is lower than that of the previous case, Eq. (6). This result is essentially the same as that of Girardeau and Arnowitt,[3] who employed the variational method. The method employed by Wentzel is almost the same as ours, but his Hamiltonian is restricted and cannot be used in higher approximations as will be shown in the next section.

3. COLLECTIVE EXCITATIONS

Secular Equation

The purpose of this note is to point out that the result of the previous section, namely the appearance of an energy gap, is a consequence of the inclusion of only two-body correlations, and hence it is not sufficient to explain the phonon spectrum in liquid helium. The occurrence of an energy gap is in contradiction with a theorem by Hugenholtz and Pines,[5] as mentioned in the introduction. We show that the energy gap does vanish when appropriate higher-order terms in the Hamiltonian are taken into consideration.

The remaining terms in our Hamiltonian are $\mathcal{H}_3$ and $\mathcal{H}_4$, which are of the form

$$\begin{aligned}\mathcal{H}_3=&\tfrac{1}{2}\sum_\mathbf{k}\sum_{\mathbf{k}'}(N-\sum_\mathbf{k} a_\mathbf{k}^\dagger a_\mathbf{k})^{\frac{1}{2}}\{L(\mathbf{k}',\mathbf{k}+\mathbf{k}')(u_\mathbf{k}+v_\mathbf{k})V_\mathbf{k}+L(\mathbf{k},\mathbf{k}+\mathbf{k}')(u_{\mathbf{k}'}+v_{\mathbf{k}'})V_{\mathbf{k}'}\\ &-M(\mathbf{k},\mathbf{k}')(u_{\mathbf{k}+\mathbf{k}'}+v_{\mathbf{k}+\mathbf{k}'})V_{\mathbf{k}+\mathbf{k}'}\}(\alpha_{\mathbf{k}+\mathbf{k}'}^\dagger\alpha_\mathbf{k}\alpha_{\mathbf{k}'}+\alpha_\mathbf{k}^\dagger\alpha_{\mathbf{k}'}^\dagger\alpha_{\mathbf{k}+\mathbf{k}'})-\tfrac{1}{6}\sum_\mathbf{k}\sum_{\mathbf{k}'}(N-\sum_\mathbf{k} a_\mathbf{k}^\dagger a_\mathbf{k})^{\frac{1}{2}}\{M(\mathbf{k}',\mathbf{k}+\mathbf{k})(u_\mathbf{k}+v_\mathbf{k})V_\mathbf{k}\\ &+M(\mathbf{k},\mathbf{k}+\mathbf{k}')(u_{\mathbf{k}'}+v_{\mathbf{k}'})V_{\mathbf{k}'}-M(\mathbf{k},\mathbf{k}')(u_{\mathbf{k}+\mathbf{k}'}+v_{\mathbf{k}+\mathbf{k}'})V_{\mathbf{k}+\mathbf{k}'}\}(\alpha_\mathbf{k}^\dagger\alpha_{\mathbf{k}'}^\dagger\alpha_{-\mathbf{k}-\mathbf{k}'}^\dagger+\alpha_{-\mathbf{k}-\mathbf{k}'}\alpha_{\mathbf{k}'}\alpha_\mathbf{k}),\\ \mathcal{H}_4=&\tfrac{1}{8}\sum_\mathbf{q}\sum_\mathbf{k}\sum_{\mathbf{k}'}V_\mathbf{q}M(\mathbf{k},\mathbf{k}+\mathbf{q})M(\mathbf{k}',\mathbf{k}'+\mathbf{q})(\alpha_{\mathbf{k}+\mathbf{q}}^\dagger\alpha_{-\mathbf{k}'-\mathbf{q}}^\dagger\alpha_{\mathbf{k}'}^\dagger\alpha_{-\mathbf{k}}^\dagger+\alpha_{-\mathbf{k}}\alpha_{\mathbf{k}'}\alpha_{-\mathbf{k}'-\mathbf{q}}\alpha_{\mathbf{k}+\mathbf{q}})\\ &+\tfrac{1}{4}\sum_\mathbf{q}\sum_\mathbf{k}\sum_{\mathbf{k}'}V_\mathbf{q}M(\mathbf{k},\mathbf{k}+\mathbf{q})M(\mathbf{k}',\mathbf{k}'+\mathbf{q})\alpha_{\mathbf{k}+\mathbf{q}}^\dagger\alpha_{-\mathbf{k}}^\dagger\alpha_{\mathbf{k}'+\mathbf{q}}\alpha_{-\mathbf{k}'}+\tfrac{1}{2}\sum_\mathbf{q}\sum_\mathbf{k}\sum_{\mathbf{k}'}V_\mathbf{q}L(\mathbf{k},\mathbf{k}+\mathbf{q})L(\mathbf{k}',\mathbf{k}'+\mathbf{q})\\ &\times\alpha_{\mathbf{k}+\mathbf{q}}^\dagger\alpha_{-\mathbf{k}'-\mathbf{q}}^\dagger\alpha_{\mathbf{k}'}\alpha_\mathbf{k}+\mathcal{H}_4',\end{aligned} \tag{17}$$

where we write

$$\begin{aligned} M(\mathbf{k},\mathbf{k}')&=u_\mathbf{k}v_{\mathbf{k}'}+v_\mathbf{k}u_{\mathbf{k}'},\\ L(\mathbf{k},\mathbf{k}')&=u_\mathbf{k}u_{\mathbf{k}'}+v_\mathbf{k}v_{\mathbf{k}'},\end{aligned} \tag{18}$$

and $\mathcal{H}_4'$ expresses all the terms of the fourth power corresponding to the second and third terms in H_4, Eq. (4a). It is not necessary to consider the quantity $\mathcal{H}_4'$ further.

Now, we shall look for the eigenmodes of excitation of the form

$$\begin{aligned}\beta_\mathbf{Q}^\dagger=&\varphi_\mathbf{Q}\alpha_\mathbf{Q}^\dagger+\chi_\mathbf{Q}\alpha_{-\mathbf{Q}}\\ &+\sum_\mathbf{K}'\{\xi(\mathbf{K};\mathbf{Q})\alpha_{\mathbf{K}+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}}^\dagger+\eta(\mathbf{K};\mathbf{Q})\alpha_{-\mathbf{K}-\mathbf{Q}}\alpha_\mathbf{K}\},\end{aligned} \tag{19}$$

where $\varphi_\mathbf{Q}$, $\chi_\mathbf{Q}$, $\xi(\mathbf{K};\mathbf{Q})$ and $\eta(\mathbf{K};\mathbf{Q})$ are coefficients to be determined and satisfy the normalization equation,

$$\begin{aligned}&|\varphi_\mathbf{Q}|^2-|\chi_\mathbf{Q}|^2\\ &\quad+\sum_\mathbf{K}'\{|\xi(\mathbf{K};\mathbf{Q})|^2-|\eta(\mathbf{K};\mathbf{Q})|^2\}=1.\end{aligned} \tag{19a}$$

The prime on the summation means that each pair of $\alpha^\dagger\alpha^\dagger$ or $\alpha\alpha$ must appear only once. The coefficients are to be determined by the condition that $\beta_\mathbf{Q}^\dagger$ satisfies the equation

$$[H,\beta_\mathbf{Q}^\dagger]=\Omega_\mathbf{Q}\beta_\mathbf{Q}^\dagger. \tag{20}$$

To get this equation, some approximations are neces-

[7] N. N. Bogoliubov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 58 (1958).

sary, i.e., linearization of the equation of motion.[8] This procedure is familiar in the various calculations of collective excitations.[9]

In our case, we get the linearized equation of motion as follows[10]:

$$
\begin{aligned}
[H,\alpha_{\mathbf{Q}}^\dagger]&=E_{\mathbf{Q}}'\alpha_{\mathbf{Q}}+\textstyle\sum_{\mathbf{K}}'\{C(\mathbf{K};\mathbf{Q})\alpha_{\mathbf{K}+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}}^\dagger+D(\mathbf{K};\mathbf{Q})\alpha_{-\mathbf{K}-\mathbf{Q}}\alpha_{\mathbf{K}}\},\\
[H,\alpha_{-\mathbf{Q}}]&=-E_{\mathbf{Q}}'\alpha_{-\mathbf{Q}}-\textstyle\sum_{\mathbf{K}}'\{D(\mathbf{K};\mathbf{Q})\alpha_{\mathbf{K}+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}}^\dagger+C(\mathbf{K};\mathbf{Q})\alpha_{-\mathbf{K}-\mathbf{Q}}\alpha_{\mathbf{K}}\},\\
[H,\alpha_{\mathbf{K}+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}}^\dagger]&=(E_{\mathbf{K}+\mathbf{Q}}'+E_{\mathbf{K}}')\alpha_{\mathbf{K}+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}}^\dagger+C(\mathbf{K};\mathbf{Q})\alpha_{\mathbf{Q}}^\dagger+D(\mathbf{K};\mathbf{Q})\alpha_{-\mathbf{Q}}\\
&\quad+\textstyle\sum_{\mathbf{K}'}'\{A(\mathbf{K},\mathbf{K}';\mathbf{Q})\alpha_{\mathbf{K}'+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}'}^\dagger+B(\mathbf{K},\mathbf{K}';\mathbf{Q})\alpha_{-\mathbf{K}'-\mathbf{Q}}\alpha_{\mathbf{K}'}\},\\
[H,\alpha_{-\mathbf{K}-\mathbf{Q}}\alpha_{\mathbf{K}}]&=-(E_{\mathbf{K}+\mathbf{Q}}'+E_{\mathbf{K}}')\alpha_{-\mathbf{K}-\mathbf{Q}}\alpha_{\mathbf{K}}-D(\mathbf{K};\mathbf{Q})\alpha_{\mathbf{Q}}^\dagger-C(\mathbf{K};\mathbf{Q})\alpha_{-\mathbf{Q}}\\
&\quad-\textstyle\sum_{\mathbf{K}'}\{B(\mathbf{K},\mathbf{K}';\mathbf{Q})\alpha_{\mathbf{K}'+\mathbf{Q}}^\dagger\alpha_{-\mathbf{K}'}^\dagger+A(\mathbf{K},\mathbf{K}';\mathbf{Q})\alpha_{-\mathbf{K}'-\mathbf{Q}}\alpha_{\mathbf{K}'}\},
\end{aligned}
\tag{21}
$$

where A, B, C, and D are given by the following expressions.

$$
\begin{aligned}
A(\mathbf{K},\mathbf{K}';\mathbf{Q})&=V_{\mathbf{Q}}M(\mathbf{K},\mathbf{K}+\mathbf{Q})M(\mathbf{K}',\mathbf{K}'+\mathbf{Q})+V_{\mathbf{K}-\mathbf{K}'}L(\mathbf{K},\mathbf{K}')L(\mathbf{K}+\mathbf{Q},\mathbf{K}'+\mathbf{Q})\\
&\quad+V_{\mathbf{K}+\mathbf{K}'+\mathbf{Q}}L(\mathbf{K},\mathbf{K}+\mathbf{Q})L(\mathbf{K}',\mathbf{K}'+\mathbf{Q}),\\
B(\mathbf{K},\mathbf{K}';\mathbf{Q})&=V_{\mathbf{Q}}M(\mathbf{K},\mathbf{K}+\mathbf{Q})M(\mathbf{K}',\mathbf{K}'+\mathbf{Q})+V_{\mathbf{K}-\mathbf{K}'}M(\mathbf{K},\mathbf{K}')M(\mathbf{K}+\mathbf{Q},\mathbf{K}'+\mathbf{Q})\\
&\quad+V_{\mathbf{K}+\mathbf{K}'+\mathbf{Q}}M(\mathbf{K},\mathbf{K}+\mathbf{Q})M(\mathbf{K}',\mathbf{K}'+\mathbf{Q}),\\
C(\mathbf{K};\mathbf{Q})&=(\bar{N})^{\frac12}\{V_{\mathbf{K}}(u_{\mathbf{K}}-v_{\mathbf{K}})L(\mathbf{Q},\mathbf{K}+\mathbf{Q})+V_{\mathbf{K}+\mathbf{Q}}(u_{\mathbf{K}+\mathbf{Q}}-v_{\mathbf{K}+\mathbf{Q}})L(\mathbf{K},\mathbf{Q})-V_{\mathbf{Q}}(u_{\mathbf{Q}}-v_{\mathbf{Q}})M(\mathbf{K},\mathbf{K}+\mathbf{Q})\},\\
D(\mathbf{K};\mathbf{Q})&=-(\bar{N})^{\frac12}\{V_{\mathbf{K}}(u_{\mathbf{K}}-v_{\mathbf{K}})M(\mathbf{Q},\mathbf{K}+\mathbf{Q})+V_{\mathbf{K}+\mathbf{Q}}(u_{\mathbf{K}+\mathbf{Q}}-v_{\mathbf{K}+\mathbf{Q}})M(\mathbf{K},\mathbf{Q})+V_{\mathbf{Q}}(u_{\mathbf{Q}}-v_{\mathbf{Q}})M(\mathbf{K},\mathbf{K}+\mathbf{Q})\},
\end{aligned}
\tag{22}
$$

and

$$\bar{N}\equiv N-\textstyle\sum_{\mathbf{k}} v_{\mathbf{k}}^2=N-K. \tag{22a}$$

$\mathcal{H}_4'$ contributes to A and B only in lower-order terms in Ω (normalization volume) than those given by (22), and the commutator with the factor $[N-\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}]^{\frac12}$ in $\mathcal{H}_3$ is also of lower order in Ω.

From Eq. (20), we obtain the secular equation for the coefficients φ, χ, ξ and η. It is more convenient to introduce the following quantities:

$$
\begin{aligned}
x_{\mathbf{Q}}&\equiv\varphi_{\mathbf{Q}}+\chi_{\mathbf{Q}},\\
y_{\mathbf{Q}}&\equiv\varphi_{\mathbf{Q}}-\chi_{\mathbf{Q}},\\
X(\mathbf{K};\mathbf{Q})&\equiv\xi(\mathbf{K};\mathbf{Q})+\eta(\mathbf{K};\mathbf{Q}),\\
Y(\mathbf{K};\mathbf{Q})&\equiv\xi(\mathbf{K};\mathbf{Q})-\eta(\mathbf{K};\mathbf{Q}).
\end{aligned}
\tag{23}
$$

Then, the equations can be written

$$
\begin{aligned}
\Omega_{\mathbf{Q}}x_{\mathbf{Q}}&=E_{\mathbf{Q}}'y_{\mathbf{Q}}+\textstyle\sum_{\mathbf{K}}'(C+D)(\mathbf{K};\mathbf{Q})Y(\mathbf{K};\mathbf{Q}),\\
\Omega_{\mathbf{Q}}X(\mathbf{K};\mathbf{Q})&=(E_{\mathbf{K}+\mathbf{Q}}'+E_{\mathbf{K}}')Y(\mathbf{K};\mathbf{Q})+(C+D)(\mathbf{K};\mathbf{Q})\\
&\quad\times y_{\mathbf{Q}}+\textstyle\sum_{\mathbf{K}'}'(A+B)(\mathbf{K},\mathbf{K}';\mathbf{Q})Y(\mathbf{K}';\mathbf{Q}),
\end{aligned}
\tag{24a}
$$

$$
\begin{aligned}
\Omega_{\mathbf{Q}}y_{\mathbf{Q}}&=E_{\mathbf{Q}}'x_{\mathbf{Q}}+\textstyle\sum_{\mathbf{K}}'(C-D)(\mathbf{K};\mathbf{Q})X(\mathbf{K};\mathbf{Q}),\\
\Omega_{\mathbf{Q}}Y(\mathbf{K};\mathbf{Q})&=(E_{\mathbf{K}+\mathbf{Q}}'+E_{\mathbf{K}}')X(\mathbf{K};\mathbf{Q})+(C-D)(\mathbf{K};\mathbf{Q})\\
&\quad\times x_{\mathbf{Q}}+\textstyle\sum_{\mathbf{K}'}'(A-B)(\mathbf{K},\mathbf{K}';\mathbf{Q})X(\mathbf{K}';\mathbf{Q}),
\end{aligned}
\tag{24b}
$$

where the prime on the summation means that only one pair of $(-\mathbf{K},\mathbf{K}+\mathbf{Q})$ should appear.

As seen from (22), A, B and C, D are symmetric for the replacements of $\mathbf{K}\leftrightarrow-\mathbf{K}-\mathbf{Q}$ and $\mathbf{K}'\leftrightarrow-\mathbf{K}'-\mathbf{Q}$ and $\mathbf{K}\leftrightarrow\mathbf{K}'$. Thus, only the symmetric parts of $X(\mathbf{K};\mathbf{Q})$ and $Y(\mathbf{K};\mathbf{Q})$ for the replacement $\mathbf{K}\leftrightarrow-\mathbf{K}-\mathbf{Q}$ have nonvanishing contributions to the summations, and we can take $X(\mathbf{K};\mathbf{Q})$ and $Y(\mathbf{K};\mathbf{Q})$ as symmetric for this replacement. That is, we can replace the summation $\sum'$ by the unrestricted summation $\frac12\sum$.

Solution of Secular Equation for Small Q

Since our interest is in the vicinity of $Q=0$, we can expand all quantities in power series of Q; for example,

$$
\begin{aligned}
A(\mathbf{K},\mathbf{K}';\mathbf{Q})&=A(\mathbf{K},\mathbf{K}';0)+\delta^{(1)}A(\mathbf{K},\mathbf{K}';\mathbf{Q})\\
&\quad+\delta^{(2)}A(\mathbf{K},\mathbf{K}';\mathbf{Q})+\cdots,
\end{aligned}
\tag{25}
$$

and

$$X(\mathbf{K};\mathbf{Q})=X(\mathbf{K};0)+X^{(1)}(\mathbf{K};\mathbf{Q})+X^{(2)}(\mathbf{K};\mathbf{Q})+\cdots,$$

where each term is constant, linear, and quadratic in Q, respectively.

The zero-order equation can be written as

$$
\begin{aligned}
\Omega_0x_0&=E_0'y_0+\tfrac12\textstyle\sum_{\mathbf{K}}(C+D)(\mathbf{K};0)Y(\mathbf{K};0),\\
\Omega_0X(\mathbf{K};0)&=2E_{\mathbf{K}}'Y(\mathbf{K};0)+(C+D)(\mathbf{K};0)y_0\\
&\quad+\tfrac12\textstyle\sum_{\mathbf{K}'}(A+B)(\mathbf{K},\mathbf{K}';0)Y(\mathbf{K}';0),
\end{aligned}
\tag{26a}
$$

$$
\begin{aligned}
\Omega_0y_0&=E_0'x_0+\tfrac12\textstyle\sum_{\mathbf{K}}(C-D)(\mathbf{K};0)X(\mathbf{K};0),\\
\Omega_0Y(\mathbf{K};0)&=2E_{\mathbf{K}}'X(K;0)+(C-D)(\mathbf{K};0)x_0\\
&\quad+\tfrac12\textstyle\sum_{\mathbf{K}'}(A-B)(\mathbf{K},\mathbf{K}';0)X(\mathbf{K}';0).
\end{aligned}
\tag{26b}
$$

[8] Generally, $[H,\beta_{\mathbf{Q}}^\dagger]$ contains higher-order terms which can be written as products of two operators, one appearing in the definition of $\beta_{\mathbf{Q}}^\dagger$ and the other not. Linearization can be achieved by replacing this latter operator by its ground-state expectation value. In our notation, this procedure corresponds to writing $[H,\beta_{\mathbf{Q}}^\dagger]$ as a normal product form and picking up only the linear terms in α^+, α, $\alpha^\dagger\alpha^\dagger$, and $\alpha\alpha$.

[9] See, for example, K. Sawada, Phys. Rev. **119**, 2090 (1960) for general formulations; K. Sawada, Phys. Rev. **106**, 372 (1957) for plasma excitations in the free electron gas; P. W. Anderson, Phys. Rev. **112**, 1900 (1958); K. Yosida, Progr. Theoret. Phys. (Kyoto) **21**, 731 (1959); and G. Rickayzen, Phys. Rev. **115**, 795 (1959) for collective excitations in superconductors.

[10] If H contains the quadratic nondiagonal terms of $\alpha^+\alpha^+$, that is, if $u_{\mathbf{K}}$, $v_{\mathbf{K}}$ were not determined as in §2, there appear constant terms in $[H,\alpha^+\alpha^+]$ and $[H,\alpha\alpha]$, which means these modes are unstable. This is the reason why we should have eliminated the "dangerous" terms in §2.

Here each coefficient has the following value,

$$(A+B)(\mathbf{K},\mathbf{K}';0)=(2V_0+V_\mathbf{K}+V_{\mathbf{K}'})M(\mathbf{K},\mathbf{K})M(\mathbf{K}',\mathbf{K}')+(V_{\mathbf{K}-\mathbf{K}'}+V_{\mathbf{K}+\mathbf{K}'})L(\mathbf{K},\mathbf{K})L(\mathbf{K}',\mathbf{K}'),$$
$$(C+D)(\mathbf{K};0)=2(\bar{N})^{\frac{1}{2}}(u_0-v_0)[(V_0+V_\mathbf{K})M(\mathbf{K},\mathbf{K})+V_\mathbf{K}L(\mathbf{K},\mathbf{K})], \quad (27a)$$

$$(A-B)(\mathbf{K},\mathbf{K}';0)=V_{\mathbf{K}-\mathbf{K}'}+V_{\mathbf{K}+\mathbf{K}'},$$
$$(C-D)(\mathbf{K};0)=2(\bar{N})^{\frac{1}{2}}(u_0+v_0)V_\mathbf{K}. \quad (27b)$$

We can easily show that Eqs. (26a) and (26b) have a solution $\Omega_0=0$, $y_0=Y(\mathbf{K};0)=0$; that is, we can show that the equations

$$E_0'x_0+(\bar{N})^{\frac{1}{2}}(u_0+v_0)\sum_\mathbf{K}V_\mathbf{K}X(\mathbf{K};0)=0,$$
$$2E_\mathbf{K}'X(\mathbf{K};0)+2(\bar{N})^{\frac{1}{2}}(u_0+v_0)V_\mathbf{K}x_0+\tfrac{1}{2}\sum_{\mathbf{K}'}(V_{\mathbf{K}-\mathbf{K}'}+V_{\mathbf{K}+\mathbf{K}'})X(\mathbf{K}';0)=0 \quad (28)$$

can be satisfied by the following solution

$$X(\mathbf{K};0)=\alpha(uv)_\mathbf{K},$$
$$x_0=-\alpha(\bar{N})^{\frac{1}{2}}(u_0+v_0)J_0/E_0', \quad (29)$$

where α is an arbitrary parameter to be determined by the normalization condition (19a). This is easily verified by substituting (29) into (28) and using Eqs. (10)–(13).

Thus, the excitation energy Ω_Q vanishes when $Q\rightarrow 0$; that is, there appears no energy gap. The next step is to show that Ω_Q is linear in Q for small Q. To do so, we must investigate the first- and second-order terms of Eq. (24). The first-order terms in Q of Eq. (24) can be written as

$$\Omega_\mathrm{Q}^{(1)}x_0=E_0'y_\mathrm{Q}^{(1)}+\tfrac{1}{2}\sum_\mathbf{K}(C+D)(\mathbf{K};0)\times Y^{(1)}(\mathbf{K};\mathbf{Q}),$$
$$\Omega_\mathrm{Q}^{(1)}X(\mathbf{K};0)=2E_\mathbf{K}'Y^{(1)}(\mathbf{K},\mathbf{Q})+(C+D)(\mathbf{K};0)\times y_\mathrm{Q}^{(1)}+\tfrac{1}{2}\sum_{\mathbf{K}'}(A+B)(\mathbf{K},\mathbf{K}';0)Y^{(1)}(\mathbf{K}',\mathbf{Q}), \quad (30a)$$

$$0=E_0'x_\mathrm{Q}^{(1)}+\tfrac{1}{2}\sum_\mathbf{K}(C-D)(\mathbf{K};0)X^{(1)}(\mathbf{K};\mathbf{Q})+\delta^{(1)}E_\mathrm{Q}x_0+\tfrac{1}{2}\sum_\mathbf{K}\delta^{(1)}(C-D)(\mathbf{K};\mathbf{Q})X(\mathbf{K};0),$$
$$0=2E_{\mathbf{K}'}X^{(1)}(\mathbf{K};\mathbf{Q})+(C-D)(\mathbf{K};0)x_\mathrm{Q}^{(1)}+\tfrac{1}{2}\sum_{\mathbf{K}'}\times(A-B)(\mathbf{K},\mathbf{K}';0)X^{(1)}(\mathbf{K}';\mathbf{Q})+\delta^{(1)}E_{\mathbf{K}+\mathbf{Q}}'\times X(\mathbf{K};0)+\delta^{(1)}(C-D)(\mathbf{K};\mathbf{Q})x_0+\tfrac{1}{2}\sum_{\mathbf{K}'}\times\delta^{(1)}(A-B)(\mathbf{K},\mathbf{K}';\mathbf{Q})X(\mathbf{K}';0), \quad (30b)$$

where

$$\delta^{(1)}(C-D)(\mathbf{K};\mathbf{Q})=(\bar{N})^{\frac{1}{2}}(u_0+v_0)\delta^{(1)}V_{\mathbf{K}+\mathbf{Q}},$$
$$\delta^{(1)}(A-B)(\mathbf{K},\mathbf{K}';\mathbf{Q})=\delta^{(1)}V_{\mathbf{K}+\mathbf{K}'+\mathbf{Q}}, \quad (31)$$

and

$$\delta^{(1)}E_\mathrm{Q}'=0,$$

since E_Q' is an even function of Q. Substituting these values and the zero-order solution (29) into (30b), we can see that Eq. (30b) has a solution

$$x_\mathrm{Q}^{(1)}=0,$$

and

$$X^{(1)}(\mathbf{K};\mathbf{Q})=\tfrac{1}{2}\alpha\delta^{(1)}(uv)_{\mathbf{K}+\mathbf{Q}}. \quad (32)$$

The value of $\Omega_\mathrm{Q}^{(1)}$ can be determined from Eq. (30a) and the second-order equation of (24b), which can be written as

$$\Omega_\mathrm{Q}^{(1)}y_\mathrm{Q}^{(1)}=E_0'x_\mathrm{Q}^{(2)}+\tfrac{1}{2}\sum_\mathbf{K}(C-D)(\mathbf{K};0)\times X^{(2)}(\mathbf{K};\mathbf{Q})+p,$$
$$\Omega_\mathrm{Q}^{(1)}Y^{(1)}(\mathbf{K};\mathbf{Q})=2E_\mathbf{K}'X^{(2)}(\mathbf{K};\mathbf{Q})+(C-D)(\mathbf{K};0)\times x_\mathrm{Q}^{(2)}+\tfrac{1}{2}\sum_{\mathbf{K}'}(A-B)(\mathbf{K},\mathbf{K}';0)\times X^{(2)}(\mathbf{K}';\mathbf{Q})+P(\mathbf{K};\mathbf{Q}), \quad (33)$$

where p and $P(\mathbf{K};\mathbf{Q})$ are given by

$$p=\delta^{(2)}E_\mathrm{Q}'x_0+\tfrac{1}{2}\sum_\mathbf{K}\{\delta^{(1)}(C-D)(\mathbf{K};\mathbf{Q})\times X^{(1)}(\mathbf{K};\mathbf{Q})+\delta^{(2)}(C-D)(\mathbf{K};\mathbf{Q})X(\mathbf{K};0)\},$$
$$P(\mathbf{K};\mathbf{Q})=\delta^{(1)}E_{\mathbf{K}+\mathbf{Q}}'X^{(1)}(\mathbf{K};\mathbf{Q})+\delta^{(2)}E_{\mathbf{K}+\mathbf{Q}}'X(\mathbf{K};0)+\delta^{(2)}(C-D)(\mathbf{K};0)x_0+\tfrac{1}{2}\sum_{\mathbf{K}'}\{\delta^{(1)}(A-B)\times(\mathbf{K},\mathbf{K}';\mathbf{Q})X^{(1)}(\mathbf{K}';\mathbf{Q})+\delta^{(2)}(A-B)\times(\mathbf{K},\mathbf{K}';\mathbf{Q})X(\mathbf{K}';0)\}. \quad (34)$$

Solving the first equation of (33) and (30a) with respect to $x_\mathrm{Q}^{(2)}$ and $y_\mathrm{Q}^{(1)}$ and substituting them into the second equation of (33) and (30a), we obtain

$$2E_\mathbf{K}'X^{(2)}(\mathbf{K};\mathbf{Q})-[(C-D)(\mathbf{K};0)/2E_0']\sum_{\mathbf{K}'}(C-D)(\mathbf{K}';0)X^{(2)}(\mathbf{K}';\mathbf{Q})+\tfrac{1}{2}\sum_{\mathbf{K}'}(A-B)(\mathbf{K},\mathbf{K}';0)X^{(2)}(\mathbf{K}';\mathbf{Q})$$
$$=\Omega_\mathrm{Q}^{(1)}[Y^{(1)}(\mathbf{K};\mathbf{Q})+[(C-D)(\mathbf{K};0)/2E_0'^2]\sum_{\mathbf{K}'}(C+D)(\mathbf{K}';0)Y^{(1)}(\mathbf{K}';\mathbf{Q})]$$
$$-\left[P(\mathbf{K};\mathbf{Q})-\frac{(C-D)(\mathbf{K};0)}{E_0'}p+\left(\frac{\Omega_\mathrm{Q}^{(1)}}{E_0'}\right)^2(C-D)(\mathbf{K};0)x_0\right], \quad (35)$$

and

$$2E_\mathbf{K}'Y^{(1)}(\mathbf{K};\mathbf{Q})-[(C+D)(\mathbf{K};0)/2E_0']\sum_{\mathbf{K}'}(C+D)(\mathbf{K}';0)Y^{(1)}(\mathbf{K}';\mathbf{Q})+\tfrac{1}{2}\sum_{\mathbf{K}'}(A+B)(\mathbf{K},\mathbf{K}';0)Y^{(1)}(\mathbf{K}';\mathbf{Q})$$
$$=\Omega_\mathrm{Q}^{(1)}\left[X(\mathbf{K};0)-\frac{(C+D)(\mathbf{K};0)}{E_0'}x_0\right]. \quad (36)$$

Equation (35) is an integral equation for $X^{(2)}(\mathbf{K};\mathbf{Q})$, whose homogeneous part vanishes if we put $X^{(2)}(\mathbf{K};\mathbf{Q}) \propto (uv)_\mathbf{K}$, as is seen from the discussion of the zeroth-order equation (28). If we put $2E_\mathbf{K}'X^{(2)}(\mathbf{K};\mathbf{Q})=f(\mathbf{K})$, the left-hand side of Eq. (35) can be written as

$$f(\mathbf{K})-\sum_{\mathbf{K}'}\Gamma(\mathbf{K},\mathbf{K}')f(\mathbf{K}'),$$

where the integral kernel $\Gamma(\mathbf{K},\mathbf{K}')$ is of the form

$$\Gamma(\mathbf{K},\mathbf{K}')=-[(C-D)(\mathbf{K};0)(C-D)(\mathbf{K}';0)/4E_0'E_{\mathbf{K}'}'] +[(A-B)(\mathbf{K},\mathbf{K}';0)/4E_{\mathbf{K}'}'].$$

The integral equation with the transposed kernel

$$f(\mathbf{K})-\sum_{\mathbf{K}'}\Gamma(\mathbf{K}',\mathbf{K})f(\mathbf{K}')=0$$

has an eigenfunction $f(\mathbf{K})=(uv)_\mathbf{K}$. Therefore, the right-hand side of Eq. (35) should be orthogonal to this eigenfunction, in order that the inhomogeneous equation (35) has a solution. Thus, we derive a relation between $\Omega_\mathbf{Q}^{(1)}$ and $Y^{(1)}(\mathbf{K};\mathbf{Q})$:

$$\Omega_\mathbf{Q}^{(1)}\{\sum_\mathbf{K}(uv)_\mathbf{K}Y^{(1)}(\mathbf{K};\mathbf{Q})+(1/2E_0')\sum_\mathbf{K}(C-D)(\mathbf{K};0) \times(uv)_\mathbf{K}\sum_{\mathbf{K}'}(C+D)(\mathbf{K}';0)Y^{(1)}(\mathbf{K}';\mathbf{Q})\} =\sum_\mathbf{K}(uv)_\mathbf{K}P(\mathbf{K};\mathbf{Q})+\frac{1}{E_0'}\left(\frac{\Omega_\mathbf{Q}^{(1)2}}{E_0'}-p\right) \times\sum_\mathbf{K}(C-D)(\mathbf{K};0)(uv)_\mathbf{K}. \quad (37)$$

It is easily seen from (36) that $Y^{(1)}(\mathbf{K};\mathbf{Q})$ is proportional to $\Omega_\mathbf{Q}^{(1)}$, and if we put $Y^{(1)}(\mathbf{K};\mathbf{Q})=\Omega_\mathbf{Q}^{(1)}Y_\mathbf{K}'$ (the Q dependence of $Y_\mathbf{K}'$ is unnecessary), Eqs. (36) and (37) become

$$2E_\mathbf{K}'Y_\mathbf{K}'-[(C+D)(\mathbf{K};0)/2E_0']\sum_{\mathbf{K}'}(C+D) \times(\mathbf{K}';0)Y_{\mathbf{K}'}'+\tfrac{1}{2}\sum_{\mathbf{K}'}(A+B)(\mathbf{K},\mathbf{K}';0)Y_{\mathbf{K}'}' =X(\mathbf{K};0)-[(C+D)(\mathbf{K};0)/E_0']x_0, \quad (36')$$

and

$$(\Omega_\mathbf{Q}^{(1)})^2\{\sum_\mathbf{K}(uv)_\mathbf{K}Y_\mathbf{K}'+(1/2E_0')\sum_\mathbf{K}(C-D)(\mathbf{K};0) \times(uv)_\mathbf{K}\sum_{\mathbf{K}'}(C+D)(\mathbf{K}';0)Y_{\mathbf{K}'}'-(1/E_0')\sum_\mathbf{K} \times(C-D)(\mathbf{K};0)(uv)_\mathbf{K}\}=\sum_\mathbf{K}(uv)_\mathbf{K}P(\mathbf{K};\mathbf{Q}) -(p/E_0')\sum_\mathbf{K}(C-D)(\mathbf{K};0)(uv)_\mathbf{K}. \quad (37')$$

If we solve (36′) with respect to $Y_{\mathbf{K}'}'$ and substitute it into (37′), we get the value of $\Omega_\mathbf{Q}^{(1)}$. The right-hand side of (37′) has a nonvanishing value, which is very complicated, but is proportional to Q^2, and so we can say that $\Omega_\mathbf{Q}^{(1)}$ does not vanish and is proportional to Q. Since $\Omega_\mathbf{Q}=\Omega_0+\Omega_\mathbf{Q}^{(1)}+\Omega_\mathbf{Q}^{(2)}+\cdots$ and $\Omega_0=0$, $\Omega_\mathbf{Q}$ is proportional to Q for small Q; thus, we obtain a phonon spectrum again. The numerical value of the proportionality constant, which corresponds to the value of the sound velocity, can in principle be obtained from a solution of the integral equation for $Y_\mathbf{K}'$, (36′). Unfortunately, however, this cannot be done in practice and need not be done if we are satisfied with the qualitative result that the energy gap appearing in §2 vanishes in higher approximations and the excitation spectrum is a phononlike one.

4. CONCLUSIONS

We have shown that the energy gap obtained by Girardeau and Arnowitt is only an apparent one, and the energy spectrum is proportional to momentum for small momentum without an energy gap. We have constructed the eigenmodes of excitations using the approximation of linearization of equation of motion, in which the third- and fourth-power terms with respect to the quasi-particles are taken into consideration appropriately.

We can show that the excitation energy for this mode becomes zero when the momentum Q tends to zero, and is proportional to Q for small Q, although the proportionality constant cannot be evaluated.

We have not discussed the ground-state energy which must be modified in this method; the calculation of it is an interesting problem and will require knowledge of the excitation spectrum for all momenta. This appears complex even if we employ some approximations such as the low-density limit used by Girardeau[6] in his calculation of the ground-state energy U_0', (9a).

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APPENDIX[11]

We want to show that in the Hamiltonian (1) one can put $[N-\sum_{\mathbf{k}\neq0}a_\mathbf{k}^\dagger a_\mathbf{k}]^{\frac{1}{2}}$ for $a_0^\dagger$ or a_0 in the case of $\langle a_0^\dagger a_0\rangle\gg1$.

First, we consider the following complete orthonormal set of wave functions in which the total number N of particles is fixed,

$$\Psi_{N;\{\eta_\mathbf{k}\}}=\frac{1}{[(N-\sum_\mathbf{k}\eta_\mathbf{k})!]^{\frac{1}{2}}}(a_0^\dagger)^{N-\sum_\mathbf{k}\eta_\mathbf{k}} \times\prod_{\mathbf{k}\neq0}\frac{(a_\mathbf{k}^\dagger)^{\eta_\mathbf{k}}}{(\eta_\mathbf{k}!)^{\frac{1}{2}}}\bigg|0\bigg), \quad (A1)$$

where $|0)$ denotes the vacuum in which there are no particles.

Next, we consider the following orthonormal set of wave functions

$$\Phi_{\{\eta_\mathbf{k}\}}=\prod_{k\neq0}\frac{(a_k^\dagger)^{\eta_\mathbf{k}}}{(\eta_\mathbf{k}!)^{\frac{1}{2}}}\bigg|0\bigg), \quad (A2)$$

which is in one-to-one correspondence to $\Psi_{N;\{\eta_\mathbf{k}\}}$ if $\sum_\mathbf{k}\eta_\mathbf{k}\leqslant N$. This is complete in the space V which is

[11] *Note added in proof.* This procedure was given by N. Fukuda in an article published in *Nuclear Physics* (Kyoritsu, Tokyo, Japan, 1959), Vol. 1. Because this article is written in Japanese, we present the essential argument briefly.

constructed by excluding the zero momentum state from the original space.

If we construct the Hamiltonian H' (or any operator) in this V space such that

$$(\Psi_{N;\,\{\eta_{k'}\}}, H\Psi_{N;\,\{\eta_k\}}) = (\Phi_{\{\eta_{k'}\}}, H'\Phi_{\{\eta_k\}}), \qquad \text{(A3)}$$

the operator $a_0^\dagger$ or a_0 in H can be seen to be replaced by $[N-\sum_k \eta_k+\epsilon]^{\frac{1}{2}}$, where ϵ is a finite number. For example,

$$f(a_k^\dagger,a_k)a_0^\dagger a_0 \to f(a_k^\dagger,a_k)(N-\textstyle\sum_k \eta_k) = f(a_k^\dagger,a_k)[N-\textstyle\sum_k a_k^\dagger a_k],$$

$$f(a_k^\dagger,a_k)a_0^2 \to f(a_k^\dagger,a_k) \times[(N-\textstyle\sum_k \eta_k)(N-\textstyle\sum_k \eta_k-1)]^{\frac{1}{2}},$$

$$(a_0^\dagger)^2 f(a_k^\dagger,a_k) \to [(N-\textstyle\sum_k \eta_k+1)(N-\textstyle\sum_k \eta_k+2)]^{\frac{1}{2}} \times f(a_k^\dagger,a_k).$$

In any case, ϵ can be neglected compared with $N-\sum_k \eta_k$ if $N-\sum_k \eta_k$ is of the order of N, so that we obtain the Hamiltonian (4), (4a).

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Test of Global Symmetry in Pion-Baryon Interactions by K^-+p Reactions*

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Under the hypothesis that the K-meson interactions do not mask the symmetries of the pion-baryon interactions appreciably, the branching ratios of the K^-+p reactions are studied to test the validity of global symmetry. The T^{-1}-matrix formalism of Matthews and Salam is adopted to calculate the branching ratios. The new Dalitz-Tuan solutions for $\bar{K}N$ scattering lengths, which incorporate the (K^+,K^0) mass difference and the new branching ratios of the various K^-+p reactions, presented at Kiev, are adopted in the analysis. The errors in the experimental branching ratios are so chosen as to satisfy the Amati-Vitale inequality. It is found that the a^- and b^+ (also a^+, though poorly) Dalitz-Tuan solutions can explain the branching ratios for K^- captured at rest. The extension of the analysis to 30-Mev incident K^- mesons under the zero-range approximation leads to very poor agreement with experiments.

I. INTRODUCTION

IT is of great interest to ascertain whether the very strong pion-baryon interactions possess any symmetry higher than charge independence. It is now clear that experiments exclude[1] the possibility of very high symmetry in both pion and K-meson interactions. So the symmetries of the pion interactions, even if they exist in the bare Lagrangian, could be distorted badly by the K-meson interactions. If so, such symmetries are not useful (except, possibly, at very high energies), since we cannot calculate accurately the consequences of strong couplings. In order to test the usefulness of the proposed symmetries of the pion interactions, we would therefore consider the possibility that the K-meson interactions may not be strong enough to break the symmetries of the pion interactions appreciably, even though, to some, this may be of academic interest only. We would apply this hypothesis specifically to the K^-+p reactions.

It was first pointed out by Amati and Vitale[2] that the low-energy K^-+p reactions provide a good tool for testing the hypotheses of the restricted and global symmetries,[3] if the K-meson interactions do not break them badly. These authors established an inequality[4] involving the branching ratios of the various K^-+p reactions on the basis of the restricted symmetry alone. This inequality will be referred to as the AV inequality in this paper. Starting with the work of Amati and Vitale, a number[5–9] of works have appeared in the literature on the same problem, with stress on the individual branching ratios for the different reaction modes, in particular on the Σ^-/Σ^+ ratio. These various attempts may be classified into two groups on the basis

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[1] A. Pais, Phys. Rev. **110**, 574 (1958).

[2] D. Amati and B. Vitale, Nuovo cimento **9**, 895 (1958).

[3] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957); J. Schwinger, Phys. Rev. **104**, 1164 (1956).

[4] See Eq. (14) of footnote 2, which reads

$$(W_{\Sigma^+\pi^-}+W_{\Sigma^-\pi^+}-4W_{\Sigma^0\pi^0})^2+4(W_{\Sigma^0\pi^0}W_{\Lambda^0\pi^0}-W_{\Sigma^+\pi^-}W_{\Sigma^-\pi^+})\geqslant 0,$$

where W_α is the branching ratio for the reaction α. This relation is referred to as the AV inequality in this paper.

[5] A. Salam, Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (unpublished).

[6] B. D. Espagnat and J. Prentki, Nuovo cimento, **15**, 130 (1960).

[7] K. Kawarabayashi, Progr. Theoret. Phys. (Kyoto) **20**, 117 (1958).

[8] M. L. Gupta, Nuovo cimento **16**, 737 (1960).

[9] M. Ross and G. L. Shaw, Bull. Am. Phys. Soc. **5**, 504 (1960). See also, Phys. Rev. **115**, 1773 (1959).