Alternative Approach to the Solution of Added Carrier Transport Problems in Semiconductors*

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A novel method of solving added carrier transport problems in semiconductors is presented. The usual procedure in treating problems of this type is to derive a continuity equation for charge carriers on the basis of carrier conservation, allowing for generation and recombination, and to solve this equation under appropriate boundary conditions. The resulting fluxes or currents are obtained from diffusion and drift current equations, which involve the concentrations and concentration gradients. In the formulation presented here, equations embodying conservation of flux (again with due allowance for generation and recombination) which incorporate the proper boundary conditions from the outset are solved in the steady-state one-dimensional case to yield a Green's function for the desired carrier fluxes directly. The method is more general than the commonly used continuity equation formulation in that the physical dimensions of the system and the diffusion lengths are not restricted to be large compared to the mean free path; in particular it is unnecessary to assume Fick's law for diffusion processes. Otherwise the method is equivalent to the continuity equation analysis. An example involving carrier generation in a plane region bounded on one side by a surface of arbitrary reflection coefficient (or recombination velocity) and on the other by a collecting *p-n* junction is worked out. The results are shown to reduce to those obtained via the continuity equation in the appropriate limiting case.

I. INTRODUCTION

In solving problems dealing with added carrier transport in semiconductors, it is customary to derive a continuity equation,[1,2] the basis of which is the computation of the gain or loss of charge carriers by diffusion, drift, generation, and recombination per unit time in an infinitesimal volume element, and to solve this equation under appropriate boundary conditions. The currents which flow are then computed from current equations which relate the average flux of carriers to the already computed concentrations and gradients by way of simple Fick's law diffusion theory and first-order Boltzmann transport theory. The validity of the results so obtained is limited by certain assumptions which are made in connection with simple random-walk theory,[3] the most important of which is that the number of free paths which a particle experiences be, on the average, large compared with unity. This, in turn, requires that the physical dimensions of the system being studied be large compared with the mean free path, and that the probability of recombination within the mean time between collisions be small in comparison with unity.

In a discussion of the scattering and absorption of light in diffuse media, Coltman, Ebbigshausen, and Altar[4] and later Longini[5] have shown how diffuse light fluxes can be calculated as a function of thickness of scattering medium by a method based upon internal multiple reflection and absorption. Certain aspects of their approach have been extended in connection with carrier recombination studies in semiconductors by McKelvey.[6] Since the basic assumptions upon which these calculations are founded, that is, conservation (with due allowance for absorption) of flux and completely random or diffuse scattering at each collision or reflection, hold under a great variety of conditions for the transport of charge carriers in semiconductors, it should be possible to formulate a mathematically analogous steady-state flux theory of carrier transport in semiconductors. This general approach forms the basis for certain rather restricted calculations involving recombination at surfaces or internal boundaries which have been made previously.[6–8] It is the purpose of this paper to formulate this general steady-state theory as it applies to semiconductors and to show the relation between the results of this theory and the results obtained in a more conventional manner from the steady-state continuity equation in a simple case of some practical interest.

II. FLUX RELATIONS IN A DIFFUSE SCATTERING MEDIUM

Coltman, Ebbigshausen, and Altar[4] consider the physical situation which is illustrated in Fig. 1. A layer of scattering material of thickness x is considered, and an additional scattering layer of thickness dx is interposed between the source and the original material. A steady-state, one-dimensional situation is assumed. A diffuse flux A (of light or of charge carriers, as the case may be) which is generated just at the surface of the layer of thickness dx and which proceeds initially to the right in Fig. 1, is incident upon the system, and gives rise to the reflected and transmitted fluxes F_1, F_1', F_2', and F_3 as shown. The distance dx is considered to be so small that the probability of multiple scattering or absorption events in that distance is negligible; in

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[1] W. Van Roosbroeck, Phys. Rev. **91**, 282 (1953).
[2] W. Van Roosbroeck, Bell System Tech. J. **29**, 560 (1950).
[3] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill Book Company, Inc., New York, 1938), p. 268ff.
[4] J. Coltman, E. G. Ebbigshausen, and W. Altar, J. Appl. Phys. **18**, 530 (1947).
[5] R. L. Longini, J. Opt. Soc. Am. **39**, 551 (1949).
[6] J. P. McKelvey, Phys. Rev. **106**, 910 (1957).
[7] J. P. McKelvey and R. L. Longini, J. Appl. Phys. **25**, 634 (1954).
[8] J. P. McKelvey and R. L. Longini, Phys. Rev. **99**, 1227 (1955).

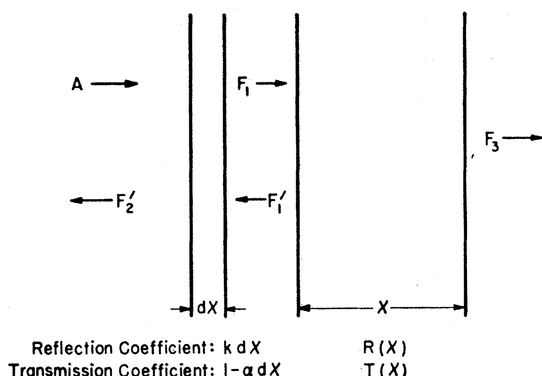

FIG. 1. Fluxes arising between a layer of bulk material of thickness x and an incremental layer of material of thickness dx which is interposed between the bulk layer and a generation source. The reflection coefficient of the thin layer is defined as kdx and the transmission coefficient as $1-\alpha dx$.

other words, the reflection coefficient and the absorption coefficient of the layer are considered to be strictly proportional to dx. According to this view, the reflection (backscattering) coefficient of this thin layer is defined to be kdx and the transmission coefficient to be $1-\alpha dx$. The parameter α includes both scattering and absorption effects, and it follows therefore that $\alpha \geqslant k$, the equality holding when there is no true absorption (or recombination), but only scattering. It is shown by Coltman *et al.*[4] that under these circumstances, differential equations for the reflection coefficient and transmission coefficient of the layer of thickness x [the reflection coefficient and transmission coefficient being defined as $R(x+dx)=F_2'/A$ and $T(x+dx)=F_3/A$, respectively] can be written, which, upon integration, yield the following results:

$$R(x)=\frac{(k/q)\sinh qx}{\cosh qx+(\alpha/q)\sinh qx}, \tag{1}$$

and

$$T(x)=[\cosh qx+(\alpha/q)\sinh qx]^{-1}, \tag{2}$$

where

$$q=(\alpha^2-k^2)^{\frac{1}{2}}. \tag{3}$$

A brief resumé of the analysis, using a rather more general method of attack[6,8] than the "multiple reflection" treatment of Coltman *et al.* is given in Appendix A.

It is necessary to identify the quantities α, k, and q with physical parameters such as the mean free path, mean carrier lifetime and the mean thermal velocity. Since the differential scattering probability kdx is assumed to be strictly proportional to dx, it is obvious that k is related inversely to the scattering mean free path λ. Likewise, if the differential "nonsurvival" coefficient α is written

$$\alpha=k+w, \tag{4}$$

then[5] w represents a differential recombination coefficient which, since the differential recombination probability wdx is assumed to be strictly proportional to dx, must be inversely proportional to the quantity $\bar{c}\tau$, where $\bar{c}$ is the mean thermal velocity $[\bar{c}=(8kT/\pi m^*)^{\frac{1}{2}}]$ and τ is the mean carrier lifetime. The constants of proportionality may be arrived at by suitable averaging calculations. A derivation of these numerical constants is given in Appendix B, and using the results obtained there to complete the definition of k and w, one may write

$$k=3/4\lambda=1/2\gamma L, \tag{5}$$

$$w=2/\bar{c}\tau=\gamma/L, \tag{6}$$

where

$$L=(\lambda\bar{c}\tau/3)^{\frac{1}{2}}=(D\tau)^{\frac{1}{2}}, \tag{7}$$

and

$$\gamma=2\lambda/3L=2(\lambda/3\bar{c}\tau)^{\frac{1}{2}}. \tag{8}$$

It follows, then, from Eqs. (3)–(6) that

$$q=(1+\gamma^2)^{\frac{1}{2}}/L. \tag{9}$$

Note that in the limit $\gamma\rightarrow 0$ (which is a necessary condition for the applicability of the continuity equation analysis), using the definitions of w and k given by (5) and (6), $q\rightarrow 1/L$. Since γ is solely a function of the ratio of λ and $\bar{c}\tau$, it is inversely proportional to the square root of the number of free paths which a carrier experiences before it is lost by recombination. One would therefore expect that the results of this flux analysis would coincide with the results of conventional diffusion theory only in the limiting case where $\gamma \ll 1$. That this is indeed the truth is shown in the following section for a simple example of some practical interest.

III. FLUX ANALYSIS OF A SURFACE REGION BOUNDED BY A COLLECTING *p-n* JUNCTION

The application of the flux analysis outlined in the preceding section will be illustrated for the case of a plane layer of semiconductor of thickness a, bounded on one side by a surface of reflection coefficient (nonrecombination probability) R_0 and on the other side by a perfectly absorbing collecting *p-n* junction.

The problem of finding the current across the junction will be solved first by the flux method, then by use of the continuity equation analysis, and a comparison of the results obtained by the two approaches will be made. The geometry which is used in formulating the flux approach is shown in Fig. 2. It is assumed that a carrier flux dA is generated isotropically at a distance x from the surface so that equal fluxes $\frac{1}{2}dA$ proceed initially to the right and to the left, and that these fluxes give rise to the reflected and transmitted fluxes dF_1, dF_1', dF_2, dF_2', and dF_3 as illustrated. These fluxes are assumed to represent fluxes of excess carriers only; the principle of detailed balancing assures that any such flux system arising from *equilibrium* carrier generation will be balanced by an equal and

opposite system caused by equilibrium recombination processes. It is assumed that a steady state has been reached, and that no electric fields act upon the carriers in the region of the crystal which is of interest. The latter assumption is equivalent to the assumption of highly extrinsic bulk semiconductor material.[1] The geometry is similar to that which is encountered in *p-n* photovoltaic devices.

Referring to Fig. 2, one can write the following relations between the fluxes which are illustrated there:

$$\begin{aligned} dF_1 &= \tfrac{1}{2}dA + R_1 dF_1' + T_1 dF_2, \\ dF_1' &= \tfrac{1}{2}dA + R_2 dF_1, \\ dF_2 &= R_0 dF_2', \\ dF_2' &= T_1 dF_1' + R_1 dF_2, \\ dF_3 &= T_2 dF_1. \end{aligned} \tag{10}$$

These simultaneous equations may be solved to yield the following expressions for the fluxes:

$$\begin{aligned} dF_1 &= (dA/2\Delta)[(1-R_0R_1)(1+R_1)+R_0T_1^2], \\ dF_1' &= (dA/2\Delta)(1-R_0R_1)(1+R_2), \\ dF_2 &= (dA/2\Delta)R_0T_1(1+R_2), \\ dF_2' &= (dA/2\Delta)T_1(1+R_2), \\ dF_3 &= (dA/2\Delta)T_2[(1-R_0R_1)(1+R_1)+R_0T_1^2], \end{aligned} \tag{11}$$

where

$$\Delta \equiv (1-R_0R_1)(1-R_1R_2)-R_0R_2T_1^2. \tag{12}$$

The flux collected by the junction is dF_3. Note, however, that according to the results of the preceding section, one may express R_1 and T_1 as $R(x)$ and $T(x)$, and R_2 and T_2 as $R(a-x)$ and $T(a-x)$, respectively, through the use of Eqs. (1) and (2). Making these substitutions and simplifying, using the mathematical identity

$$\frac{R(x)R(a-x)}{T(x)T(a-x)} = \frac{1}{T(x)T(a-x)} - \frac{1}{T(a)}, \tag{13}$$

which is easily proved from (1), (2), and (3), and writing out the differential flux dF_3 across the junction produced by the differential generated flux dA, it can be shown that

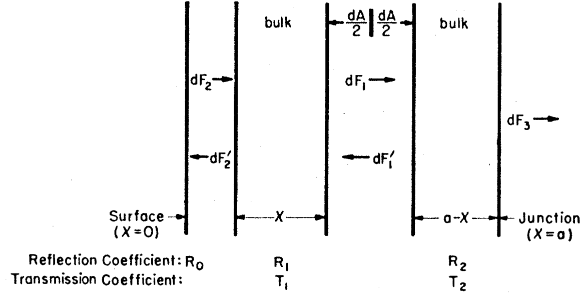

FIG. 2. Disposition of fluxes in a plane semiconductor layer bounded on one side by a surface of reflection coefficient R_0 and on the other by a perfectly absorbing *p-n* junction. These fluxes are assumed to arise from an isotropic plane source of generation at a distance x to the right of the surface. Bulk reflection and transmission coefficients are defined as illustrated.

$$dF_3 = \tfrac{1}{2}dA \frac{\left[\dfrac{1}{T(x)} - R_0\dfrac{R(x)}{T(x)}\right]\left[\dfrac{1}{T(x)} + \dfrac{R(x)}{T(x)}\right] + R_0}{\dfrac{1}{T(a)}\left[\dfrac{1}{T(x)} - R_0\dfrac{R(x)}{T(x)}\right] - R_0\dfrac{R(a-x)}{T(a-x)}}$$

$$= dA \frac{[2\gamma(1+\gamma^2)^{\frac{1}{2}}\cosh qx + (2\gamma^2+1-R_0)\sinh qx][\gamma(1+\gamma^2)^{\frac{1}{2}}\cosh qx + (1+\gamma^2)\sinh qx] + 2\gamma^2(1+\gamma^2)R_0}{[2\gamma(1+\gamma^2)^{\frac{1}{2}}\cosh qx + (2\gamma^2+1-R_0)\sinh qx][2\gamma(1+\gamma^2)^{\frac{1}{2}}\cosh qa + (1+2\gamma^2)\sinh qa] - 2\gamma R_0(1+\gamma^2)^{\frac{1}{2}}\sinh q(a-x)},$$

or

$$dF_3 = \Phi(x,\gamma)dA. \tag{14}$$

In general, the flux source strength dA will be a function of x. Since the carriers generated within a sufficiently thin elementary layer contribute to the flux across the boundaries of that layer without first-order losses by recombination, the flux dA may be related to the generation rate of carriers per unit volume, $g(x)$, by a simple conservation argument, the result being

$$dA = g(x)dx. \tag{15}$$

Substituting this result into (14) one obtains a Green's function for the junction current which may in principle be integrated over any given generation function $g(x)$ to give the total flux across the junction. Thus,

$$\begin{aligned} dF_3 &= g(x)\Phi(x,\gamma)dx, \\ F_3(\gamma) &= \int_0^a g(x)\Phi(x,\gamma)dx. \end{aligned} \tag{16}$$

In some instances, the integral must be evaluated by approximate or numerical methods, or by a computer. It is instructive to examine the rather simple case where

$$g(x) = g_0\delta(x-x_0), \quad (0 \leqslant x_0 \leqslant a). \tag{17}$$

$\delta(x)$ referring to the Dirac δ function. In this case, F_3 can be evaluated analytically, the result being

$$F_3 = g_0\Phi(x_0,\gamma). \tag{18}$$

The continuity equation applicable to this example may be written as

$$D(d^2p/dx^2)-p(x)/\tau=0, \tag{19}$$

where D ($\equiv\frac{1}{3}\lambda\bar{c}$) is the diffusion coefficient and p is the excess carrier concentration (actual concentration minus equilibrium concentration). It may be solved under the boundary conditions:

(i) $D(dp/dx)_{x=0}=sp(0)$,

where s is the surface recombination velocity[9];

(ii) $\lim_{\epsilon\to0}p(x_0-\epsilon)=\lim_{\epsilon\to0}p(x_0+\epsilon)$;

(iii) $p(a)=0$;

(iv) $\lim_{\epsilon\to0}[D(dp/dx)_{x_0-\epsilon}-D(dp/dx)_{x_0+\epsilon}]=g_0$,

to yield the excess carrier concentration. The flux at the junction may then be obtained by evaluating the quantity $-D(dp/dx)$ at $x=a$. The boundary conditions so chosen define precisely the same conditions of carrier generation as Eq. (17). The result of the continuity equation analysis subject to these boundary conditions can be shown to be

$$F=\frac{g_0}{\cosh[(a-x_0)/L]+\dfrac{(sL/D)+\tanh(x_0/L)}{1+(sL/D)\tanh(x_0/L)}\sinh[(a-x_0)/L]}. \tag{20}$$

It is to be expected that (18) reduces to (20) in the limit where the continuity equation analysis is valid. Accordingly, we shall examine the behavior of (18) in the limit $\gamma\ll1$. If in the expression for $\Phi(x_0,\gamma)$, the radicals are expanded by the binomial theorem, and if subsequently all terms in γ^2 are neglected (this assumes $\gamma^2\ll1-R_0$ as well as $\gamma\ll1$), one obtains

$$\Phi(x_0,\gamma)=\frac{[2\gamma\cosh qx_0+(1-R_0)\sinh qx_0](\gamma\cosh qx_0+\sinh qx_0)}{[2\gamma\cosh qx_0+(1-R_0)\sinh qx_0](2\gamma\cosh qa+\sinh qa)-2\gamma R_0\sinh q(a-x_0)}. \tag{21}$$

In this expression, if one divides numerator and denominator by

$$[2\gamma\cosh qx_0+(1-R_0)\sinh qx_0](\gamma\cosh qx_0+\sinh qx_0)$$

and then expresses the hyperbolic functions of qa in terms of functions of qx_0 and $q(a-x_0)$, neglecting terms in γ^2 where they arise, assuming also that $1-R_0\ll1$, one obtains the expression

$$\Phi(x_0,\gamma)=\left[\frac{2\gamma+\tanh qx_0}{\gamma+\tanh qx_0}\cosh q(a-x_0)+\frac{\tanh qx_0}{\gamma+\tanh qx_0}\,\frac{\tanh qx_0+(1-R_0)/2\gamma}{1+[(1-R_0)/2\gamma]\tanh qx_0}\sinh q(a-x_0)\right]^{-1}. \tag{22}$$

According to (9), for $\gamma\ll1$, $q\cong1/L$; likewise, for $x_0\gg\lambda$, the coefficient of $\cosh q(a-x_0)$ and the expression $\tanh qx_0(\gamma+\tanh qx_0)^{-1}$ approach unity. Therefore, under these conditions (18) agrees with (20), provided that one takes

$$s=\frac{D}{L}\frac{(1-R_0)}{2\gamma}=\frac{\bar{c}}{4}(1-R_0). \tag{23}$$

For $1-R_0\ll1$, this is in close agreement with previously published results.[7] In the case where the $\sinh qx$ terms in $\Phi(x_0,\gamma)$ may be neglected (i.e., for $x_0\ll\lambda$), inspection of (14) reveals that it is necessary to retain terms of the order of γ^2. It is easy to show that in this case (again for $\gamma\ll1$)

$$\Phi(x_0,\gamma)=\left(\frac{2}{1+R_0}\cosh qa+\frac{1}{\gamma}\frac{(1-R_0)}{(1+R_0)}\sinh qa\right)^{-1}. \qquad (x_0\ll\lambda) \tag{24}$$

This is again in agreement with the predictions of (20), provided that s is taken as $\frac{1}{2}\bar{c}(1-R_0)/(1+R_0)$; this is in exact agreement with previously obtained results[7] and in close agreement with (23) for $1-R_0\ll1$. The slight discrepancy in the expression obtained for s may be due to the fact that the assumption $1-R_0\ll1$ was used to obtain (22), but was not used to obtain (24).

It should be emphasized that the fulfillment of the condition $\gamma\ll1$ (i.e., $\lambda\ll L$) does not by itself guarantee agreement between the flux theory presented here and the classical diffusion-recombination theory as applied via the continuity equation. In addition, the physical dimensions of the system under consideration must, in general, be large in comparison with λ for the classical continuity theory to be applicable. It is apparent, for example, that (22) no longer corresponds in a simple way with (20) when $\tanh qx_0\sim\gamma$ (i.e., when $x_0\sim\lambda$). This occurs not only because the quadratic term in the numerator of $\Phi(x_0,\gamma)$ must be considered, as mentioned previously, but also because of a breakdown in the assumptions upon which the classical diffusion-recombination analysis is based—notably the assumption

[9] W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 321.

that many free paths are involved, on the average, between the generation and subsequent collection or recombination of a carrier.

In the limit where $\gamma \ll 1$, and x_0, $a \ll L$ (this special case is discussed in detail for simplicity), it is found that $\Phi(x_0,\gamma)$ reduces to

$$\Phi(x_0,\gamma)=\frac{1+R_0+(3x_0/4\lambda)[2+(1-R_0)]+(1-R_0)(9x_0^2/8\lambda^2)}{2+(3x_0/2\lambda)+(1-R_0)(3a/2\lambda)(1+3x_0/4\lambda)}, \tag{25}$$

This expression is obtained by expanding the sinh, cosh, and $(1+\gamma^2)^{\frac{1}{2}}$ functions in (14), and simplifying numerator and denominator, neglecting terms of higher order than quadratic in γ, x_0, a, or their products. If, in addition, it is assumed that x_0, a, $a-x_0 \gg \lambda$, then

$$\Phi(x_0,\gamma)=\frac{1+(3x_0/4\lambda)(1-R_0)}{1+(3a/4\lambda)(1-R_0)}. \quad (x_0,\ a,\ a-x_0 \gg \lambda) \tag{26}$$

Equation (26) agrees with the result of the continuity equation analysis, which predicts that $\Phi=(1+sx_0/D)/(1+sa/D)$, provided s is defined as in (23). If the condition x_0, a, $a-x_0 \gg \lambda$ does *not* hold, then the expression (25) must be used to compute Φ, and the results no longer correspond in any simple way to those derived by way of the continuity equation. On the other hand, if x_0 is identically zero, then (25) gives

$$\Phi(0,\gamma)=\left(\frac{2}{1+R_0}+\frac{3a}{2\lambda}\frac{(1-R_0)}{(1+R_0)}\right)^{-1}, \quad (x_0=0) \tag{27}$$

which does in this restricted case again agree with the result of the continuity equation analysis if $1-R_0 \ll 1$ and if s is defined as $\frac{1}{2}\bar{c}(1-R_0)/(1+R_0)$; this definition of s agrees closely with that set down in (23) for $1-R_0 \ll 1$.

In comparing the results of the flux analysis with those obtained from the continuity equation, it has been assumed in various contexts that $1-R_0 \ll 1$. That this is indeed the case for most experimentally realizable surface conditions, at least for germanium and silicon, has been demonstrated by surface recombination measurements.[7,10] It should be noted, however, that a boundary such as the one characterized as the "surface" in this example need not necessarily correspond to a real crystal-atmosphere interface, but may describe any boundary which is partially reflecting to carriers such as, for instance, a *p-n* junction which is maintained at arbitrary bias.

In instances where the physical dimensions of the system are not large compared with λ, where L is not very much greater than λ, where γ^2 is not small compared with $1-R_0$ or, finally, where $1-R_0$ is not small compared with unity, the use of the continuity equation must be abandoned in favor of a more general treatment, such as the one outlined here. The first of these instances may be encountered in sufficiently thin surface *p-n* junction structures, the second may arise in certain of the III-V intermetallic compounds.

It should be pointed out, parenthetically, that this example can be solved by the same general procedure even if the absorbing *p-n* junction is replaced by a partially reflecting surface of reflectance R_0'. In this case the flux crossing this boundary is given by

$$dF=\tfrac{1}{2}dA\,\frac{T_2(1-R_0')[(1+R_1)(1-R_0R_1)+R_0T_1^2]}{(1-R_0R_1)[(1-R_1R_2)(1-R_0'R_2)-R_0'R_1T_2^2]-(1-R_0'R_2)R_0R_2T_1^2-R_0R_0'T_1^2T_2^2}, \tag{28}$$

where, as usual, R_1, R_2, T_1, T_2 are to be identified with $R(x)$, $R(a-x)$, $T(x)$, $T(a-x)$, respectively, as given by (1) and (2). This expression reduces to the expression given for dF_3 in (11) and (12) for $R_0'=0$.

IV. DISCUSSION

It has been demonstrated that results concerning fluxes of added carriers in semiconductors usually obtained by solving a continuity equation and computing currents from concentrations and concentration gradients obtained thereby can be obtained by a consideration of scattering and absorption phenomena acting on the fluxes themselves. The results of the flux analysis, while as yet having been derived only for the steady-state, one-dimensional, zero-electric-field case, are nevertheless in some respects more general than those obtained from the continuity equation, in that no restrictive assumptions regarding the dimensions of the system or the average number of free paths experienced by the carriers before absorption need be made.

In familiar semiconductors, even those characterized by high carrier mobility, the mean free path is always less than 10^{-4} cm at room temperature. In very pure germanium at low temperatures ($\sim 10°$K) the mean free path may be of the order of 10^{-3} cm. It is thus clear that only if the system dimensions are very small, or if the diffusion length L is very low, will there be appreciable departure from the results derived from the continuity equation analysis. The effect of short lifetime will not be appreciable in germanium at low

[10] W. Heywang and M. Zerbst, Z. Naturforsch. **11a**, 256 (1956).

temperature if the lifetime exceeds about 10^{-9} sec, and at room temperature will not be apparent for lifetime in excess of about 10^{-10} sec.

It should be kept in mind that the evaluation of the scattering and absorption parameters is based upon a simple kinetic model which assumes strict equality of probability of forward and backward scattering, and in which no account of the effect of either applied electric fields or internal electric fields[1] (arising from the tendency of the crystal to maintain local electrical neutrality in situations where the hole and electron mobilities are unequal) has been taken. The effect of the internal fields may be expected to play an important role when large added carrier concentrations or high concentration gradients exist, but may be expected to be negligible in small-signal situations where the added carrier density is small compared to the equilibrium carrier density. There will be, on the average, no internal fields even at high added carrier concentrations in cases where the hole and electron mobilities are equal, and the flux analysis may, of course, be expected to apply unequivocally under those conditions. The desirability of extending the analysis presented here to include cases where nonnegligible electric fields are present is obvious, and indeed it is possible to effect such a generalization on a simple phenomenological basis by modifying the scattering and absorption parameters with additive terms varying linearly with the electric field. It is, however, not a simple matter to arrive at expressions for field-dependent scattering and recombination parameters on a rigorous, physically correct basis, and (although the results of the phenomenological theory agree with the results given by the continuity equation in the appropriate limit), it appears that the simple phenomenological model may not be sufficient to describe the physical situation correctly in a completely general way. It is expected that continuing research effort on this subject will yield a physically rigorous solution to the field-dependent problem.

Despite the fact that rather cumbersome algebraic procedures have been necessary in the foregoing analysis to exhibit in detail the relationship between the flux theory and the conventional analysis, it should be noted that the flux analysis is generally no more tedious nor difficult to apply in a given situation than the conventional method. The ease and precision with which surface or interfacial boundary conditions can be introduced, and the generality of the results obtained render it, where applicable, a powerful tool for the analysis of excess carrier problems in semiconductors.

APPENDIX A. REFLECTION AND TRANSMISSION COEFFICIENTS

Referring to Fig. 1, the following flux relations can be written for the fluxes illustrated therein:

$$\begin{aligned} F_1 &= A(1-\alpha dx)+kF_1'dx, \\ F_1' &= RF_1, \\ F_2' &= kAdx+(1-\alpha dx)F_1', \\ F_3 &= TF_1. \end{aligned} \tag{A.1}$$

Here kdx and $1-\alpha dx$ are assumed to be the reflection (backscattering) and transmission (survival without backscattering or recombination) coefficients, respectively, of a layer of thickness dx. The equations (A.1) may be solved to give

$$\begin{aligned} F_1 &= A\frac{1-\alpha dx}{1-kRdx}, \quad F_2' = A\left[kdx+\frac{R(1-\alpha dx)^2}{1-kRdx}\right], \\ F_1' &= AR\frac{1-\alpha dx}{1-kRdx}, \quad F_3 = AT\frac{1-\alpha dx}{1-kRdx}. \end{aligned} \tag{A.2}$$

Then (following Coltman *et al.*[4]),

$$R(x+dx)=\frac{F_2'}{A}=kdx+\frac{R(1-\alpha dx)^2}{1-kRdx},$$

and (neglecting second-order terms in dx)

$$\frac{dR}{dx}=\frac{R(x+dx)-R(x)}{dx}=k\left(1-\frac{2\alpha}{k}R+R^2\right).$$

This equation can be integrated, the constant of integration being evaluated by the requirement that $R(0)=0$. The result is Eq. (1). The transmission coefficient can be derived in a somewhat similar manner to give (2). Note that if there is no absorption ($\alpha=k$, $q=0$), then $R+T=1$, as expected.

APPENDIX B. DERIVATION OF PROPORTIONALITY CONSTANTS

In deriving Eq. (5), it is necessary to show that the average distance from which flux, entering an element of area normal to the x axis and being scattered in that element, originates, projected upon the x axis, is $2\lambda/3$. Referring to Fig. 3, one can write

$$\bar{\lambda}_x=\langle\lambda\cos\theta\rangle_{\text{av}}=\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}(\lambda\cos\theta)\cos\theta\,\lambda^2\sin\theta d\theta d\phi\Big/\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\cos\theta\,\lambda^2\sin\theta d\theta d\phi,$$

whence

$$\bar{\lambda}_x=2\lambda/3. \tag{B.1}$$

The quantity ϕ is the azimuthal coordinate about the polar (x) axis. The extra $\cos\theta$ factor in the integral is, of course, the Lambertian cosine factor, necessitated by the fact that the area subtended, and thus the flux inter-

cepted, by dS varies with angle as $\cos\theta$. The result is consistent with the conclusions of previous treatments involving this subject.[7,8] Effectively, then, so far as the fluxes are concerned, scattering events are separated, on the average, by distance $2\lambda/3$ along the x axis. The reflectivity kdx, should thus be $\frac{1}{2}$ for $dx=2\lambda/3$, assuming the scattering to be isotropic (i.e., the probability of a backward scatter equals that of a forward scatter equals $\frac{1}{2}$). Writing down this equality and solving for k, one obtains at once, $k=3/4\lambda$, or Eq. (5).

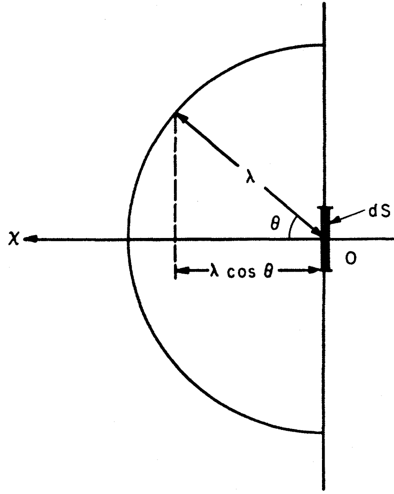

FIG. 3. Geometry assumed in calculating average separation of scattering events along the x axis.

To derive Eq. (6), it must be noted that the fraction of flux absorbed in a layer approaches asymptotically $\bar{t}/\tau$ as the layer thickness approaches zero, where $\bar{t}$ is the mean traversal time for the flux. For an elementary layer of thickness ϵ and an angle of approach θ relative to the x axis, the time of traversal is $\epsilon(v\cos\theta)^{-1}$, where v is the flux velocity. This, accordingly, is the quantity which must be averaged in order to find $\bar{t}$. The weighting factors are those which are used in classical kinetic calculations which involve particles striking a wall or streaming through an aperture; the average is thus computed according to Eq. (B.2).

$$\bar{t}(\epsilon)=\left\langle\frac{\epsilon}{v\cos\theta}\right\rangle=\int_0^\infty\int_0^{\frac{1}{2}\pi}\frac{\epsilon}{v\cos\theta}v^2\exp(-mv^2/2kT)v\cos\theta\sin\theta d\theta dv\bigg/\int_0^\infty\int_0^{\frac{1}{2}\pi}v^2\exp(-mv^2/2kT)v\cos\theta\sin\theta d\theta dv. \quad \text{(B.2)}$$

The $\cos\theta\sin\theta$ terms arise from the assumed spherical isotropy of velocities, and the factor v is included because the flux associated with an element of the velocity distribution is the particle density associated with that element times the velocity which pertains to it. The factor $v^2\exp(-mv^2/2kT)$ represents the familiar Maxwell-Boltzmann distribution of velocities. Evaluating the integrals in (B.2) and recalling from the definition of w that $w\epsilon$ also represents the fraction of flux absorbed in an elementary layer of thickness ϵ, it can be shown that

$$\bar{t}(\epsilon)=2\epsilon/\bar{c}=w\tau\epsilon, \quad \text{(B.3)}$$

whereby

$$w=2/\bar{c}\tau,$$

as represented by Eq. (6).