TABLE I. The change in g factor due to the polarization of conduction electrons in rare-earth metals; the theoretical values are calculated from Eq. (25) with $I=-0.174$ ev.

Element	Δg (theoret)	Δg (exp)
Gd	0.055	0.04 ± 0.01
Tb	0.028	0.04 ± 0.01[a]
Dy	0.018	0.017 ± 0.009[b]
Ho	0.014	0.03[c]
Er	0.011	0.05[c]
Tm	0.009	0[d]

[a] W. C. Thorburn, S. Legvold, and F. H. Spedding, Phys. Rev. **112**, 56 (1958).
[b] D. R. Behrendt, S. Legvold, and F. H. Spedding, Phys. Rev. **109**, 1544 (1958).
[c] F. H. Spedding, S. Levgold, A. H. Daane, and L. D. Jennings, in *Progress in Low-Temperature Physics*, edited by J. C. Gorter (North-Holland Publishing Company, Amsterdam, 1957), Vol. II, pp. 368–394.
[d] D. D. Davis and R. M. Bozorth, Phys. Rev. **118**, 1543 (1960).

susceptibilities. The theoretical and the experimental values agree in order of magnitude.

Many rare-earth metals and alloys undergo a spontaneous magnetic transition from ferromagnetic to antiferromagnetic ordering. However, these materials obey the Curie-Weiss law in the paramagnetic temperature region with positive Curie points. This is a good indication that the basic interaction in the materials is ferromagnetic. In such cases, Eqs. (14) and (23) should apply to the paramagnetic Curie temperatures rather than the Néel temperatures.

The indirect exchange model may apply to the transition elements as well. However, the problem is hard to analyze because of the complicated band structures.

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Current-Carrier Transport with Space Charge in Semiconductors

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Differential equations are given for a general formulation of current-carrier transport that includes space charge. Arbitrary dependences of diffusivities and magnitudes of drift velocities on electrostatic field are considered, and extension is made for applied magnetic field. Though excess electron and hole concentrations are not equal, the small-signal recombination rate depends on a single lifetime, the "diffusion-length lifetime," τ_0. The formulation is applied to one-dimensional drift with recombination for an injected pulse of electron-hole pairs. The exact electron and hole distributions are obtained in closed form for the linear small-signal case. The condition for linearity is given; it is usually the same as that for substantially unperturbed applied field, E_0. There are two principal types of solution, essentially according to whether τ_0 is larger or smaller than the dielectric relaxation time, τ_d. For $\tau_0>\tau_d$, the electron and hole distributions in not too strongly extrinsic material are ultimately similar Gaussian distributions displaced by the "polarization distance," x_P, the distance electrons and holes drift apart in time $(\tau_d^{-1}-\tau_0^{-1})^{-1}$. These distributions drift at a velocity that differs from the ambipolar velocity by an amount which, besides being small for small τ_d/τ_0, vanishes for equal mobilities. They spread, exhibiting an apparent diffusion. A "pseudodiffusivity," D_v, is defined. For $\tau_0\gg\tau_d$ and constant mobilities, D_v is proportional to $\tau_d E_0^2/\sigma_0^2$, with σ_0 the conductivity. The ambipolar diffusivity and D_v are additive. They are equal in intrinsic material for E_0 equal to kT/e divided by the Debye length $(kT\epsilon/8\pi n_i e^2)^{\frac{1}{2}}$, or 10 v/cm for silicon at 300°K. An extension to a nonlinear case involving high-level injection is given; concentration-dependent D_v and velocity function are defined. For sufficiently strongly extrinsic material and $\tau_0>\tau_d$, the minority carriers drift in a delta pulse that leads the majority carriers distributed in an exponential tail of characteristic length x_P, which may be quite large. For nonconstant mobilities and $\tau_0>\tau_d$, ambipolar velocity in the majority-carrier or "reverse" direction may occur. For $\tau_d>\tau_0$, the other principal type of solution gives distributions that in general (and for constant mobilities) drift in the reverse direction. Involving also regions of local carrier depletion, and thus generation as well as recombination, these distributions may persist for times long compared with τ_0, being attenuated then with time constant τ_d.

1. INTRODUCTION

WITH carrier injection and transport in semiconductor material of high resistivity, the widely used approximation of local electrical neutrality frequently does not apply. Thus, for various experiments and for a number of devices, including semiconductor detectors of nuclear particles, solutions are needed that take space charge into account. Extending results previously reported,[1] this paper presents a general formulation of transport with space charge, including applied magnetic field, and gives solutions for various cases of one-dimensional drift with recombination. An injected pulse of electron-hole pairs is considered. For linear small-signal cases, with relatively small perturbation of applied electrostatic field, exact solutions are obtained

[1] W. van Roosbroeck, Bull. Am. Phys. Soc. **5**, 180 (1960).

in closed form, and these are treated in detail. From the viewpoint of the mathematical methods employed, the analysis is quite similar to that of ambipolar drift with trapping,[2–4] for which were established certain Laplace transforms[3] by whose use the solutions could be obtained. On the basis of this analysis, an extension to a nonlinear case of high-level injection is presented. Trapping is not considered, but the theory could be extended to include trapping without essential change in its formal structure.

With the present treatment, the neglect of diffusion in the linear small-signal case makes it possible to obtain exact solutions for the carrier-concentration distributions that are otherwise unrestricted. At the same time, the neglect of diffusion facilitates physical interpretation of these solutions. Theory for transport with space charge has been given that is based on direct calculation of the means and second moments of the distributions for the general linear small-signal case.[5] Diffusion has been taken into account at the outset also in still another treatment,[6] in which evaluation of the distributions in closed form is accomplished under the assumptions of strongly extrinsic material and comparatively small dielectric relaxation time.

In advance of the proper derivations, it may be well to outline in descriptive terms some principal aspects of the present results concerning transport with space charge. Note, first, that local electrical neutrality is, in a sense, tantamount to infinite Coulomb forces or zero dielectric relaxation time. Effects associated with space charge are the more pronounced the larger is the dielectric relaxation time. They are thus the more pronounced the higher is the resistivity, whether this be in, say, weakly extrinsic silicon at room temperature, or in a high-purity semiconductor at low temperature, which may be quite strongly extrinsic. Note also that the problem of specifying the recombination rate with space charge, that is, with excess electron and hole concentrations that are unequal, is similar to that previously considered in connection with trapping[3]: For given equilibrium concentrations of electrons and holes, the small-signal rate is determined simply by the "diffusion-length lifetime," τ_0.

The condition that τ_0 be larger than the dielectric relaxation time, τ_d, is of course implicit in the assumption of local neutrality. Transport with space charge under this condition accordingly possesses some qualitative similarity to neutral transport. In particular, with constant mobilities, the direction of drift of the concentration disturbances is that of the minority carriers. With the finite Coulomb forces taken into account, however, it is shown that in not too strongly extrinsic material under conditions that may readily occur in practice, spreading of the distributions may be associated primarily with a field-dependent "pseudo-diffusivity," D_v, rather than with diffusion itself. This D_v is occasioned by the applied field acting against Coulomb forces to which the contribution of the equilibrium minority carriers is not inappreciable. Even with strongly extrinsic material, for which D_v is negligible, numerical estimate shows that the "polarization distance," x_P, by which the means of the electron and hole distributions are separated[5] may in practice be quite large.

Familiar descriptive considerations that tacitly entail local neutrality require quite drastic revision if τ_d is greater than τ_0. The transport is then typified by comparatively rapid recombination (rather than dielectric relaxation) in conjunction with polarization of charge essentially by the distance electrons and holes drift apart in time τ_0 (rather than τ_d). With this polarization, recombining excess current carriers of one kind cause local depletion of the carriers of the other kind, and there is generation of carriers as well as recombination. For the carriers of each type, there are adjoining regions of carrier excess and carrier depletion, whose means exhibit the polarization of charge and for which the net total number of carriers is substantially zero. The frequency of recombination and generation of the minority carriers being the greater, depletion of the majority carriers is the more pronounced. The distributions, which result essentially from drift with a multiple recombination-generation mechanism, therefore move together in the majority-carrier or "reverse" direction. For $\tau_d \gg \tau_0$, excess electrons do not recombine with excess holes, since there is no overlap of the distributions of excess carriers; and the concentration disturbances, attenuated with time constant substantially τ_d, persist for a time long compared with the lifetime, τ_0. In this case, both the polarization of charge and the attenuation are such that the dielectric relaxation time and lifetime are, in effect, interchanged.

2. FORMULATION

In this section, phenomenological differential equations that take space charge into account are first written for arbitrary dependences of the diffusivities and the magnitudes of (collinear) drift velocities on electrostatic field, and the extension for steady applied magnetic field is included. The formulation is then examined in further detail for the case of constant mobilities and no applied magnetic field. The notation is largely consistent with notation previously employed.[3,4,7,8]

The continuity equations for holes and electrons may be written as

$$\begin{aligned} \partial p/\partial t &= -e^{-1}\,\mathrm{div}\mathbf{I}_p + g - \mathcal{R}, \\ \partial n/\partial t &= e^{-1}\,\mathrm{div}\mathbf{I}_n + g - \mathcal{R}, \end{aligned} \tag{1}$$

[2] W. van Roosbroeck, Bull. Am. Phys. Soc. **2**, 152 (1957).
[3] W. van Roosbroeck, Bell System Tech. J. **39**, 515 (1960).
[4] W. van Roosbroeck, Phys. Rev. **119**, 636 (1960).
[5] J. Keilson, J. Appl. Phys. **24**, 1198 (1953).
[6] R. Gevers, Physica **21**, 888 (1955).
[7] W. van Roosbroeck, Phys. Rev. **91**, 282 (1953).
[8] W. van Roosbroeck, Phys. Rev. **101**, 1713 (1956).

in which the same volume generation and recombination functions g and $\mathcal{R}$ apply in both equations for interband excitations and no trapping. For arbitrary drift velocity functions $\mathbf{v}_p$ and $\mathbf{v}_n$ and no applied magnetic field, the current densities are

$$\begin{aligned} \mathbf{I}_p &= e\mathbf{v}_p p - eD_p \,\mathrm{grad} p, \\ \mathbf{I}_n &= e\mathbf{v}_n n + eD_n \,\mathrm{grad} n. \end{aligned} \qquad (2)$$

For the homogeneous semiconductor (with uniform net fixed-charge concentration), space charge is associated only with the inbalance in the hole and electron concentration increments Δp and Δn, and (with uniform dielectric constant ϵ) Poisson's equation is

$$\mathrm{div}\mathbf{E} = (4\pi e/\epsilon)(\Delta p - \Delta n). \qquad (3)$$

Lamellar electrostatic field generally obtains as an excellent approximation, the effect of time-dependent magnetic field that may be associated with the transport generally being negligible. Thus, the equations

$$\mathrm{curl}\mathbf{E} = 0, \quad \mathbf{E} = -\mathrm{grad} V, \qquad (4)$$

hold. It follows readily from Eqs. (1) and (3) that the total current density $\mathbf{I}$, which includes the displacement current density, is solenoidal.[9] A field equation of Maxwell relates $\mathbf{I}$ to the curl of the magnetic field:

$$\mathbf{I} = \mathbf{I}_p + \mathbf{I}_n + (\epsilon/4\pi)\partial\mathbf{E}/\partial t = (c/4\pi)\,\mathrm{curl}\mathbf{H}. \qquad (5)$$

Since, with Gaussian units, c is the speed of light, and since (for uniform magnetic permeability) $\mathrm{div}\mathbf{H}$ is zero, it is evident that the contribution to $\mathbf{H}$ from the transport can be neglected, unless $\mathbf{I}$ is very large.

For steady uniform applied magnetic field, the current-density equations must be suitably modified; the other equations still hold. Consistently with results previously given,[8] for magnetic field applied in the direction of the z axis, the direction of the unit vector $\mathbf{k}$, the current densities are given by

$$\begin{aligned} \mathbf{I}_p - (\mathbf{I}_p \times \mathbf{k}) \tan\theta_p &= \mathbf{I}_p^*, \\ \mathbf{I}_n - (\mathbf{I}_n \times \mathbf{k}) \tan\theta_n &= \mathbf{I}_n^*, \end{aligned} \qquad (6)$$

where θ_p and θ_n are the Hall angles for holes and for electrons. These equations may readily be solved[10] for $\mathbf{I}_p$ and $\mathbf{I}_n$. In the present context, the definitions

$$\begin{aligned} \mathbf{I}_p^* \equiv{}& e(\mathbf{v}_{pt} p - D_{pt}\,\mathrm{grad} p)\cdot(\mathbf{ii}+\mathbf{jj}) \\ &+ e(\mathbf{v}_{pl} p - D_{pl}\,\mathrm{grad} p)\cdot\mathbf{kk}, \\ \mathbf{I}_n^* \equiv{}& e(\mathbf{v}_{nt} n + D_{nt}\,\mathrm{grad} n)\cdot(\mathbf{ii}+\mathbf{jj}) \\ &+ e(\mathbf{v}_{nl} n + D_{nl}\,\mathrm{grad} n)\cdot\mathbf{kk} \end{aligned} \qquad (7)$$

are employed for the case of tensor ellipsoids that are ellipsoids of rotation about the magnetic field. The second subscripts in Eqs. (7) are used to denote the phenomenological transverse or longitudinal velocity and diffusion-constant functions. Equations (6) and (7) are readily specialized for the case of small Hall angles.

[9] W. van Roosbroeck, Bell System Tech. J. **29**, 560 (1950).

[10] See Eqs. (11) of reference 8.

The current densities $\mathbf{I}_p^*$ and $\mathbf{I}_n^*$ defined by Eqs. (7) represent the "forces" from which the transport originates. The velocities and diffusion constants in these definitions do not include the effects of the Lorentz forces. For the transverse and longitudinal quantities equal, as in the case of small Hall angles, $\mathbf{I}_p^*$ and $\mathbf{I}_n^*$ are formally the same as the $\mathbf{I}_p$ and $\mathbf{I}_n$ for no applied magnetic field of Eqs. (2). With applied magnetic field but no diffusion, $\mathbf{I}_p$ and $\mathbf{I}_n$ of Eqs. (6) are current densities that result from the actual velocities, the cross-product terms in the equations being the terms that arise from the Lorentz forces. From Eqs. (6), velocity tensors that involve the Hall angles may be written, and, with the diffusion terms included, diffusivity tensors as well. It is implicit in these equations that the current densities do not change appreciably in times comparable with the relaxation times for conductivity.[11]

The continuity equations may be written out in detail by substituting for the current densities in Eqs. (1). Useful simplifications result from the assumption that the drift-velocity functions of Eqs. (2) are collinear with the electrostatic field, or that those of Eqs. (7) are collinear with the transverse and longitudinal components of the field. This assumption applies for the drift currents along a symmetry axis of the crystal, or, at least at room temperature, in general as a reasonable approximation.[12] With this assumption and with applied magnetic field taken into account for the case of small Hall angles, the following continuity equations result:

$$\begin{aligned} &\partial\Delta p/\partial t - \mathrm{div}(D_p\,\mathrm{grad}\Delta p) - g + \mathcal{R} \\ &\quad = -\mathrm{div}(p\mathbf{v}_p) - \theta_p\,\mathrm{div}(p\mathbf{v}_p\times\mathbf{k}) \\ &\quad = -\nu_p(\Delta p - \Delta n) - \mathbf{v}_p\cdot\mathrm{grad}\Delta p - p(v_p - E dv_p/dE) \\ &\quad \times \mathrm{div}(\mathbf{E}/E) - \theta_p\{(v_p/E)[\mathrm{grad}\Delta p, \mathbf{E}, \mathbf{k}] \\ &\qquad\qquad + p(v_p - E dv_p/dE)\,\mathrm{curl}(\mathbf{E}/E)\cdot\mathbf{k}\}, \qquad (8) \\ &\partial\Delta n/\partial t - \mathrm{div}(D_n\,\mathrm{grad}\Delta n) - g + \mathcal{R} \\ &\quad = \mathrm{div}(n\mathbf{v}_n) + \theta_n\,\mathrm{div}(n\mathbf{v}_n\times\mathbf{k}) \\ &\quad = -\nu_n(\Delta n - \Delta p) + \mathbf{v}_n\cdot\mathrm{grad}\Delta n + n(v_n - E dv_n/dE) \\ &\quad \times \mathrm{div}(\mathbf{E}/E) + \theta_n\{(v_n/E)[\mathrm{grad}\Delta n, \mathbf{E}, \mathbf{k}] \\ &\qquad\qquad + n(v_n - E dv_n/dE)\,\mathrm{curl}(\mathbf{E}/E)\cdot\mathbf{k}\}. \end{aligned}$$

Here, the heavy brackets denote scalar triple products

[11] For analysis of space-charge effects at frequencies comparable with the collision frequencies (as for certain experiments on plasma resonance), the respective left-hand members of Eqs. (6) should include the derivative of the current density, $\mathbf{I}_p$ or $\mathbf{I}_n$, with respect to time measured in units of the mean relaxation time for conductivity due to holes or electrons. See Dresselhaus, Kip, and Kittel, Phys. Rev. **98**, 368, **100**, 618 (1955); W. P. Allis, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21; E. Groschwitz and K. Siebertz, Z. Naturforsch. **11a**, 482 (1956); E. Groschwitz, *ibid.* **12a**, 529 (1957); P. A. Wolff, Phys. Rev. **112**, 66 (1958).

[12] The maximum departure from collinearity does not exceed a few degrees in germanium at room temperature. See M. Shibuya, Phys. Rev. **99**, 1189 (1955); W. Sasaki, M. Shibuya, and K. Mizuguchi, J. Phys. Soc. Japan **13**, 456 (1958); S. H. Koenig, Proc. Phys. Soc. (London) **A73**, 959 (1959); E. G. S. Paige, Proc. Phys. Soc. (London) **A75**, 174 (1960).

and ν_p and ν_n are defined by

$$\begin{aligned}\nu_p &\equiv 4\pi e(dv_p/dE)p/\epsilon,\\ \nu_n &\equiv 4\pi e(dv_n/dE)n/\epsilon.\end{aligned}\tag{9}$$

With the velocity magnitudes v_p and v_n known functions of the magnitude E of the field, Eqs. (3) and (8) provide three differential equations in the dependent variables Δp, Δn, and $\mathbf{E}$ or V.

The second forms of the right-hand members of Eqs. (8) apply specifically for $\mathbf{v}_p$ and $\mathbf{v}_n$ collinear with $\mathbf{E}$. The divergences of $\mathbf{v}_p$ and $\mathbf{v}_n$ may then be written so that they give contributions involving $\mathrm{div}\mathbf{E}$ and contributions involving $\mathrm{div}(\mathbf{E}/E)$; note that the unit vector $\mathbf{E}/E$ gives the direction field of the electrostatic field, which depends essentially on flow geometry. Eliminating $\mathrm{div}\mathbf{E}$ by means of Eq. (3) gives the terms involving the frequencies ν_p and ν_n. These are reciprocals of dielectric relaxation times associated, respectively, with holes and with electrons. Their sum is the reciprocal of the actual dielectric relaxation time, τ_d, or

$$\tau_d=(\nu_p+\nu_n)^{-1}.\tag{10}$$

For constant mobilities, ν_p and ν_n reduce to $4\pi\sigma_p/\epsilon$ and $4\pi\sigma_n/\epsilon$, which are proportional to the conductivities due to holes and to electrons.

The factors that multiply $\mathrm{div}(\mathbf{E}/E)$ and that multiply $\mathrm{curl}(\mathbf{E}/E)\cdot\mathbf{k}$ vanish for constant mobilities. For flow in one Cartesian dimension, $\mathrm{div}(\mathbf{E}/E)$ is zero. For cylindrical or spherical symmetry and outwardly directed field, $\mathrm{div}(\mathbf{E}/E)$ equals the reciprocal of the radius r or $2/r$, respectively, and negative signs apply for inwardly directed field. For all of these geometries, $\mathrm{curl}(\mathbf{E}/E)$ is zero. The terms in $\mathrm{div}(\mathbf{E}/E)$ are clearly carrier-generation or -depletion terms that occur for nonconstant mobilities in any flow geometry other than the parallel unidirectional one. The terms in $\mathrm{curl}(\mathbf{E}/E)\cdot\mathbf{k}$ are related terms for applied magnetic field and flow geometries other than the three simple ones considered.

From the specialization for constant mobilities and no applied magnetic field, connections with an earlier treatment[7] of the neutral case will now be exhibited, and differential equations derived that are especially suited for certain theoretical applications. For this specialization, the continuity equations may be written as

$$\begin{aligned}\partial\Delta p/\partial t &= D_p\,\mathrm{div\,grad}\Delta p-\mu_p\mathbf{E}\cdot\mathrm{grad}\Delta p\\ &\qquad -\mu_p p\,\mathrm{div}\mathbf{E}-\Delta p/\tau_p,\\ \partial\Delta n/\partial t &= D_n\,\mathrm{div\,grad}\Delta n+\mu_n\mathbf{E}\cdot\mathrm{grad}\Delta n\\ &\qquad +\mu_n n\,\mathrm{div}\mathbf{E}-\Delta n/\tau_n,\end{aligned}\tag{11}$$

in which g has been omitted and $\mathcal{R}$ written as $\Delta p/\tau_p$ and as $\Delta n/\tau_n$, with τ_p and τ_n lifetime functions for holes and for electrons. The differential equation

$$\sigma\mathbf{E}+(\epsilon/4\pi)\partial\mathbf{E}/\partial t=\mathbf{I}-e(D_n\,\mathrm{grad}\Delta n-D_p\,\mathrm{grad}\Delta p)\tag{12}$$

follows readily from Eqs. (2) and (5). In view of Poisson's equation, it is well to employ Δm and Δq defined by

$$\begin{aligned}\Delta m &\equiv \tfrac{1}{2}(\Delta p+\Delta n),\\ \Delta q &\equiv \tfrac{1}{2}(\Delta p-\Delta n),\end{aligned}\tag{13}$$

as concentration variables. With these, and by eliminating $\mathrm{div}\mathbf{E}$ by multiplying Eqs. (11), respectively, by σ_n and σ_p and adding, substituting for $\mathbf{E}$ from Eq. (12), and making use of

$$\begin{aligned}&\mathrm{div}D\,\mathrm{grad}\Delta m\\ &\quad =D\,\mathrm{div\,grad}\Delta m+e^2\sigma^{-2}\mu_n\mu_p(D_n-D_p)\\ &\qquad \times[(n-p)\,\mathrm{grad}\Delta m+(n+p)\,\mathrm{grad}\Delta q]\cdot\mathrm{grad}\Delta m,\end{aligned}\tag{14}$$

$$\begin{aligned}&\mathrm{div}D'\,\mathrm{grad}\Delta q\\ &\quad =D'\,\mathrm{div\,grad}\Delta q-e^2\sigma^{-2}\mu_n\mu_p(D_n+D_p)\\ &\qquad \times[(n-p)\,\mathrm{grad}\Delta m+(n+p)\,\mathrm{grad}\Delta q]\cdot\mathrm{grad}\Delta q,\end{aligned}$$

where D and D' are defined by

$$\begin{aligned}D &\equiv (D_p\sigma_n+D_n\sigma_p)/\sigma=kT\mu_n\mu_p(n+p)/\sigma,\\ D' &\equiv (D_p\sigma_n-D_n\sigma_p)/\sigma=kT\mu_n\mu_p(n-p)/\sigma,\end{aligned}\tag{15}$$

the continuity equation

$$\begin{aligned}&\partial\Delta m/\partial t-\mathrm{div}D\,\mathrm{grad}\Delta m\\ &\quad +e\sigma^{-2}\mu_n\mu_p(n-p)[\mathbf{I}-(\epsilon/4\pi)\partial\mathbf{E}/\partial t]\cdot\mathrm{grad}\Delta m\\ &\qquad +\sigma^{-1}(\sigma_n/\tau_p+\sigma_p/\tau_n)\Delta m\\ &=-\sigma^{-1}(\sigma_n-\sigma_p)\partial\Delta q/\partial t+\mathrm{div}D'\,\mathrm{grad}\Delta q\\ &\quad -e\sigma^{-2}\mu_n\mu_p(n+p)[\mathbf{I}-(\epsilon/4\pi)\partial\mathbf{E}/\partial t]\cdot\mathrm{grad}\Delta q\\ &\qquad -\sigma^{-1}(\sigma_n/\tau_p-\sigma_p/\tau_n)\Delta q\end{aligned}\tag{16}$$

results. This equation is the generalization of the ambipolar continuity equation[7] previously derived for the neutral case. In a formal sense, the latter results from Eq. (16) if ϵ is set equal to zero; Δq is then zero also (and $\Delta p=\Delta n$ and $\tau_p=\tau_n$ follow), since Poisson's equation is, in the present notation,

$$\Delta q=(\epsilon/8\pi e)\,\mathrm{div}\mathbf{E}.\tag{17}$$

The quantity D is the concentration-dependent ambipolar diffusivity previously derived. The velocity for Δm which multiplies $\mathrm{grad}\Delta m$ is the same as the ambipolar drift velocity except that the factor $\mathbf{I}$ is replaced by $\mathbf{I}$ minus the displacement current density. The quantity D' may consistently be identified as a diffusivity for the concentration inbalance Δq, and that which multiplies $\mathrm{grad}\Delta q$, as a drift velocity for Δq. There is a correspondence between D' and the velocity for Δm which is the same as that between D and the velocity for Δq: The former two quantities contain as a factor $(n-p)$ where the latter two contain $(n+p)$. Note also that the recombination terms on the left-hand side of Eq. (16) transposed to the right combine, as may be expected, with terms there to give $(-\mathcal{R})$, which is the contribution to $\partial\Delta m/\partial t$ from recombination; there is no contribution to $\partial\Delta q/\partial t$ from recombination.

Another differential equation, one that gives $\partial\Delta q/\partial t$, may be obtained by subtracting the second of Eqs.

(11) from the first. This procedure is tantamount to writing the equation that expresses the solenoidal property of **I**, as by taking the divergence of Eq. (12) and introducing $\partial\Delta q/\partial t$ from the time derivative of Eq. (17). With Eq. (17) used also to eliminate div**E**, the result obtained is

$$e^{-1}\,\mathrm{div}\mathbf{I}=2\partial\Delta q/\partial t-(D_n+D_p)\,\mathrm{div}\,\mathrm{grad}\Delta q-(\mu_n-\mu_p)\mathbf{E}\cdot\mathrm{grad}\Delta q+(8\pi\sigma/\epsilon)\Delta q+(D_n-D_p)\,\mathrm{div}\,\mathrm{grad}\,\Delta m+(\mu_n+\mu_p)\mathbf{E}\cdot\mathrm{grad}\Delta m=0. \quad (18)$$

For the neutral case, the last three terms on the right together equal zero if div**E** is restored by replacing the first of these terms by $e^{-1}\sigma$ div**E**.

A pair of differential equations obtained as the linear combinations of Eqs. (16) and (18) that correspond to $\partial(\mu_n\Delta p\pm\mu_p\Delta n)/\partial t$ is of advantage for certain applications. The equations are

$$\begin{aligned}
&\partial\Delta m/\partial t+[(b-1)/(b+1)]\partial\Delta q/\partial t\\
&\quad=2\bar{\mu}[(kT/e)\,\mathrm{div}\,\mathrm{grad}\Delta m-\mathbf{E}\cdot\mathrm{grad}\Delta q-q\,\mathrm{div}\mathbf{E}]\\
&\qquad-[b/(b+1)\tau_p+1/(b+1)\tau_n]\Delta m\\
&\qquad\qquad-[b/(b+1)\tau_p-1/(b+1)\tau_n]\Delta q,\\
&[(b-1)/(b+1)]\partial\Delta m/\partial t+\partial\Delta q/\partial t\\
&\quad=2\bar{\mu}[(kT/e)\,\mathrm{div}\,\mathrm{grad}\Delta q-\mathbf{E}\cdot\mathrm{grad}\Delta m-m\,\mathrm{div}\mathbf{E}]\\
&\qquad-[b/(b+1)\tau_p-1/(b+1)\tau_n]\Delta m\\
&\qquad\qquad-[b/(b+1)\tau_p+1/(b+1)\tau_n]\Delta q,
\end{aligned} \quad (19)$$

with $m\equiv\frac{1}{2}(p+n)$, $q\equiv\frac{1}{2}(p-n)$, $b\equiv\mu_n/\mu_p$, and $\bar{\mu}\equiv\mu_n\mu_p/(\mu_n+\mu_p)$. Note that the coefficient of div gradΔm and of div gradΔq is the diffusivity $D_i\equiv 2D_nD_p/(D_n+D_p)$ for intrinsic material. The recombination terms on the right in the respective equations are, as may be expected, equal to $(-\mathcal{R})$ and $[(b-1)/(b+1)](-\mathcal{R})$. As written with div**E**, Eqs. (19) are symmetrical, since one results from the other upon interchange of m and q and of Δm and Δq. The equations that Eqs. (19) reduce to for the neutral case do not possess this symmetry.[13]

The two concentration variables and **E** or V are the dependent variables, and three differential equations are Eqs. (16), (17), and (18) or Eqs. (17) and (19). If **I** must be determined from boundary conditions, then it is well to eliminate **I** from Eq. (16) by use of Eq. (12). But **I** is retained and Eq. (12) advantageously used instead of Eq. (17) if, as is often the case, **I** is a known function of the space coordinates and time.

3. TRANSPORT OF AN INJECTED PULSE

3.1 Drift in the Linear Small-Signal Case

3.11 The Exact Solutions

From Eqs. (8), linear small-signal continuity equations for one-dimensional drift with recombination, neglecting diffusion and for no applied magnetic field, are

$$\begin{aligned}
\partial\Delta p/\partial t&=(\nu_p-\nu_{nr})\Delta n-(\nu_p+\nu_{pr})\Delta p-v_p\partial\Delta p/\partial x,\\
\partial\Delta n/\partial t&=-(\nu_n+\nu_{nr})\Delta n+(\nu_n-\nu_{pr})\Delta p+v_n\partial\Delta n/\partial x.
\end{aligned} \quad (20)$$

Here, the volume generation term has been omitted; the dielectric relaxation frequencies and the drift velocities have values that apply for thermal equilibrium; and ν_{nr} and ν_{pr} are decay constants for "linear recombination" defined in accordance with[14]

$$\mathcal{R}=(p_0\Delta n+n_0\Delta p)/(n_0+p_0)\tau_0\equiv\nu_{nr}\Delta n+\nu_{pr}\Delta p, \quad (21)$$

where τ_0 is the diffusion-length lifetime. Equation (21) indicates that (for the linear small-signal case) it is the concentration $(p_0\Delta n+n_0\Delta p)/(n_0+p_0)$ to which a lifetime applies, and that this lifetime is τ_0. This result is based on the hypothesis of negligible trapping, that is, that the recombination centers, present in comparatively small concentration, have a trapping transient of negligible amplitude and also of negligible duration. Note that in regions where there is carrier depletion, the recombination function $\mathcal{R}$ may be negative, which implies generation of electron-hole pairs

To solve Eqs. (20) for a pulse of electron-hole pairs injected into a doubly-infinite filament, for which a suitable technique is that of the bilateral or two-sided Laplace transform with respect to the distance variable, a particular dimensionless formulation is employed. This involves independent variables

$$X\equiv x/L, \quad U\equiv t/\tau, \quad (22)$$

and reduced concentrations

$$\Delta P\equiv\Delta p/(\mathcal{P}/L), \quad \Delta N\equiv\Delta n/(\mathcal{P}/L), \quad (23)$$

where the distance and time units L and τ are given by

$$\begin{aligned}
&L\equiv(v_n+v_p)\tau, \quad \tau\equiv(|\nu^2|)^{\frac{1}{2}},\\
&\nu^2\equiv 4(\nu_n-\nu_{pr})(\nu_p-\nu_{nr})\\
&\quad=\frac{4n_i^2}{(n_0+p_0)^2}\left[\frac{4\pi e(n_0+p_0)}{\epsilon}\frac{dv_n}{dE}-\tau_0^{-1}\right]\times\left[\frac{4\pi e(n_0+p_0)}{\epsilon}\frac{dv_p}{dE}-\tau_0^{-1}\right],
\end{aligned} \quad (24)$$

subject to the restriction $\nu^2\neq 0$. The reduced differential equations that result are

$$\begin{aligned}
\partial\Delta P/\partial U&=\tfrac{1}{2}[(\lambda-\kappa)\Delta N-(\zeta-\kappa)\Delta P-(1-\alpha)\partial\Delta P/\partial X],\\
\partial\Delta N/\partial U&=\tfrac{1}{2}[-(\zeta+\kappa)\Delta N+(\lambda+\kappa)\Delta P+(1+\alpha)\partial\Delta N/\partial X],
\end{aligned} \quad (25)$$

[13] For this case, the transport terms of the right-hand members are, respectively, $(-e^{-1}\,\mathrm{div}\mathbf{I}')$ and $(b-1)/(b+1)$ times $(-e^{-1}\,\mathrm{div}\mathbf{I}'')$, where $\mathbf{I}'$ and $\mathbf{I}''$ are current densities defined in reference 3.

[14] See reference 3, pp. 573, 574. The expression for $\mathcal{R}$ follows from the linearization of the general (steady-state) expression and use of the last form for τ_0 of Eqs. (65) of this reference.

in which α, κ, λ, and ζ are the parameters

$$\begin{aligned}\alpha&\equiv(v_n-v_p)/(v_n+v_p),\\ \kappa&\equiv(\nu_n-\nu_p+\nu_{nr}-\nu_{pr})\tau\\ &=[\nu_n-\nu_p-(n_0-p_0)/(n_0+p_0)\tau_0]\tau, \quad (26)\\ \lambda&\equiv(\nu_n+\nu_p-\nu_{nr}-\nu_{pr})\tau=(\tau_d^{-1}-\tau_0^{-1})\tau,\\ \zeta&\equiv(\nu_n+\nu_p+\nu_{nr}+\nu_{pr})\tau=(\tau_d^{-1}+\tau_0^{-1})\tau.\end{aligned}$$

It is clear from the second form for ν^2 of Eqs. (24) that ν is real for sufficiently large or sufficiently small lifetime τ_0, and imaginary for an intermediate range in which, for constant mobilities, the relative change in τ_0 equals the mobility ratio. Thus, with differing electron and hole mobilities, imaginary ν may obtain for τ_0 of the order of the dielectric relaxation time, τ_d. The relationships between parameters,

$$\kappa^2\pm1=\lambda^2=\zeta^2-4\tau^2/\tau_d\tau_0, \quad (27)$$

in which the upper and lower signs apply, respectively, for real and imaginary ν, follow readily from the definitions of Eqs. (24) and (26). The equation to the left shows that κ and λ are not independent parameters. The one to the right combines the identities

$$\tfrac{1}{2}(\zeta+\lambda)\tau_d\equiv\tau\equiv\tfrac{1}{2}(\zeta-\lambda)\tau_0. \quad (28)$$

Note that ζ is always positive, and, for no recombination, λ and ζ are equal. Positive or negative λ corresponds, respectively, to τ_d smaller or larger than τ_0. It is evident that ν is real or imaginary according to whether the quantities

$$\tfrac{1}{2}(\lambda-\kappa)=(\nu_p-\nu_{nr})\tau,\quad \tfrac{1}{2}(\lambda+\kappa)=(\nu_n-\nu_{pr})\tau \quad (29)$$

are of the same or of opposite sign. It is readily shown that imaginary ν imposes no restriction on λ and implies either $1<\kappa<\infty$ for n-type material or $-\infty<\kappa<-1$ for p-type material, while real ν imposes no restriction on κ and implies either $1<\lambda<\infty$ or $-\infty<\lambda<-1$ for positive or negative $(\tau_0-\tau_d)$, respectively.

Solutions of Eqs. (25) for a Gaussian initial distribution of electron-hole pairs, obtained in Sec. A.1 of the Appendix by use of two-sided Laplace transforms previously derived,[3] provide solutions for the limiting case of the injected delta pulse. These are, for the pulse injected at $X=0$,

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}=\{\exp[\kappa X-\tfrac{1}{2}(\zeta-\alpha\kappa)U]\}\Bigg\{\begin{pmatrix}\delta[\frac{1}{2}(1-\alpha)U-X]\\ \delta[X+\frac{1}{2}(1+\alpha)U]\end{pmatrix}+\frac{1}{2}\Bigg[\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}I_0([\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}})$$
$$+\begin{pmatrix}X+\frac{1}{2}(1+\alpha)U\\ -X+\frac{1}{2}(1-\alpha)U\end{pmatrix}[\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{-\frac{1}{2}}I_1([\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}})\Bigg]\times\mathbf{1}[\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]\Bigg\} \quad (30)$$

for ν real, and

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}=\{\exp[\kappa X-\tfrac{1}{2}(\zeta-\alpha\kappa)U]\}\Bigg\{\begin{pmatrix}\delta[\frac{1}{2}(1-\alpha)U-X]\\ \delta[X+\frac{1}{2}(1+\alpha)U]\end{pmatrix}+\frac{1}{2}\Bigg[\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}J_0([\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}})$$
$$+\begin{pmatrix}-X-\frac{1}{2}(1+\alpha)U\\ X-\frac{1}{2}(1-\alpha)U\end{pmatrix}[\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{-\frac{1}{2}}J_1([\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}})\Bigg]\times\mathbf{1}[\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]\Bigg\} \quad (31)$$

for ν imaginary.

The terms in ΔP and ΔN with the delta functions $\delta[\frac{1}{2}(1-\alpha)U-X]=L\delta(v_pt-x)$ and $\delta[X+\frac{1}{2}(1+\alpha)U]=L\delta(x+v_nt)$ represent, respectively, pulses of holes and lectrons at distances from the origin corresponding to the particle-drift displacements. The continuous contributions, in which I_0, I_1, J_0, and J_1 denote Bessel functions in the notation of Watson, are confined to the interval $-\frac{1}{2}(1+\alpha)U\leq X\leq\frac{1}{2}(1-\alpha)U$ or $-v_nt\leq x\leq v_pt$, the step function

$$\begin{aligned}\mathbf{1}[\tfrac{1}{4}U^2-(X+\tfrac{1}{2}\alpha U)^2]&=\mathbf{1}[(\tfrac{1}{2}(1-\alpha)U-X)(X+\tfrac{1}{2}(1+\alpha)U)]\\ &=\mathbf{1}[(v_pt-x)(x+v_nt)]\end{aligned}$$

being, respectively, zero and unity for negative and positives value of its argument. Thus, the Bessel functions contribute only for positive values of their argument.

It is of particular advantage to transform Eqs. (30) and (31) by eliminating the reduced distance X in accordance with

$$\begin{aligned}X&\equiv-\tfrac{1}{2}(\cos\Theta+\alpha)U\\ &=[\sin^2\tfrac{1}{2}\Theta-\tfrac{1}{2}(1+\alpha)]U\\ &=[-\cos^2\tfrac{1}{2}\Theta+\tfrac{1}{2}(1-\alpha)]U,\quad U>0, \quad (32)\end{aligned}$$

in favor of Θ, an angle variable that specifies relative location within the range covered by the distributions. This procedure gives

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}=\{\exp[-\tfrac{1}{2}(\zeta+\kappa\cos\Theta)U]\}\Bigg\{\begin{pmatrix}[\frac{1}{2}(\pi-\Theta)U]^{-1}\delta(\pi-\Theta)\\ [\frac{1}{2}\Theta U]^{-1}\delta\Theta\end{pmatrix}$$
$$+\frac{1}{2}\Bigg[\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}I_0(\tfrac{1}{2}U\sin\Theta)+\begin{pmatrix}\tan\frac{1}{2}\Theta\\ \cot\frac{1}{2}\Theta\end{pmatrix}I_1(\tfrac{1}{2}U\sin\Theta)\Bigg]\times\mathbf{1}[\Theta(\pi-\Theta)]\Bigg\} \quad (33)$$

for ν real, and

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}=\{\exp[-\tfrac{1}{2}(\zeta+\kappa\cos\Theta)U]\}\left\{\begin{pmatrix}[\frac{1}{2}(\pi-\Theta)U]^{-1}\delta(\pi-\Theta)\\ [\frac{1}{2}\Theta U]^{-1}\delta\Theta\end{pmatrix}\right.$$
$$\left.+\frac{1}{2}\left[\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}J_0(\tfrac{1}{2}U\sin\Theta)-\begin{pmatrix}\tan\frac{1}{2}\Theta\\ \cot\frac{1}{2}\Theta\end{pmatrix}J_1(\tfrac{1}{2}U\sin\Theta)\right]\times 1[\Theta(\pi-\Theta)]\right\} \quad (34)$$

for ν imaginary. The use of Θ as a variable implies the step function of Eqs. (30) and (31), while the step function of Eqs. (33) and (34) simply restricts Θ as defined by Eq. (32) to the interval $0\leq\Theta\leq\pi$. The particle-drift displacements $x=-v_nt$ and $x=v_pt$ correspond to $X=-\frac{1}{2}(1+\alpha)U$ and $X=\frac{1}{2}(1-\alpha)U$ and to $\Theta=0$ and $\Theta=\pi$, respectively. The total range in X is equal to U.

Some details of the evaluation of integrals of the concentrations over the drift range as well as integrals for the means or first moments are given in Sec. A.2 of the Appendix. It is verified that (with no trapping) the fractions F_p and F_n at given elapsed time of holes and electrons initially injected are both $\exp[-\frac{1}{2}(\zeta-\lambda)U]=\exp(-t/\tau_0)$, as may be expected. The means of the distributions of Eqs. (33) and (34) are then found to be given by

$$\begin{pmatrix}\langle X_p\rangle\\ \langle X_n\rangle\end{pmatrix}\equiv\int_{-\frac{1}{2}(1+\alpha)U}^{\frac{1}{2}(1-\alpha)U}X\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}dX\Big/\exp(-t/\tau_0)$$
$$=\tfrac{1}{2}[\kappa/\lambda-\alpha]U-\left[\begin{pmatrix}\kappa-\lambda\\ \kappa+\lambda\end{pmatrix}\Big/2\lambda^2\right]\times[1-\exp(-\lambda U)], \quad (35)$$

whence

$$\langle X_p\rangle-\langle X_n\rangle=\lambda^{-1}[1-\exp(-\lambda U)]. \quad (36)$$

This result shows that, for λ positive, the difference of the means of the distributions approaches asymptotically, with time constant $(\tau_d^{-1}-\tau_0^{-1})^{-1}$, a "polarization distance" x_P given by[15]

$$x_P=L/\lambda=(v_n+v_p)/(\tau_d^{-1}-\tau_0^{-1}). \quad (37)$$

It is clear that x_P is essentially the distance electrons and holes drift apart in the dielectric relaxation time τ_d, provided $\tau_d\ll\tau_0$ holds, and that x_P is increased by recombination.

For λ negative, the right-hand member of Eq. (36) may be written as $|\lambda|^{-1}[\exp(|\lambda|U)-1]$ and the means of the distributions ultimately separate at an exponentially increasing rate, so that an x_P does not apply. As is shown in further detail in Sec. 3.12 in connection with specific illustrative cases, the physical interpretation has to do with the circumstance that (with $\lambda<0$) the distributions include regions of carrier depletion as well as regions of carrier excess, and both means ultimately lie outside the drift range,[16] even though the distributions themselves do not.

If an x_P applies, then Eqs. (35) show that the means of the distributions ultimately exhibit the common drift velocity $\frac{1}{2}(\kappa/\lambda-\alpha)(v_n+v_p)$. This velocity differs from the ambipolar drift velocity[7] which, by eliminating the contributions from div**E** (which are proportional to $\Delta n-\Delta p$) between Eqs. (20) and then introducing the neutrality condition $\Delta n=\Delta p$, is found to be given by

$$\begin{aligned}v_0&=(\nu_nv_p-\nu_pv_n)/(\nu_n+\nu_p)\\&=\tfrac{1}{2}[(\nu_n-\nu_p)/(\nu_n+\nu_p)-\alpha](v_n+v_p)\\&=(v_pn_0dv_n/dE-v_np_0dv_p/dE)/\\&\qquad(n_0dv_n/dE+p_0dv_p/dE).\end{aligned} \quad (38)$$

With the definitions of Eqs. (26), the common drift velocity and v_0 are clearly equal if there is no recombination. The correction to v_0 is given by[17]

$$\tfrac{1}{2}(\kappa/\lambda-\alpha)(v_n+v_p)-v_0=\frac{n_i^2}{n_0+p_0}\frac{dv_n/dE-dv_p/dE}{n_0dv_n/dE+p_0dv_p/dE}\frac{\tau_d/\tau_0}{1-\tau_d/\tau_0}, \quad (39)$$

which is evidently a small correction[5] for $\tau_d\ll\tau_0$, vanishing for no recombination. It vanishes also if the differential mobilities dv_n/dE and dv_p/dE are equal. Note that, according to Eq. (38), with nonconstant mobilities an ambipolar velocity v_0 may occur whose direction is opposite to that normally associated with the conductivity type.[18]

3.12 Approximate Solutions and Illustrative Cases

Principal physical interest attaches to cases of real ν. Cases of imaginary ν apply over a range of approximate equality of τ_0 and τ_d that is generally quite limited, their occurrence depending on differing electron and hole mobilities. For real ν, approximate solutions of two main types are here considered, the first being for sufficiently large times and not too strongly extrinsic material, and the second for extrinsic material of sufficiently high resistivity. These correspond, respectively, to large and to small values of the argument of the Bessel functions.

[15] Equation (37) written for constant mobilities is consistent with Eq. (14) of reference 5.

[16] This conclusion follows from Eq. (35) since, from Eqs. (29), $\kappa+|\lambda|>0$ and $\kappa-|\lambda|<0$ hold for ν real and $\lambda<-1$.

[17] For constant mobilities, this result gives the corresponding correction $en_i^2(\mu_n^2-\mu_p^2)\tau_d/\sigma_0(n_0+p_0)(\tau_0-\tau_d)$ to the ambipolar pseudomobility,[7] and this correction can be shown to be consistent with Eq. (16) of reference 5.

[18] A. C. Prior, Proc. Phys. Soc. (London) **A76**, 465 (1960).

For U large and Θ not too close to the limits of the interval to which it is confined, approximation[19] of the Bessel functions in the solution of Eqs. (33) for ν real gives

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}\sim\begin{pmatrix}\exp[-\tfrac{1}{2}(\zeta-\kappa)U]\times[\tfrac{1}{2}(\pi-\Theta)U]^{-1}\delta(\pi-\Theta)\\ \exp[-\tfrac{1}{2}(\zeta+\kappa)U]\times[\tfrac{1}{2}\Theta U]^{-1}\delta\Theta\end{pmatrix}$$
$$+\tfrac{1}{2}(\pi U\sin\Theta)^{-\frac{1}{2}}\begin{pmatrix}\lambda-\kappa+\tan\tfrac{1}{2}\Theta\\ \lambda+\kappa+\cot\tfrac{1}{2}\Theta\end{pmatrix}\exp[-\tfrac{1}{2}(\zeta-\sin\Theta+\kappa\cos\Theta)U]\times 1[\Theta(\pi-\Theta)]. \quad (40)$$

Since U is large, the continuous distributions can be appreciable only for Θ in a certain range about the value Θ_m for which the exponential factor (for given U) has a maximum, and for which

$$\tan\Theta_m=-\kappa^{-1},\quad \sin\Theta_m=(\kappa^2+1)^{-\frac{1}{2}}=|\lambda|^{-1},$$
$$\zeta-\sin\Theta_m+\kappa\cos\Theta_m=\zeta-|\lambda|, \quad (41)$$
$$\lambda-\kappa+\tan\tfrac{1}{2}\Theta_m=\lambda+\kappa+\cot\tfrac{1}{2}\Theta_m=\lambda+|\lambda|,$$

can easily be shown to hold. It is well, therefore, to write the approximate solution, Eqs. (40), as an expansion about the maximum. Equations (41) indicate that the solution will assume either of two distinct forms, according to whether λ is positive or negative.

Positive λ implies $\lambda>1$, since ν is real. For this case, lifetime exceeds the dielectric relaxation time, and the last of Eqs. (41) indicates that the continuous hole and electron distributions from Eqs. (40) are approximately the same. Thus, the fractions of holes and of electrons in the respective delta pulses must be relatively small, and similar continuous distributions then result whose means are displaced by the polarization distance x_P, with x_P small compared with the range covered by the distributions.

This consideration provides the condition for large t. From Eqs. (40) and by use of Eqs. (26), the decay constants for the delta pulses of holes and of electrons are $\nu_p+\nu_{pr}$ and $\nu_n+\nu_{nr}$. Since the decay constant for the total number of carriers is τ_0^{-1}, the decay constants for the fractions of holes and of electrons in the delta pulses are $\nu_p+\nu_{pr}-\tau_0^{-1}=\nu_p-\nu_{nr}$ and $\nu_n+\nu_{nr}-\tau_0^{-1}=\nu_n-\nu_{pr}$, the second forms following by use of Eq. (21). From the definition of ν^2 in Eqs. (24), it is clear that the assumption of real ν and positive λ ensures that these decay constants are positive. The condition of large time is thus

$$t\gg(\nu_p-\nu_{nr})^{-1},\quad t\gg(\nu_n-\nu_{pr})^{-1}. \quad (42)$$

This provides also $t\gg\tfrac{1}{2}[(\nu_p-\nu_{nr})^{-1}+(\nu_n-\nu_{pr})^{-1}]$, the condition that the argument $\tfrac{1}{2}U\sin\Theta_m=U/2\lambda$ of the Bessel functions for $\Theta=\Theta_m$ is sufficiently large.[20] One of Eqs. (42) requires that t be large compared to a time at least equal to the dielectric relaxation time associated with the minority carriers. Thus originates for this case of positive λ, the previously stated requirement that the material be not too strongly extrinsic.

With negligible fractions of the carriers in the delta pulses and with the variables $\Delta\Theta$ and $\Delta x=L\Delta X$ introduced in accordance with

$$\Delta\Theta\equiv\Theta-\Theta_m,$$
$$\Delta X\equiv X-X_m\sim(U/2\lambda)\Delta\Theta, \quad (43)$$
$$X_m\equiv-\tfrac{1}{2}(\cos\Theta_m+\alpha)U=\tfrac{1}{2}(\kappa/\lambda-\alpha)U,$$

Eqs. (40) and (41) give

$$\Delta p\sim\Delta n\sim(\mathcal{P}/L)\Delta P$$
$$\sim(\mathcal{P}/L)(\lambda^3/\pi U)^{\frac{1}{2}}\exp[-t/\tau_0-\tfrac{1}{4}\lambda U\Delta\Theta^2]$$
$$=\tfrac{1}{2}\mathcal{P}(\pi D_v t)^{-\frac{1}{2}}\exp[-t/\tau_0-\Delta x^2/4D_v t], \quad (44)$$

and, by use also of Eq. (37),

$$\Delta p-\Delta n=(\mathcal{P}/L)(\Delta P-\Delta N)$$
$$\sim(\mathcal{P}/L)(\lambda^5/\pi U)^{\frac{1}{2}}\Delta\Theta\exp[-t/\tau_0-\tfrac{1}{4}\lambda U\Delta\Theta^2]$$
$$=\tfrac{1}{4}\pi^{-\frac{1}{2}}\mathcal{P}(D_v t)^{-\frac{3}{2}}x_P\Delta x\exp[-t/\tau_0-\Delta x^2/4D_v t]$$
$$\sim-x_P\partial\Delta p/\partial x, \quad (45)$$

with

$$D_v\equiv\tfrac{1}{4}(v_n+v_p)^2\tau/\lambda^3$$
$$=(\nu_n-\nu_{pr})(\nu_p-\nu_{nr})(v_n+v_p)^2/(\tau_d^{-1}-\tau_0^{-1})^3. \quad (46)$$

The similar hole and electron distributions for this case—whose displacement by the polarization distance x_P is verified by Eq. (45)—are thus Gaussian distributions that are attenuated by the lifetime decay factor, $\exp(-t/\tau_0)$, drift at the common drift velocity, $\tfrac{1}{2}(\kappa/\lambda-\alpha)(v_n+v_p)$, and spread, exhibiting an apparent diffusion with "pseudodiffusivity" D_v. For Gaussian distributions, as is shown in Sec. 3.3, the pseudodiffusivity D_v and the ambipolar diffusivity D_0 are additive.

For constant mobilities μ_n and μ_p, the pseudodiffusivity is proportional to the square of the applied field, E_0. If also recombination is negligible, then D_v is proportional to $\tau_d E_0^2/\sigma_0^2$, being given by

$$D_v=\mu_n\mu_p(\sigma_i/\sigma_0)^2(\epsilon/4\pi\sigma_0)E_0^2, \quad (47)$$

where σ_i is the value for intrinsic material of the conductivity σ_0. In its dependence on conductivity, this D_v is largest for material of minimum conductivity, namely,[7] p-type material of conductivity $2b^{\frac{1}{2}}\sigma_i/(b+1)$. In given semiconductor material, D_v may be larger or

[19] Use is made of: $I_0(z)\sim I_1(z)\sim(2\pi z)^{-\frac{1}{2}}\exp z$ for $|z|$ large.

[20] The asymptotic expansions of I_0 and I_1 give $U/2\lambda\gg\tfrac{1}{8}$ and $U/2\lambda\gg\tfrac{3}{8}$. Hence $U\gg\lambda$ or $t\gg(\tau_d^{-1}-\tau_0^{-1})/2(\nu_n-\nu_{pr})(\nu_p-\nu_{nr})$ is essentially the condition required. With $\tau_d^{-1}=\nu_n+\nu_p$ and $\tau_0^{-1}=\nu_{nr}+\nu_{pr}$, this reduces to the form given.

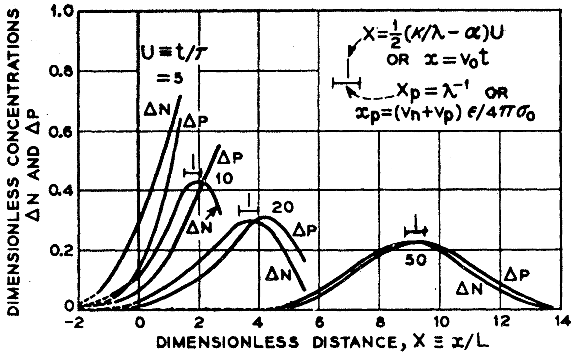


FIG. 1. Continuous concentration distributions at different times of electrons and holes from an injected neutral delta pulse for a case of drift with space charge for which diffusion-length lifetime is larger than the dielectric relaxation time. The assumptions of conductivity due to electrons 10 times that due to holes, constant mobilities with mobility ratio 2.63 (as for silicon at 300°K), and negligible recombination give $\alpha=0.449$, $\kappa=1.422$, and $\lambda=\zeta=1.738$. The delta pulse of holes at the end of the drift range to the right is attenuated by the factor $\exp(-0.158U)$, and that of electrons to the left by $\exp(-1.58U)$.

smaller than D_0, depending on the applied field. From Eq. (47), D_v is equal to $D_0=kT\mu_n\mu_p(n_0+p_0)/\sigma_0$ for the applied field

$$E_0=(\sigma_0/\sigma_i)[4\pi kT(n_0+p_0)/\epsilon]^{\frac{1}{2}}. \tag{48}$$

For intrinsic material, this field is $(8\pi kTn_i/\epsilon)^{\frac{1}{2}}$, which is equal to kT/e divided by the Debye length[9] $L_D \equiv (kT\epsilon/8\pi n_i e^2)^{\frac{1}{2}}$. Thus, for intrinsic silicon at 300°K, D_v equals D_0 for a field of only 10 v/cm. For silicon of minimum conductivity at 300°K, the field is about 6% less, being $[2b^{\frac{1}{2}}/(b+1)]^{\frac{1}{2}}$ times that for intrinsic material.[21]

Figure 1 illustrates a case of drift in an n-type semiconductor for which the Gaussian approximation of Eqs. (44) to (46) applies. The continuous distributions, calculated from the exact solutions of Eqs. (33), are shown at different times following injection of the neutral pulse at the origin. Equilibrium conductivity due to electrons is assumed to be 10 times conductivity due to holes, constant mobilities are assumed with a mobility ratio equal to that for silicon at 300°K, and recombination is neglected. For this case, the time unit τ equals $\epsilon/8\pi e(\mu_n\mu_p)^{\frac{1}{2}}n_i$ and is thus of the order of the dielectric relaxation time in intrinsic material. The continuous distributions, shown at reduced times $U\equiv t/\tau$ equal to 5, 10, 20, and 50, illustrate the approach to the Gaussian approximation. Since they cover a total range in reduced distance equal to the reduced time, the ones for $U=50$, for example, extend in principle from about plus 14 in reduced distance $X\equiv x/L$ off scale to minus 36. Evaluated specifically for n-type silicon at 300°K, 50τ is about 14 μsec; and for an applied field of 10 v/cm, the distributions at time 50τ are about a half-millimeter from the origin. This distance is, of course, proportional to the applied field. The applied field for which D_v equals D_0 is about 17 v/cm for this n-type silicon, and for fields in excess of this, the pseudodiffusivity predominates.[22]

A delta pulse of majority electrons adjoins each continuous electron distribution shown, off scale to the left, and a delta pulse of minority holes adjoins the abrupt front of each hole distribution. The majority carriers appear comparatively rapidly in the continuous distribution, since the delta pulse of these carriers in the extrinsic material is attenuated with time constant substantially τ_d. This is a reduced time for the present case of about 0.6. The delta pulse of minority carriers is attenuated more slowly, with decay constant ν_p; only 80% of the excess holes are in the continuous distribution shown for U equal to 10. The continuous distribution of minority carriers leads that of majority carriers, and the latter is the first to exhibit a relative maximum.

The case of negative λ is that of dielectric relaxation time greater than the lifetime. With ν real, negative λ implies $\lambda<-1$. For this case, as the last of Eqs. (41) indicates, the continuous hole and electron distributions for large U from Eqs. (40) are odd functions of $\Delta\Theta$ or Δx, being given by

$$\begin{pmatrix}\Delta p\\ \Delta n\end{pmatrix}\sim\tfrac{1}{2}(\mathcal{P}/L)(|\lambda|^3/\pi U)^{\frac{1}{2}}\begin{pmatrix}|\lambda|+\kappa\\ -|\lambda|+\kappa\end{pmatrix}\Delta\Theta\exp[-t/\tau_d-\tfrac{1}{4}|\lambda|U\Delta\Theta^2]$$
$$=\frac{2\mathcal{P}\tau^3}{(v_n+v_p)^2}\left[\frac{(\tau_0^{-1}-\tau_d^{-1})^5}{\pi t^3}\right]^{\frac{1}{2}}\begin{pmatrix}[p_0/(n_0+p_0)][\tau_0^{-1}-(4\pi e/\epsilon)(n_0+p_0)dv_p/dE]\\ [-n_0/(n_0+p_0)][\tau_0^{-1}-(4\pi e/\epsilon)(n_0+p_0)dv_n/dE]\end{pmatrix}$$
$$\times\Delta x\exp[-t/\tau_d-\Delta x^2/4D_v't], \tag{49}$$

with

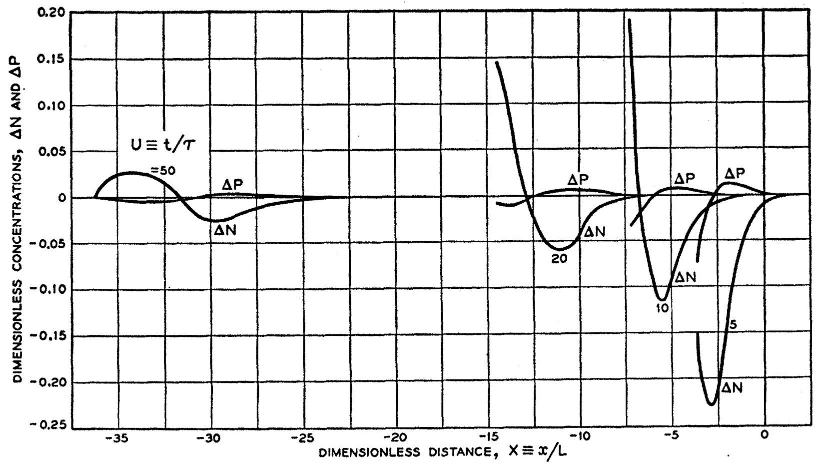
$$D_v'\equiv\tfrac{1}{4}(v_n+v_p)^2\tau/|\lambda|^3$$
$$=(\nu_{nr}-\nu_p)(\nu_{pr}-\nu_n)(v_n+v_p)^2/(\tau_0^{-1}-\tau_d^{-1})^3. \tag{50}$$

That the decay term in the exponent involves τ_d rather than τ_0 follows from Eqs. (28) and the second of Eqs. (41). From Eq. (24), the second factors in brackets in the matrix are both positive, since ν^2 is positive and λ negative. Thus, as shown by the first factors in brackets in the matrix, the hole and electron distributions in the present approximation are everywhere proportional but

[21] For silicon at 300°K, as in reference 3, the values 1500 and 570 cm²/v sec are used for μ_n and μ_p, and 1.316×10^{10}/cm³ for n_i.

[22] From Eq. (48), the field for $D_v=D_0$ equals the factor $(\sigma_0/\sigma_i)\times[(n_0+p_0)/2n_i]^{\frac{1}{2}}$ times the corresponding field for intrinsic material. The factor is 1.738 for the example considered.

FIG. 2. Continuous concentration distributions at different times of electrons and holes from an injected neutral delta pulse for a case of drift with space charge for which dielectric relaxation time is larger than the diffusion-length lifetime. The assumptions of equilibrium electron concentration 10 times equilibrium hole concentration, constant mobilities with mobility ratio 2.63 (as for silicon at 300°K), and negligible dielectric relaxation give $\alpha=0.449$, $\kappa=-1.422$, and $\lambda=-\zeta=-1.738$. The delta pulse of holes at the end of the drift range to the right is attenuated by the factor $\exp(-1.58U)$, and that of electrons to the left by $\exp(-0.158U)$.



of opposite sign, the majority-carrier distribution generally having the greater amplitude. Furthermore, each distribution includes a region of carrier depletion as well as a symmetrically equal region of carrier excess. The common point x_m of zero concentration is given by X_m equal to $\frac{1}{2}(\kappa/|\lambda|-\alpha)U$, or

$$x_m=\tfrac{1}{2}(\kappa/|\lambda|-\alpha)(v_n+v_p)t. \qquad (51)$$

It is easily shown that the concentration inbalance is given by

$$\Delta p-\Delta n\sim$$

$$-x_P'\frac{\partial}{\partial x}\{\tfrac{1}{2}(\pi D_v't)^{-\frac{1}{2}}\exp[-t/\tau_d-\Delta x^2/4D_v't]\}, \qquad (52)$$

with

$$x_P'\equiv L/|\lambda|=(v_n+v_p)/(\tau_0^{-1}-\tau_d^{-1}), \qquad (53)$$

a result that represents a certain formal analog of Eq. (45) for positive λ: The definitions of x_P' and D_v' result from those of x_P and D_v if λ is replaced by $|\lambda|$, that is, if τ_0 and τ_d are interchanged. Thus, x_P' is, for $\tau_d\gg\tau_0$, substantially $(v_n+v_p)\tau_0$, the distance electrons and holes drift apart in a lifetime, and x_P' is increased by dielectric relaxation. Since the distributions are not Gaussian (but proportional to the gradient of a Gaussian distribution), D_v' cannot properly be construed as a pseudodiffusivity.

Comparison with the case of positive λ furnishes the condition of large time, which is

$$t\gg(\nu_{nr}-\nu_p)^{-1}, \quad t\gg(\nu_{pr}-\nu_n)^{-1}. \qquad (54)$$

For the present case, this condition does not entail the requirement that the fractions of holes and of electrons in the delta pulses be negligible. The reason is that, while the hole and electron delta pulses are themselves attenuated with decay constants $\nu_p+\nu_{pr}$ and $\nu_n+\nu_{nr}$, the fractions in the pulses increase, their decay constants $\nu_p-\nu_{nr}$ and $\nu_n-\nu_{pr}$ being negative. This increase of the fractions in the pulses is essentially a consequence of the vanishing of the distance integrals of the concentrations in the continuous distributions. One of Eqs. (54) requires that t be large compared with a time at least equal to the majority-carrier recombination time $\nu_{nr}^{-1}=(n_0/p_0+1)\tau_0$ or $\nu_{pr}^{-1}=(p_0/n_0+1)\tau_0$. Thus originates, for this case of negative λ, the requirement that the material be not too strongly extrinsic.

Figure 2 illustrates a case of drift for negative λ in an n-type semiconductor for which the approximation of Eqs. (49) applies. This case is otherwise largely similar to that of positive λ of Fig. 1: Equilibrium electron concentration is assumed to be 10 times the equilibrium hole concentration, so that the parameter ζ is the same; constant mobilities are assumed, as before; and, for this case, the dielectric relaxation times are neglected. The continuous distributions, shown at reduced times 5, 10, 20, and 50, indicate, consistently with Eq. (51) for x_m, that drift occurs in the majority-carrier direction, opposite to that normally associated with the conductivity type. With τ equal to $[(n_0+p_0)/2n_i]\tau_0$ or $1.738\tau_0$, a lifetime of 1 μsec gives about 87 μsec for 50τ, and for an applied field of 10 v/cm in silicon at 300°K, the distributions at time 50τ are about 1.14 cm from the origin. Since the corresponding diffusion distance[23] $(D_0t)^{\frac{1}{2}}$ is only about 0.04 cm, the effect of diffusion is quite negligible for this lifetime.

Thus, somewhat paradoxically, if lifetime is sufficiently short, concentration disturbances drift under applied field that are not subject to decay according to the lifetime, but to decay according to a dielectric relaxation time that may be considerably larger. The continuous distributions of ΔP and ΔN are initially both negative for this case; for U small, Eqs. (29) and (33) give $\frac{1}{2}(\lambda\mp\kappa)$ or $(\nu_p-\nu_{nr})\tau<0$ and $(\nu_n-\nu_{pr})\tau<0$ for their initial amplitudes. Thus, with a frequency ν_{nr} of electron recombination that exceeds the hole dielectric relaxation frequency ν_p, the excess electrons (from the delta pulse) cause hole depletion or negative

[23] The value of D_0 is 15.6 cm^2/sec.

ΔP before their charge is neutralized by holes. Negative ΔN results similarly. These negative distributions give negative $\mathcal{R}$ and carrier-pair generation; with no trapping, the same function $\mathcal{R}$ applies to both electrons and holes. Regions in which the distributions are positive then appear, but nowhere are the distributions of both carriers positive together. Recombination from a region in which the distribution of given carriers is positive results in depletion of the other carriers only. The amplitude of the distribution of majority carriers is the larger because the frequency of recombination and generation of these carriers is the smaller. Drift accordingly occurs in the majority-carrier direction. A multiple recombination-generation process accounts for the distributions' progressive changes in shape, shown in Fig. 2, with the approach to the approximation for large U of Eq. (49). The case of this figure involving no decay factor, the distributions decrease in amplitude simply because they spread, with a distance for large U between extrema of a given distribution equal to[24] $(8D_v't)^{\frac{1}{2}}$. For negligible dielectric relaxation and large U, there is substantially no net recombination or generation; as is readily seen, $\mathcal{R}$ is zero for the approximation (written for small τ_d^{-1}) of Eq. (49). That the hole distributions shown result, however, from the regions of electron depletion is reflected in the lag of the maximum of each of these distributions behind the minimum of the corresponding electron distribution. As the figure shows, this lag is substantially independent of U and corresponds to a reduced distance of about 0.6. By formal analogy with the case of positive λ, this lag is x_P', the distance electrons and holes drift apart in time τ_0. The reduced distance of lag of the figure, for which $\tau_0=\tau/1.738\sim 0.6\tau$ holds, is thus explained.

Approximate solutions for U are small are, from Eqs. (33) and (34),

$$\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}\sim\{\exp[-\tfrac{1}{2}(\zeta+\kappa\cos\Theta)U]\}\left\{\begin{pmatrix}[\frac{1}{2}(\pi-\Theta)U]^{-1}\delta(\pi-\Theta)\\ [\frac{1}{2}\Theta U]^{-1}\delta\Theta\end{pmatrix}+\frac{1}{2}\left[\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}\pm\tfrac{1}{2}U\begin{pmatrix}\sin^2\frac{1}{2}\Theta\\ \cos^2\frac{1}{2}\Theta\end{pmatrix}\right]\times\mathbf{1}[\Theta(\pi-\Theta)]\right\}$$

$$=\begin{pmatrix}\{\exp[-(\nu_p+\nu_{pr})t]\}\times\delta(v_pt/L-X)\\ \{\exp[-(\nu_n+\nu_{nr})t]\}\times\delta(X+v_nt/L)\end{pmatrix}+\{\exp[-(\nu_n+\nu_{nr})(v_pt/L-X)\tau-(\nu_p+\nu_{pr})(X+v_nt/L)\tau]\}$$

$$\times\begin{pmatrix}(\nu_p-\nu_{nr})\tau\pm\frac{1}{4}(X+v_nt/L)\\ (\nu_n-\nu_{pr})\tau\pm\frac{1}{4}(v_pt/L-X)\end{pmatrix}\times\mathbf{1}[(v_pt-x)(x+v_nt)]. \quad (55)$$

Here, the double signs in the continuous contributions refer to the sign of ν^2. The second forms of the solutions are obtained by writing the exponent in the first forms in terms of $\sin^2\frac{1}{2}\Theta$ and $\cos^2\frac{1}{2}\Theta$ and by use of Eqs. (26) and (32). It is easily seen that the magnitudes of the terms with the double signs cannot exceed $\frac{1}{4}U$, and hence must be small compared with $\frac{1}{2}$, since $U\gg 2$ is the condition[25] on which the approximation depends.

In practice, small U would generally be a consequence of large time unit τ. If recombination may be neglected compared to dielectric relaxation, then large τ implies large dielectric relaxation time for intrinsic material, that is, high intrinsic resistivity, as may readily be obtained with sufficiently low temperature. Since intrinsic resistivity at low temperature may, indeed, be extremely high, comparatively small concentration of impurities would give material that is quite strongly extrinsic and still of quite high resistivity.

For such n-type material, with $\nu_n\gg\nu_p$ and negligible recombination, Eqs. (55) give

$$\begin{aligned}\Delta p&=(\mathcal{P}/L)\Delta P\sim\mathcal{P}\delta(v_pt-x)\\ \Delta n&=(\mathcal{P}/L)\Delta N\sim\mathcal{P}\{\exp(-\nu_nt)\delta(x+v_nt)+\nu_n(v_n+v_p)^{-1}\\ &\quad\times\exp[-\nu_n(v_pt-x)/(v_n+v_p)]\\ &\qquad\times\mathbf{1}[(v_pt-x)(x+v_nt)]\},\end{aligned}\quad (56)$$

and a similar result holds for p-type material. Thus, for this case, the minority carriers drift at the minority-carrier velocity in an unattenuated delta pulse, which leads the majority carriers distributed in an exponential tail of characteristic length substantially equal to the polarization distance x_P. This exponential tail terminates at an attenuated delta pulse of majority carriers that drifts with the majority-carrier velocity, and it is easily verified that this attenuated pulse accounts for the cut-off portion of the tail. Equations (56) accordingly represent a consistent approximation; it is, moreover, also easily verified from them that the difference of the means of the distributions (including the delta pulses) approaches x_P with time constant τ_d, as follows in general for negligible recombination from Eqs. (36) and (37).

For an illustrative numerical estimate of a fairly large x_P, consider silicon of resistivity 10^7 ohm-cm at 77.4°K, for which v_n+v_p is[26] 1.09×10^5 cm/sec for $E_0=10$ v/cm. Since τ_d in seconds for silicon is 1.06×10^{-12} times the resistivity in ohm-cm, an x_P of about 1.2 cm results. This x_P is proportional to E_0 and to the resistivity.

Another case of small U associated with high intrinsic resistivity is that of recombination with negli-

[24] This corresponds to a reduced distance of $(2U/|\lambda|^3)^{\frac{1}{2}}$, or 4.36 in Fig. 2 for $U=50$.

[25] This condition follows from the MacLaurin's expansion of I_0 or J_0.

[26] Electron and hole mobilities at 77.4°K of 9000 and 1900 cm²/v sec from thermal scattering (in high-purity material) are used. See E. Conwell, Proc.IRE 40, 1327 (1952), Fig. 2.

gible dielectric relaxation. For n-type material with $n_0 \gg p_0$, which implies $\nu_{pr} \gg \nu_{nr}$, Eqs. (55) give

$$\begin{aligned}\Delta p &= (\mathcal{P}/L)\Delta P \sim \mathcal{P}\exp(-\nu_{pr}t)\times\delta(v_pt-x),\\ \Delta n &= (\mathcal{P}/L)\Delta N \sim \mathcal{P}\{\delta(x+v_nt)-\nu_{pr}(v_n+v_p)^{-1}\\ &\quad\times\exp[-\nu_{pr}(x+v_nt)/(v_n+v_p)]\\ &\quad\times 1[(v_pt-x)(x+v_nt)]\}\end{aligned} \tag{57}$$

for this case; a similar result holds for p-type material. The majority carriers accordingly drift at the majority-carrier velocity in an unattenuated delta pulse of excess carriers with an exponential tail of carrier depletion. It is readily seen that this depletion region and delta pulse approach equivalence; the integrals over the entire range of the electron and hole concentrations approach zero, both being equal to $\mathcal{P}\exp(-\nu_{pr}t)$. This quantity corresponds to the cut-off portion of the exponential tail, which terminates at the attenuated minority-carrier delta pulse. The characteristic length of the exponential tail is substantially the distance x_P' electrons and holes drift apart in a lifetime. As for the case of negative λ and large U, if lifetime is sufficiently short, a substantially unattenuated majority-carrier concentration disturbance drifts under applied field.

The cases of small U show that carriers of opposite charge can be completely separated, excess carriers of only one charge occurring at any given point. This consideration provides a condition for the substantial constancy of the field assumed in the calculation. By integrating $\partial E/\partial x$ from Poisson's equation over a small interval in x that includes the unattenuated delta pulse, the magnitude of the change in field is found to equal $4\pi e/\epsilon$. Because of over-all neutrality, this change in field is, of course, balanced by an equal and opposite change obtained by integrating over the remainder of the range. The condition that the maximum change in field be small compared with the applied field E_0 is accordingly $\mathcal{P} \ll \epsilon E_0/4\pi e$ or

$$\beta \equiv 4\pi e\mathcal{P}/\epsilon E_0 = e\mathcal{P}/\tau_d I \ll 1. \tag{58}$$

3.2 Reformulation of the Drift Problem

Results of the analysis of the drift of an injected pulse suggest new variables in terms of which the linear differential equations might advantageously be written. Such reformulation will now be considered, and conclusions that were arrived at will be discussed in connection with it. This reformulation will serve also as basis for analysis of nonlinear cases. It consists in use of Θ and U as independent variables (instead of X and U), and

$$\begin{aligned}\Delta M &\equiv \tfrac{1}{2}(\Delta P+\Delta N),\\ \Delta Q &\equiv \tfrac{1}{2}(\Delta P-\Delta N)\end{aligned} \tag{59}$$

as dependent variables. With, from Eq. (32),

$$\begin{aligned}\frac{\partial \Delta P}{\partial X} &= \frac{2}{U\sin\Theta}\frac{\partial\Delta P}{\partial\Theta},\\ \left(\frac{\partial\Delta P}{\partial U}\right)_X &= \left(\frac{\partial\Delta P}{\partial U}\right)_\Theta + \frac{\cos\Theta+\alpha}{U\sin\Theta}\frac{\partial\Delta P}{\partial\Theta},\end{aligned} \tag{60}$$

and similar equations for ΔN, Eqs. (25) for drift with no diffusion give

$$\begin{aligned}\frac{\partial\Delta M}{\partial U} &= -U^{-1}\cot\Theta\frac{\partial\Delta M}{\partial\Theta} - U^{-1}\csc\Theta\frac{\partial\Delta Q}{\partial\Theta}\\ &\quad -\tfrac{1}{2}(\zeta-\lambda)\Delta M+\kappa\Delta Q,\\ \frac{\partial\Delta Q}{\partial U} &= -U^{-1}\csc\Theta\frac{\partial\Delta M}{\partial\Theta} - U^{-1}\cot\Theta\frac{\partial\Delta Q}{\partial\Theta} - \tfrac{1}{2}(\zeta+\lambda)\Delta Q.\end{aligned} \tag{61}$$

From Eqs. (28), the term $[-\frac{1}{2}(\zeta-\lambda)\Delta M]$ is associated with decay with lifetime τ_0 of the dimensionless total concentration ΔM, while the term $[-\frac{1}{2}(\zeta+\lambda)\Delta Q]$ is associated with decay according to the dielectric relaxation time τ_d of the dimensionless concentration inbalance ΔQ. Which one of these decays may actually be exhibited as such through an exponential decay factor depends on the particular nature of the solution: The Gaussian approximation for large U with λ or $(\tau_0-\tau_d)$ positive involves $\exp(-t/\tau_0)$, while the corresponding approximation for negative λ involves $\exp(-t/\tau_d)$. Thus, for large U, it is well to rewrite Eqs. (61) for dependent variables

$$\Delta\mathcal{M} \equiv \exp(t/\tau_0)\times\Delta M, \quad \Delta\mathcal{Q} \equiv \exp(t/\tau_0)\times\Delta Q, \tag{62}$$

if λ is positive and dependent variables $\exp(t/\tau_d)\times\Delta M$ and $\exp(t/\tau_d)\times\Delta Q$ if λ is negative. In the former case, the dielectric relaxation term $[-\frac{1}{2}(\zeta+\lambda)\Delta Q]$ then gives rise to $[-\lambda\Delta\mathcal{Q}]$. This term may be shown to be associated with the pseudodiffusivity as follows: With similar electron and hole distributions that cover a range large compared with the polarization distance x_P separating their means, $|\Delta Q| \ll \Delta M$ holds everywhere. Also, with fixed separation of the means, $\Delta\mathcal{Q}$ changes relatively slowly; its value is, consistently also with Eq. (45), substantially that which results from setting $\partial\Delta\mathcal{Q}/\partial U$ equal to zero. These considerations give

$$\begin{aligned}\Delta\mathcal{Q} &\sim -(\lambda U)^{-1}(\csc\Theta\,\partial\Delta\mathcal{M}/\partial\Theta+\cot\Theta\,\partial\Delta\mathcal{Q}/\partial\Theta)\\ &\sim -(\lambda U)^{-1}\left[\csc\Theta\frac{\partial\Delta\mathcal{M}}{\partial\Theta} - (\lambda U)^{-1}\cot\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta\mathcal{M}}{\partial\Theta}\right].\end{aligned} \tag{63}$$

This approximation for $\Delta\mathcal{Q}$ may now be substituted in the differential equation for $\Delta\mathcal{M}$, which then takes the form

$$\partial\Delta\mathcal{M}/\partial U = \lambda^{-1}U^{-2}\partial^2\Delta\mathcal{M}/\partial\Theta^2 \tag{64}$$

for large U. It is easily seen that this is the differential equation for the result of Eqs. (44) to (46), which exhibits the lifetime decay factor and the pseudodiffusivity D_v.

In the corresponding analysis for negative λ, with use of $\exp(t/\tau_d)\times\Delta M$ and $\exp(t/\tau_d)\times\Delta Q$ as dependent variables, the term $[-\frac{1}{2}(\zeta+\lambda)\Delta Q]$ is eliminated from the equation for $\partial\Delta Q/\partial U$. Consistently with this circumstance and from Eqs. (49), in this case it is $\kappa\Delta Q-|\lambda|\Delta M$ that is relatively small in magnitude; and $\exp(t/\tau_d)$ times this quantity changes relatively slowly. The former condition is equivalent to $(\nu_{nr}-\nu_p)\Delta n+(\nu_{pr}-\nu_n)\Delta p$ being substantially zero: The hole depletion rate determined by the excess of the electron recombination frequency ν_{nr} over the hole dielectric relaxation frequency ν_p is balanced algebraically by the corresponding electron depletion rate. That is, depletion is matched by generation and dielectric relaxation. If dielectric relaxation is negligible, then the net recombination rate $\mathfrak{R}$ is substantially zero, as was pointed out in connection with Fig. 2.

While this brief exploration of consistencies in another formulation in itself adds nothing materially new, its motivation has been the heuristic value of the analysis it entails for extensions to nonlinear cases. It has appeared, for example, that if independent variables X and U are employed, then the assumption (for positive λ) of slowly varying $\Delta\mathfrak{Q}$ seems to lead to a wrong pseudodiffusivity,[27] an apparent inconsistency the reason for which is not yet entirely evident.

3.3 A Nonlinear Case

If the strength of an injected pulse of current carriers is increased so that the condition for substantially unperturbed applied field is no longer met, then the transport is significantly modified. If λ is positive, so that there is no carrier depletion, then injection of the pulse results in locally decreased field through mutually consistent space-charge and conductivity-modulation mechanisms: It is readily seen that the condition of Eq. (58) for relatively small decrease in field according to Poisson's equation is also essentially the condition that the maximum relative increase in conductivity from the continuous distribution of Eqs. (56) be small.[28] Field decreased over a certain finite region exhibits a minimum within this region, and in a neighborhood of this minimum the divergence of the field is small and substantial neutrality obtains. The minimum field may be relatively quite small, particularly with injection in material of high resistivity. Since diffusion is predominant in a near-neutral region of small field, transport in its early stages following injection may occur principally through this mechanism, with distributions that are approximately Gaussian in shape. The amplitude of the distributions is then approximately $\mathcal{P}/2(\pi D_i t)^{\frac{1}{2}}$ and the dispersion, $(D_i t)^{\frac{1}{2}}$, with D_i the ambipolar diffusivity for intrinsic material. Comparing this amplitude with the majority-carrier concentration gives an estimate of the time over which modulation nonlinearity may persist, and the dispersion may be calculated for this time. The time and dispersion so calculated will be appreciably larger than the dielectric relaxation time and the polarization distance, respectively, if $\mathcal{P}$ is sufficiently large and the resistivity not too small.

An extension for high-level injection using certain simplifying assumptions now follows. In the present context, effects associated with the difference in mobilities are comparatively minor, at least for semiconductors like germanium and silicon. With the assumption of equal mobilities and with $\operatorname{div}\mathbf{E}$ eliminated by use of Eq. (17), Eqs. (19) reduce to[29]

$$\begin{aligned}\partial\Delta m/\partial t&=D\partial^2\Delta m/\partial x^2-\mu[E\partial\Delta q/\partial x+(8\pi e/\epsilon)q\Delta q]-\mathfrak{R},\\ \partial\Delta q/\partial t&=D\partial^2\Delta q/\partial x^2-\mu[E\partial\Delta m/\partial x+(8\pi e/\epsilon)m\Delta q]\end{aligned}\qquad(65)$$

for the transport in one dimension, in which μ denotes the common mobility and $D=kT\mu/e$ the diffusivity. With no trapping, the recombination function $\mathfrak{R}$ may properly be written as the steady-state function[30] $(p_0\Delta n+n_0\Delta p+\Delta n\Delta p)/[\tau_{p0}(n+n_1)+\tau_{n0}(p+p_1)]$, where τ_{n0} and τ_{p0} are the respective limiting lifetimes in strongly extrinsic p- and n-type materials.

The angle variable Θ and other dimensionless variables are now introduced in accordance with Eqs. (22), (23), (24), and (32); the length and time units reduce to

$$\begin{aligned}L&\equiv2\mu E_0\tau=2\mu I\tau/\sigma_0,\\ \tau&\equiv(n_0+p_0)/[2n_i|4\pi e\mu(n_0+p_0)/\epsilon-\tau_0^{-1}|]\end{aligned}\qquad(66)$$

for the present case. Use of Eqs. (60) then gives

$$\begin{aligned}\frac{\partial\Delta M}{\partial U}&=\frac{\mathfrak{D}}{\lambda^3U^2}\csc\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta M}{\partial\Theta}-\frac{E/E_0}{U}\csc\Theta\frac{\partial\Delta Q}{\partial\Theta}\\ &\quad-\frac{\cot\Theta}{U}\frac{\partial\Delta M}{\partial\Theta}-\tfrac{1}{2}(\zeta-\lambda)\Delta M+\kappa\Delta Q-\beta\Delta Q^2\\ &\quad-(\tau L/\mathcal{P})\delta\mathfrak{R},\\ \frac{\partial\Delta Q}{\partial U}&=\frac{\mathfrak{D}}{\lambda^3U^2}\csc\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta Q}{\partial\Theta}-\frac{E/E_0}{U}\csc\Theta\frac{\partial\Delta M}{\partial\Theta}\\ &\quad-\frac{\cot\Theta}{U}\frac{\partial\Delta Q}{\partial\Theta}-\tfrac{1}{2}(\zeta+\lambda)\Delta Q-\beta\Delta M\Delta Q,\end{aligned}\qquad(67)$$

in which

$$\mathfrak{D}\equiv4D\lambda^3\tau/L^2=D/D_v\qquad(68)$$

[27] Pseudodiffusivity D_v times $(\sigma_0/\sigma_i)^2$ results for no recombination and equal mobilities.

[28] Equation (58) is equivalent to $e\mathcal{P}\ll I\tau_d$. If λ is negative, then, from Eqs. (57), relatively small maximum conductivity change from the continuous distribution implies $e\mathcal{P}\ll I\tau_0$.

[29] Note that $(\tau_p^{-1}-\tau_n^{-1})\Delta m+(\tau_p^{-1}+\tau_n^{-1})\Delta q=0$ is equivalent to $\Delta n/\tau_n=\Delta p/\tau_p$.

[30] See reference 3, p. 573 and Eqs. (71).

is the diffusivity expressed in units of the pseudo-diffusivity D_v (in the linear limit) given by Eq. (46). The parameter β is defined by Eq. (58), and κ, λ, and ζ by Eqs. (26). In the first equation, recombination in the linear limit accounts for the term in ΔM and part of the term in ΔQ, while

$$\delta\mathcal{R}\equiv\mathcal{R}-\nu_{nr}\Delta n-\nu_{pr}\Delta p$$
$$=\frac{[\tau_{p0}(n_1-p_0)+\tau_{n0}(p_1-n_0)]\Delta m^2+2[\tau_{p0}p_0-\tau_{n0}n_0]\Delta m\Delta q-[\tau_{p0}(n_1+p_0)+\tau_{n0}(p_1+n_0)]\Delta q^2}{(n_0+p_0)\tau_0[(n_0+p_0)\tau_0+\tau_{p0}(\Delta m-\Delta q)+\tau_{n0}(\Delta m+\Delta q)]} \quad (69)$$

is the nonlinear contribution to the recombination function $\mathcal{R}$.

With λ positive, it is well to employ the dependent variables of Eqs. (62) that take recombinative decay (in the linear limit) into account. Equations (67) then transform into

$$\begin{aligned}\frac{\partial\Delta\mathfrak{M}}{\partial U}&=\frac{\mathfrak{D}}{\lambda^3U^2}\csc\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}-\frac{E/E_0}{U}\csc\Theta\frac{\partial\Delta\mathfrak{Q}}{\partial\Theta}\\&\quad-\frac{\cot\Theta}{U}\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}+(\kappa-\beta\Delta Q)\Delta\mathfrak{Q}\\&\quad-(\tau L/\mathcal{P})[\exp(t/\tau_0)]\delta\mathcal{R},\\\frac{\partial\Delta\mathfrak{Q}}{\partial U}&=\frac{\mathfrak{D}}{\lambda^3U^2}\csc\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta\mathfrak{Q}}{\partial\Theta}-\frac{E/E_0}{U}\csc\Theta\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}\\&\quad-\frac{\cot\Theta}{U}\frac{\partial\Delta\mathfrak{Q}}{\partial\Theta}-(\lambda+\beta\Delta M)\Delta\mathfrak{Q}.\end{aligned} \quad (70)$$

The nonlinear extension of Eq. (64), the differential equation in $\Delta\mathfrak{M}$ for the Gaussian approximation, can now be obtained by assuming large U, slowly changing $\Delta\mathfrak{Q}$ small compared with $\Delta\mathfrak{M}$, and $\Theta\sim\Theta_m$, where Θ_m, given in Eqs. (41), is the value for the maximum in the linear case. The field may be eliminated from Eqs. (70) by use of

$$E\sim(I+2eD\partial\Delta q/\partial x)/\sigma\sim I/\sigma, \quad (71)$$

which follows from Eq. (12). It is readily seen that the field given by Eq. (71) is approached asymptotically, essentially exponentially with time constant $\epsilon/4\pi\sigma$ if the concentrations (on which σ depends) do not change appreciably in this time. From Eq. (71), E/E_0 may be replaced by σ_0/σ. Then, for U large and $\Delta\mathfrak{Q}\ll\Delta\mathfrak{M}$, solving for $\Delta\mathfrak{Q}$ in the second of Eqs. (70) after setting $\partial\Delta\mathfrak{Q}/\partial U$ equal to zero gives

$$\begin{aligned}\Delta\mathfrak{Q}&\sim-\frac{\csc\Theta}{(\lambda+\beta\Delta M)U}\\&\quad\times\left[(\sigma_0/\sigma)\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}-\frac{\cos\Theta}{U}\frac{\partial}{\partial\Theta}\frac{\sigma_0/\sigma}{\lambda+\beta\Delta M}\csc\Theta\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}\right]\\&\sim-\frac{(\sigma_0/\sigma)\lambda}{(\lambda+\beta\Delta M)U}\left(\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}+\frac{\kappa}{(\lambda+\beta\Delta M)U}\frac{\partial^2\Delta\mathfrak{M}}{\partial\Theta^2}\right),\end{aligned} \quad (72)$$

the second form applying, from Eqs. (41), for $\Theta\sim\Theta_m$; and substituting this $\Delta\mathfrak{Q}$ in the first of Eqs. (70) results in

$$\begin{aligned}\frac{\partial\Delta\mathfrak{M}}{\partial U}&\sim U^{-2}\left[\frac{\mathfrak{D}}{\lambda^3}+\frac{(\sigma_0/\sigma)^2}{\lambda+\beta\Delta M}+\frac{(\kappa-\beta\Delta Q)(\sigma_0/\sigma)}{(\lambda+\beta\Delta M)^2}\cos\Theta\right]\csc\Theta\frac{\partial}{\partial\Theta}\csc\Theta\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}\\&\quad-U^{-1}\left[\cot\Theta+\frac{(\kappa-\beta\Delta Q)(\sigma_0/\sigma)}{\lambda+\beta\Delta M}\csc\Theta\right]\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}-(\tau L/\mathcal{P})[\exp(t/\tau_0)]\delta\mathcal{R}\\&\sim U^{-2}\left[\frac{\mathfrak{D}}{\lambda}+\frac{\lambda^2(\sigma_0/\sigma)^2}{\lambda+\beta\Delta M}-\frac{\kappa\lambda(\kappa-\beta\Delta Q)(\sigma_0/\sigma)}{(\lambda+\beta\Delta M)^2}\right]\frac{\partial^2\Delta\mathfrak{M}}{\partial\Theta^2}-U^{-1}\left[-\kappa+\frac{\lambda(\kappa-\beta\Delta Q)(\sigma_0/\sigma)}{\lambda+\beta\Delta M}\right]\frac{\partial\Delta\mathfrak{M}}{\partial\Theta}\\&\quad-(\tau L/\mathcal{P})[\exp(t/\tau_0)]\delta\mathcal{R}.\end{aligned} \quad (73)$$

Note the relationships

$$\begin{aligned}\kappa-\beta\Delta Q&=[-8\pi e\mu q/\epsilon+q_0/m_0\tau_0]\tau,\\\lambda+\beta\Delta M&=\lambda[1+\Delta m/m_0(1-\tau_d/\tau_0)],\end{aligned} \quad (74)$$

which follow by use of Eqs. (26) and (58). Also, with τ given by the second of Eqs. (66), κ and λ reduce (for positive λ) to[31]

$$\kappa=-q_0/n_i,\quad \lambda=m_0/n_i. \quad (75)$$

It is easily verified that, near equilibrium, the coefficient of the first derivative in the second form in Eq. (73) vanishes, which implies drift at the ambipolar drift velocity. Thus, near equilibrium, only the second-derivative contribution remains; and, with $\lambda^2-\kappa^2=1$ from the first of Eqs. (27), the terms inside the brackets that follow $\mathfrak{D}/\lambda$ in the second form reduce to $1/\lambda$. Thus,

[31] For negative λ, the signs of the expressions on the right are changed.

consistently with Eq. (64), these terms give the pseudodiffusivity of Eq. (46). A generalized concentration-dependent pseudodiffusivity may accordingly be defined from Eqs. (46) and (73) as

$$D_v \equiv (\sigma_0/\sigma)[\lambda(\sigma_0/\sigma) - \kappa(\kappa - \beta\Delta Q)/(\lambda + \beta\Delta M)] \times (\mu E_0)^2 \tau / \lambda(\lambda + \beta\Delta M). \quad (76)$$

For sufficiently large injection level,[32] this D_v approaches zero inversely as the cube of the conductivity; note that, from Eq. (72), ΔQ approaches zero also. As comparison with the D_v for the linear case of Eq. (47) indicates, this behavior of the large-signal D_v results from approximate proportionality to the large-signal dielectric relaxation time divided by the square of the conductivity, or times the square of the (decreased) local field.

In accordance with Eq. (73), an effective diffusivity is simply the sum of the (constant) diffusivity D and the pseudodiffusivity D_v of Eq. (76). Note that the correction[33] to the diffusivity of order τ_d/τ_0 associated with the departure from local electrical neutrality is absent. It appears, therefore, that this correction depends on differing electron and hole diffusion constants; its vanishing for equal diffusion constants is readily verified.

A velocity function may be evaluated from the term in $\partial\Delta\mathfrak{M}/\partial\Theta$ in Eq. (73). In the first form, the contribution involving $\cot\Theta$ is canceled by transforming from $(\partial\Delta\mathfrak{M}/\partial U)_\Theta$ to $(\partial\Delta\mathfrak{M}/\partial U)_X$ in accordance with the second of Eqs. (60). From Eq. (32), the operator $U^{-1}\csc\Theta\partial/\partial\Theta$ is equal to $\frac{1}{2}\partial/\partial X$, so that the contribution with $\partial\Delta\mathfrak{M}/\partial\Theta$ as factor that remains may be written as $(-\frac{1}{2})(\sigma_0/\sigma)(\kappa-\beta\Delta Q)(\lambda+\beta\Delta M)^{-1}\partial\Delta\mathfrak{M}/\partial X$. The velocity function v is accordingly given by

$$v = \tfrac{1}{2}(\sigma_0/\sigma)(\kappa - \beta\Delta Q)(\lambda + \beta\Delta M)^{-1}(L/\tau) = e\mu^2 \frac{n - p - (n_0 - p_0)\tau_d/\tau_0}{1 - \tau_d/\tau_0} \frac{I}{\sigma^2}, \quad (77)$$

in which τ_d and τ_0 are the equilibrium dielectric relaxation time and diffusion-length lifetime. The final form of Eq. (77) is obtained by replacing L/τ by $2\mu I/\sigma_0$ and by use of Eqs. (74). For small τ_d, this velocity reduces to the ambipolar velocity previously derived.[7] The correction to the ambipolar velocity involving τ_d/τ_0 vanishes in the small-signal limit for this case of equal mobilities. Thus, the small-signal correction[5] depends on differing mobilities, as pointed out in connection with Eq. (39). The correction may occur for equal mobilities if there are appreciable concentrations of injected carriers with space charge.

[32] The condition $\Delta\sigma \sim \sigma$, which is $\Delta m/m_0 \gg 1$, subsumes the condition $\beta\Delta M \gg \lambda$, which is $\Delta m/m_0 \gg 1 - \epsilon/4\pi e\mu(n_0+p_0)\tau_0$. For positive λ, this expression on the right is positive.

[33] See reference 5, Eq. (18).

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APPENDIX A

A.1 Solutions for Drift of an Injected Pulse

The two-sided Laplace transform of $f(X,U)$ with respect to X is defined by

$$\mathfrak{L}\{f(X,U)\} \equiv \int_{-\infty}^{\infty} e^{-s\gamma} f(\gamma,U)d\gamma \equiv F(s,U), \quad (78)$$

and application of this transform to Eqs. (25) gives

$$\begin{aligned} \partial\mathfrak{L}\{\Delta P\}/\partial U &= \tfrac{1}{2}((\lambda-\kappa)\mathfrak{L}\{\Delta N\} - [\zeta - \kappa + (1-\alpha)s]\mathfrak{L}\{\Delta P\}), \\ \partial\mathfrak{L}\{\Delta N\}/\partial U &= \tfrac{1}{2}(-[\zeta+\kappa-(1+\alpha)s]\mathfrak{L}\{\Delta N\} + (\lambda+\kappa)\mathfrak{L}\{\Delta P\}). \end{aligned} \quad (79)$$

The general solution of Eqs. (79) is

$$\mathfrak{L}\{\Delta P\} = \sum_{j=1}^{2} A_{pj} e^{-N_j U}, \quad \mathfrak{L}\{\Delta N\} = \sum_{j=1}^{2} A_{nj} e^{-N_j U}, \quad (80)$$

in which the dimensionless decay constants are readily found to be given by

$$\binom{N_1}{N_2} = \tfrac{1}{2}\{\zeta - \alpha s \pm [(s-\kappa)^2 \pm 1]^{\frac{1}{2}}\}. \quad (81)$$

The double sign inside the radical—here and in what follows—relates only to sign of ν^2. With the ratios A_{nj}/A_{pj} fixed by

$$\frac{A_{nj}}{A_{pj}} = \frac{-N_j + \frac{1}{2}[\zeta - \kappa + (1-\alpha)s]}{\frac{1}{2}(\lambda-\kappa)} = \frac{\frac{1}{2}(\lambda+\kappa)}{-N_j + \frac{1}{2}[\zeta+\kappa-(1+\alpha)s]}, \quad j=1,2, \quad (82)$$

the four constants A_{pj} and A_{nj} are determined by the transforms $\mathfrak{L}\{\Delta P_1\}$ and $\mathfrak{L}\{\Delta N_1\}$ of the initial concentrations. If these concentrations are equal, then

$$\begin{aligned} \begin{pmatrix} A_{p1}/\mathfrak{L}\{\Delta P_1\} \\ A_{p2}/\mathfrak{L}\{\Delta P_1\} \end{pmatrix} &= \tfrac{1}{2}\{1 \pm (s-\lambda)/[(s-\kappa)^2 \pm 1]^{\frac{1}{2}}\}, \\ \begin{pmatrix} A_{n1}/\mathfrak{L}\{\Delta P_1\} \\ A_{n2}/\mathfrak{L}\{\Delta P_1\} \end{pmatrix} &= \tfrac{1}{2}\{1 \mp (s+\lambda)/[(s-\kappa)^2 \pm 1]^{\frac{1}{2}}\} \end{aligned} \quad (83)$$

results. Thus, with $\mathfrak{L}\{\Delta P_1\}$ given by

$$\mathfrak{L}\{\Delta P_1\} = \mathfrak{L}\{\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1}\exp(-X^2/4a^2)\} = \exp(a^2s^2) \quad (84)$$

as the transform of a Gaussian initial distribution, the transforms of the concentrations are

$$\begin{pmatrix}\mathcal{L}\{\Delta P\}\\ \mathcal{L}\{\Delta N\}\end{pmatrix}=\{\exp[a^2s^2+\tfrac{1}{2}U(\alpha s-\zeta)]\}\{\cosh(\tfrac{1}{2}U[(s-\kappa)^2\pm1]^{\frac{1}{2}})\mp(s\mp\lambda)[(s-\kappa)^2\pm1]^{-\frac{1}{2}}\times\sinh(\tfrac{1}{2}U[(s-\kappa)^2\pm1]^{\frac{1}{2}})\}. \tag{85}$$

By use of the identity

$$a^2s^2+\tfrac{1}{2}U(\alpha s-\zeta)\equiv a^2(\kappa^2\mp1)-\tfrac{1}{2}U(\zeta-\alpha\kappa)+(2a^2\kappa+\tfrac{1}{2}\alpha U)(s-\kappa)+a^2[(s-\kappa)^2\pm1], \tag{86}$$

and transform formulas previously derived,[3] the solution for the initial Gaussian distribution is found to be

$$\begin{aligned}\Delta P=&\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1}\{\exp[a^2(\kappa^2\mp1)+\kappa X-\tfrac{1}{2}(\zeta-\alpha\kappa)U]\}\\ &\times\Big\{\exp[-(X+2a^2\kappa-\tfrac{1}{2}(1-\alpha)U)^2/4a^2]-\tfrac{1}{2}(\lambda-\kappa)\int_{X+2a^2\kappa+\frac{1}{2}\alpha U}^{\infty}{I_0 \atop J_0}\{[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}}\}\\ &\times\{\exp[-(\gamma+\tfrac{1}{2}U)^2/4a^2]-\exp[-(\gamma-\tfrac{1}{2}U)^2/4a^2]\}d\gamma\\ &\mp\tfrac{1}{2}\int_{X+2a^2\kappa+\frac{1}{2}\alpha U}^{\infty}[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{-\frac{1}{2}}{I_1 \atop J_1}\{[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}}\}\\ &\quad\times\{(X+2a^2\kappa+\tfrac{1}{2}\alpha U-\gamma)\exp[-(\gamma+\tfrac{1}{2}U)^2/4a^2]-(X+2a^2\kappa+\tfrac{1}{2}\alpha U+\gamma)\exp[-(\gamma-\tfrac{1}{2}U)^2/4a^2]\}d\gamma\Big\},\\ \Delta N=&\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1}\{\exp[a^2(\kappa^2\mp1)+\kappa X-\tfrac{1}{2}(\zeta-\alpha\kappa)U]\}\\ &\times\Big\{\exp[-(X+2a^2\kappa+\tfrac{1}{2}(1+\alpha)U)^2/4a^2]-\tfrac{1}{2}(\lambda+\kappa)\int_{X+2a^2\kappa+\frac{1}{2}\alpha U}^{\infty}{I_0 \atop J_0}\{[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}}\}\\ &\times\{\exp[-(\gamma+\tfrac{1}{2}U)^2/4a^2]-\exp[-(\gamma-\tfrac{1}{2}U)^2/4a^2]\}d\gamma\\ &\pm\tfrac{1}{2}\int_{X+2a^2\kappa+\frac{1}{2}\alpha U}^{\infty}[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{-\frac{1}{2}}{I_1 \atop J_1}\{[\gamma^2-(X+2a^2\kappa+\tfrac{1}{2}\alpha U)^2]^{\frac{1}{2}}\}\\ &\quad\times\{(X+2a^2\kappa+\tfrac{1}{2}\alpha U+\gamma)\exp[-(\gamma+\tfrac{1}{2}U)^2/4a^2]-(X+2a^2\kappa+\tfrac{1}{2}\alpha U-\gamma)\exp[-(\gamma-\tfrac{1}{2}U)^2/4a^2]\}d\gamma\Big\},\end{aligned} \tag{87}$$

in which the upper and lower signs and functions apply respectively for real and imaginary ν. The limiting solutions of Eqs. (30) and (31) for the injected delta pulse involve the step-function factor as a result of the requirement that, for contributions in the limit of zero a, the Gaussian factors in the integrands of Eqs. (87) be centered at values within the range of integration.

A.2 Integrals over the Drift Range

The fractions of carriers initially injected that remain after given elapsed time are given by

$$\begin{aligned}\begin{pmatrix}F_p\\ F_n\end{pmatrix}&\equiv\mathcal{P}^{-1}\int_{-v_nt}^{v_pt}\begin{pmatrix}\Delta p\\ \Delta n\end{pmatrix}dx=\int_{-\frac{1}{2}(1+\alpha)U}^{\frac{1}{2}(1-\alpha)U}\begin{pmatrix}\Delta P\\ \Delta N\end{pmatrix}dX\\ &=\{\exp[-\tfrac{1}{2}\zeta U]\}\{\lambda(\kappa^2\pm1)^{-\frac{1}{2}}\sinh[\tfrac{1}{2}(\kappa^2\pm1)^{\frac{1}{2}}U]\\ &\qquad+\cosh[\tfrac{1}{2}(\kappa^2\pm1)^{\frac{1}{2}}U]\}\\ &=\exp[-\tfrac{1}{2}(\zeta-\lambda)U]=\exp(-t/\tau_0).\end{aligned} \tag{88}$$

To establish this result, Θ is first introduced as variable of integration from Eqs. (32) to (34). Then, comparison with a solution for ΔP and the corresponding F_p previously derived[34] gives the expression in the second line, in which the double sign refers to the sign of ν^2. That this expression is also F_n is evident from the observations that it involves κ only through its square and that, in Eqs. (33) and (34), ΔP and ΔN are transformed into each other if Θ is replaced by $\pi-\Theta$ and κ by its negative. Since the expression is an even function of $(\kappa^2\pm1)^{\frac{1}{2}}=|\lambda|$, this quantity may be replaced by λ, and the first form of the third line follows. The final form then follows from the second of Eqs. (28).

With Eqs. (32) to (34), the means $\langle X_p\rangle$ and $\langle X_n\rangle$ of Eqs. (35) are given by

$$\begin{aligned}\begin{pmatrix}\langle X_p\rangle\\ \langle X_n\rangle\end{pmatrix}=\tfrac{1}{2}U\begin{pmatrix}(1-\alpha)\exp[-\tfrac{1}{2}(\lambda-\kappa)U]\\ -(1+\alpha)\exp[-\tfrac{1}{2}(\lambda+\kappa)U]\end{pmatrix}-\tfrac{1}{8}U^2[\exp(-\tfrac{1}{2}\lambda U)]\int_0^{\pi}\exp(-\tfrac{1}{2}\kappa U\cos\Theta)\\ \times\left\{\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}I_0(\tfrac{1}{2}U\sin\Theta)+\begin{pmatrix}\tan\frac{1}{2}\Theta\\ \cot\frac{1}{2}\Theta\end{pmatrix}I_1(\tfrac{1}{2}U\sin\Theta)\right\}\sin\Theta(\cos\Theta+\alpha)d\Theta\end{aligned} \tag{89}$$

[34] The ΔP of Eq. (158) of reference 3 with ξ replaced by λ is (with a differing definition of Θ) formally the same as the ΔP of Eqs. (33) and (34) of the present paper. Hence F_p is obtained by replacing ξ by λ in Eq. (164) of reference 3.

for ν real; for ν imaginary, I_0 is replaced by J_0 and I_1 by $(-J_1)$. The contribution to the integral from the Bessel function of order zero and with α as a factor, similar to an integral previously evaluated by transforming it to a Gegenbauer's integral,[35] is, by use also of the first of Eqs. (27),

$$-\tfrac{1}{8}\alpha\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}U^2[\exp(-\tfrac{1}{2}\lambda U)]\int_0^\pi \exp(-\tfrac{1}{2}\kappa U\cos\Theta)I_0(\tfrac{1}{2}U\sin\Theta)\sin\Theta d\Theta=-\tfrac{1}{4}\alpha\begin{pmatrix}1-\kappa/\lambda\\ 1+\kappa/\lambda\end{pmatrix}U(1-e^{-\lambda U}). \quad (90)$$

Evaluated by means of the same transformation, a second contribution is

$$-\tfrac{1}{8}U^2\begin{pmatrix}\lambda-\kappa\\ \lambda+\kappa\end{pmatrix}[\exp(-\tfrac{1}{2}\lambda U)]\int_0^\pi \exp(-\tfrac{1}{2}\kappa U\cos\Theta)I_0(\tfrac{1}{2}U\sin\Theta)\sin\Theta\cos\Theta d\Theta$$
$$=\tfrac{1}{2}(\kappa/\lambda)\begin{pmatrix}1-\kappa/\lambda\\ 1+\kappa/\lambda\end{pmatrix}U[\tfrac{1}{2}(1+e^{-\lambda U})-(1-e^{-\lambda U})/\lambda U]. \quad (91)$$

Equations (90) and (91) with J_0 in the integrand in place of I_0 hold for ν imaginary. For the contributions from the Bessel functions of order one, use is made of

$$(\tan\tfrac{1}{2}\Theta)^{\pm1}\sin\Theta(\cos\Theta+\alpha)=\pm\sin^2\Theta\mp(1\mp\alpha)(1\mp\cos\Theta). \quad (92)$$

The transformation gives

$$-\tfrac{1}{8}U^2[\exp(-\tfrac{1}{2}\lambda U)]\int_0^\pi \exp(-\tfrac{1}{2}\kappa U\cos\Theta)I_1(\tfrac{1}{2}U\sin\Theta)\sin^2\Theta d\Theta=-\tfrac{1}{2}\lambda^{-2}U[\tfrac{1}{2}(1+e^{-\lambda U})-(1-e^{-\lambda U})/\lambda U], \quad (93)$$

and, for ν imaginary, with $(-J_1)$ in the integrand in place of I_1, the sign of the right-hand member is changed. Also, essentially as previously derived,[35] the result

$$-\tfrac{1}{8}U^2[\exp(-\tfrac{1}{2}\lambda U)]\int_0^\pi \exp(-\tfrac{1}{2}\kappa U\cos\Theta)I_1(\tfrac{1}{2}U\sin\Theta)\begin{pmatrix}1-\cos\Theta\\ 1+\cos\Theta\end{pmatrix}d\Theta$$
$$=\tfrac{1}{4}U\left\{2\begin{pmatrix}\exp[-\tfrac{1}{2}(\lambda-\kappa)U]\\ \exp[-\tfrac{1}{2}(\lambda+\kappa)U]\end{pmatrix}-\begin{pmatrix}1+\kappa/\lambda\\ 1-\kappa/\lambda\end{pmatrix}-\begin{pmatrix}1-\kappa/\lambda\\ 1+\kappa/\lambda\end{pmatrix}\exp(-\lambda U)\right\} \quad (94)$$

follows; note that replacing $(1-\cos\Theta)$ by $(1+\cos\Theta)$ simply replaces κ by $(-\kappa)$, as may be seen by replacing Θ by $\pi-\Theta$. Equation (94) with $(-J_1)$ in the integrand in place of I_1 holds for ν imaginary. Equations (89) exhibit expressions for the contributions from the delta pulses, and the results for the various integrals may be combined with these; with Eqs. (92) taken into consideration, Eqs. (35) readily follow, and these equations hold whether ν be real or imaginary.

[35] See reference 3, Appendix C.