Kapitza Resistance between Helium and Metals in the Normal and Superconducting States

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The contribution of the conduction electrons of a metal to the heat flow across a helium-metal interface has been calculated. It is found that the "totally" reflected phonons from the fluid play an important role in the transfer mechanism as had been predicted previously. The dominant term in the heat flow is proportional to $T^3\Delta T$, in agreement with the experimentally observed value on lead. However, the numerical agreement is poor. The reasons for this are discussed. In the superconducting state it is shown why this heat transfer becomes inoperative. Several interesting consequences of this calculation are given. In particular, it is predicted that a phonon-drag effect may be observed between the conduction electrons in the metal and the phonons in the fluid and vice versa. Also, it is shown that the variation of the Kapitza resistance with applied magnetic field can help to distinguish between the various contributions to the heat flow.

IN 1941 Kapitza[1] observed that when a solid was heated while immersed in liquid helium a discontinuous jump occurred in the temperature in crossing the solid-liquid interface. The thermal resistance which gives rise to this discontinuity has become known as the Kapitza resistance. The earlier attempts to understand this effect were based on the properties of the superfluid,[2,3] He II.

However, Lee and Fairbank[4] made the important observation that the effect occurred when He II was replaced by He^3 and, indeed, it occurs also with He I. Moreover, there has been a considerable amount of evidence to show that such a temperature discontinuity occurs at the interface of any two materials and that this discontinuity becomes more marked as the temperature is lowered.[5] Khalatnikov[6] in 1952 gave a plausible explanation of the effect. This theory showed that the acoustic mismatch at the interface between the solid and the liquid severely limited the flow of heat across the interface. The theory depended upon the elastic properties of the liquid and solid and not the peculiar properties of the superfluid. It gave a value for the heat flow which was appreciable, although considerably less than the experimentally observed values. While the agreement between experiment and theory was not good, it was clear that Khalatnikov's mechanism would always be present and that any other process of transferring heat across the boundary would only act in parallel with it. Some of the details and numerical values in Khalatnikov's theory have been criticized in some recent papers[7,8] but these merely stress the fact that this mechanism is not sufficient in itself to explain the phenomenon.

Khalatnikov's idea was elaborated upon in an earlier paper by the present author and extended to the general problem of the transfer of heat between dissimilar solids.[9] In this paper the particular problem of the transfer of heat between a metal and liquid helium was also discussed. It was shown there that because of the vast difference between the acoustic velocities in helium and practically all solids, a large fraction of the phonons in the fluid would be "totally reflected" by the solid surface. These "totally reflected" phonons create a surface disturbance which in the case of a metal could interact directly with the conduction electrons. Hence, reflection would be no longer total, but energy would be transferred directly to or from the conduction electron. The amplitude of this surface disturbance was found to be large, and thus this mechanism could be expected to be an important one. It was suggested, therefore, that one should expect a difference in the Kapitza resistance between liquid helium and a metal in the normal state and one in the superconducting state. For, in the superconducting state, the phonon-electron interaction necessary for this mechanism vanishes and thus the heat flow would be reduced. Challis[10] has observed that this difference does indeed occur between liquid helium and lead in the normal and superconducting state.

The purpose of this paper, then, is to develop the intuitive ideas of the earlier paper and present a detailed calculation of the effect. In a later paper it is intended to show the role the anharmonic terms in the elastic properties of the solid play, and the role the static imperfections play in scattering energy out of the surface disturbance; and hence the contribution of these to the Kapitza resistance.

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OUTLINE OF APPROACH

In this paper we will consider only the contribution of the conduction electrons to the heat transfer. The contribution of the refracted phonons is treated adequately in references 6 and 9. We consider a semi-infinite solid metal bounded by liquid helium. We calculate first the amplitude of the disturbance created in the solid by phonons of the fluid. This disturbance is treated, next, as a perturbation upon the energy states of the conduction electrons of the solid. The energy scattered to or from the electron system due to this perturbation is then calculated, and hence the heat flow across the surface of the metal in the normal state.

We show then why it is that in the superconducting state this mechanism becomes inoperative and discuss briefly the effect of band structure and magnetic fields upon the Kapitza resistance.

AMPLITUDE OF SURFACE DISTURBANCE IN SOLID

Consider a semi-infinite metal bounded at $z=0$ by liquid helium. Let the solid lie in the region of positive z. Let Lame's constant and the modulus of rigidity of the liquid and solid be λ_1 and μ_1, and λ_2 and μ_2, respectively. The displacement vector $\mathbf{s}(\mathbf{r})$ in the solid or liquid must satisfy the appropriate wave equation for the medium:

$$\partial^2\mathbf{s}(\mathbf{r})/\partial^2 t = C_l^2 \operatorname{grad}\operatorname{div}\mathbf{s}(\mathbf{r}) - C_t^2 \operatorname{curl}\operatorname{curl}\mathbf{s}(\mathbf{r}), \quad (1.1)$$

where C_l is the velocity of longitudinal waves $=[(\lambda+2\mu)/\rho]^{\frac{1}{2}}$, C_t is the velocity of transverse waves $=(\mu/\rho)^{\frac{1}{2}}$, and ρ is the density of the medium.

It is convenient to write the vector $\mathbf{s}(\mathbf{r})$ in terms of a scalar and a vector potential ϕ and ψ, respectively.

$$\mathbf{s}(\mathbf{r}) = \operatorname{grad}\phi(\mathbf{r},t) + \operatorname{curl}\psi(\mathbf{r},t). \quad (1.2)$$

In the liquid the modulus of rigidity μ_1 is zero and there are no transverse waves. The displacement is then purely longitudinal and may be expressed in terms of ϕ_1 alone. We expand ϕ_1 in a three-dimensional Fourier series

$$\phi_1(\mathbf{r},t) = \sum_{\mathbf{q}} \alpha_{\mathbf{q}} \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega_{\mathbf{q}} t)] + \text{c.c.} \quad (1.3)$$

In the solid we set

$$\mathbf{s}_2(\mathbf{r}) = \operatorname{grad}\phi_2(\mathbf{r},t) + \operatorname{curl}\psi_2(\mathbf{r},t), \quad (1.4)$$

where

$$\phi_2(\mathbf{r},t) = \sum_{\mathbf{q}'} \beta_{\mathbf{q}'} \exp[i(\mathbf{q}'\cdot\mathbf{r} - \omega_{\mathbf{q}'} t)] + \text{c.c.},$$

and

$$\psi_2(\mathbf{r},t) = \sum_{\mathbf{q}''} \xi_{\mathbf{q}''}\{\gamma_{\mathbf{q}''} \exp[i(\mathbf{q}''\cdot\mathbf{r} - \omega_{\mathbf{q}''} t)] + \text{c.c.}\};$$

$\xi_{\mathbf{q}''}$ is a unit vector.

At the interface four boundary conditions must be satisfied. The normal components of the displacement in each medium must be equal and the resultant normal and tangential stresses at the boundary must be zero at all points, and at all times

$$s_z^{(1)} = s_z^{(2)}, \quad (1.5)$$

and

$$\begin{aligned} \sigma_{zz} &= \sigma_{zz}^{(1)} - \sigma_{zz}^{(2)} = 0, \\ \sigma_{xz} &= \sigma_{xz}^{(1)} - \sigma_{xz}^{(2)} = 0, \\ \sigma_{yz} &= \sigma_{yz}^{(1)} - \sigma_{yz}^{(2)} = 0, \end{aligned} \quad (1.6)$$

where

$$\sigma_{ij}^{(1)} = \lambda_1\delta_{ij}(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) + 2\mu_1\epsilon_{ij},$$

and

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial s_i}{\partial j} + \frac{\partial s_j}{\partial i}\right),$$

with $i, j = x, y, z$.

Utilizing (2), (3), and (4) we get the relations

$$q_x = q_x' = q_x'', \quad q_y = q_y' = q_y'', \quad (1.7)$$

and from the wave equation

$$\alpha^2 q^2 = (q')^2, \quad \beta^2 q^2 = (q'')^2, \quad (1.8)$$

where

$$\alpha = C_l^{(1)}/C_l^{(2)}; \quad \beta = C_t^{(1)}/C_t^{(2)}. \quad (1.9)$$

From (7), (8), and (9) we get

$$q_z' = q(\alpha^2 - \sin^2\theta)^{\frac{1}{2}} = i\eta_1, \quad (1.10)$$

$$q_z'' = q(\beta^2 - \sin^2\theta)^{\frac{1}{2}} = i\eta_2, \quad (1.11)$$

where θ is the angle $\mathbf{q}$ makes with the normal to the solid surface.

For many solids α and β are about 0.1 so that for a large range of θ, q_z', and q_z'' are imaginary.

In this case the wave is not propagated in the second medium, but instead a disturbance is created which decays exponentially away from the surface. After some calculation, one obtains solutions in the two media of the form

$$\phi_1 = \sum_{\mathbf{q}}\{b_{\mathbf{q}} \exp[i(q_x x + q_y y)] + b_{\mathbf{q}}^* \exp[-i(q_x x + q_y y)]\} \cos(q_z z + \epsilon), \quad (1.12)$$

$$\phi_2 = \sum_{\mathbf{q}} f(\theta)\{b_{\mathbf{q}} \exp[i(q_x x + q_y y)] + b_{\mathbf{q}}^* \exp[-i(q_x x + q_y y)]\} \exp(-\eta_1 z), \quad (1.13)$$

$$\psi_2 = \sum_{\mathbf{q}} g(\theta)\xi_{\mathbf{q}}\{b_{\mathbf{q}} \exp[i(q_x x + q_y y)] + b_{\mathbf{q}}^* \exp[-i(q_x x + q_y y)]\} \exp(-\eta_2 z). \quad (1.14)$$

$f(\theta)$ is given by

$$\frac{1}{f^2(\theta)}=\left[\frac{\beta^4(\sin^2\theta-\alpha^2)}{\cos^2\theta(2\sin^2\theta-\beta^2)^2}+\left(\frac{\rho_2}{\rho_1}\right)^2\left\{1-\frac{2\sin^2\theta}{\beta^2}\left(1+\frac{2[(\alpha^2-\sin^2\theta)(\beta^2-\sin^2\theta)]^{\frac{1}{2}}}{2\sin^2\theta-\beta^2}\right)\right\}\right]^2, \tag{1.15}$$

$$g(\theta)=f(\theta)\left[\frac{-2\sin\theta(\alpha^2-\sin^2\theta)^{\frac{1}{2}}}{2\sin^2\theta-\beta^2}\right], \tag{1.16}$$

$$\tan\epsilon=\frac{-\beta^2(\sin^2\theta-\alpha^2)^{\frac{1}{2}}}{\cos\theta(2\sin^2\theta-\beta^2)\left(\frac{\rho_2}{\rho_1}\right)\left[1-\frac{2\sin^2\theta}{\beta^2}\left\{1+\frac{2[(\alpha^2-\sin^2\theta)(\beta^2-\sin^2\theta)]^{\frac{1}{2}}}{2\sin^2\theta-\beta^2}\right\}\right]}. \tag{1.17}$$

ξ_q is a unit vector lying perpendicular to the plane containing $\mathbf{q}$ and the normal to the surface.

The phonon field may be quantized in the usual way.[11] We let b_q and b_q^* play the role of annihilation and creation operators, respectively. Their matrix elements are

$$\langle\mathfrak{N}_q+1|b_q^*|\mathfrak{N}_q\rangle=\left\{\frac{2\hbar(\mathfrak{N}_q+1)}{\omega_q q^2 V\rho_1}\right\}^{\frac{1}{2}},$$
$$\langle\mathfrak{N}_q-1|b_q|\mathfrak{N}_q\rangle=\left\{\frac{2\hbar\mathfrak{N}_q}{\omega_q q^2 V\rho_1}\right\}^{\frac{1}{2}}. \tag{1.18}$$

In thermal equilibrium the mean value of $\mathfrak{N}_q$ is

$$\mathfrak{N}_q=\langle\mathfrak{N}_q\rangle_{\mathrm{av}}=\frac{1}{\exp(\hbar\omega_q/kT)}. \tag{1.19}$$

In obtaining the expression (1.18), we have ignored the contribution to the phonon Hamiltonian of the disturbance created in the solid. This is a small part of the total Hamiltonian if the volume of the liquid, V contains many complete waves of the phonon.

ELECTRON-PHONON INTERACTION

For this problem we take a simple model of a metal. We consider the electrons as moving in a constant potential, $V_0(\mathbf{r})$ determined by the ionic lattice and consider the interaction between the electrons and the phonons caused by a change of density of this ionic lattice. The electron Hamiltonian is

$$\sum_i\left[\frac{\hbar^2}{2m}\nabla_i^2+eV_0(\mathbf{r})\right]=\mathfrak{IC}_e. \tag{2.1}$$

We expand $V_0(\mathbf{r})$ in powers of the deviation of the density from its equilibrium value,

$$V(\mathbf{r})=V_0(\mathbf{r})+\rho\left(\frac{\partial V}{\partial\rho}\right)_0\left(\frac{\Delta\rho}{\rho}\right)+\cdots, \tag{2.2}$$

$$V(\mathbf{r})=V_0(\mathbf{r})+V_1(\Delta\rho/\rho)+\cdots, \tag{2.3}$$

where $V_1=(\partial V/\partial\rho)_0\rho$. The energy eV_1 is of the order of magnitude of the Fermi energy, i.e., 1 to 10 ev and

$$\Delta\rho/\rho=-(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}). \tag{2.4}$$

The unperturbed Hamiltonian has eigenstates consistent with the boundary condition that V_0 vanishes at $z=0$:

$$\phi_{\mathbf{k}}(\mathbf{r})=(2^{\frac{1}{2}}/\Omega^{\frac{1}{2}})\exp[i(k_x x+k_y y)]\sin(k_z z+\delta); \tag{2.5}$$

Ω is the volume of the metal and δ is a small phase factor.

The matrix elements are then obtained from (2.3), (2.4), and (1.13):

$$\begin{aligned}&\langle\mathbf{k}_1\cdots\mathbf{k}'\cdots\mathbf{k}_n|eV_1(\partial\rho/\rho)|\mathbf{k}_1\cdots\mathbf{k}\cdots\mathbf{k}_n\rangle\\&=\frac{2eV_1}{\Omega}\sum_q\alpha^2 f(\theta)\int\exp[-i(k_x'x+k_y'y)]\sin(k_z'+\delta')\\&\quad\times\exp[i(k_x x+k_y y)]\sin(k_z z+\delta)\\&\quad\times q^2\{b_q\exp[i(q_x x+q_y y)]\\&\quad+b_q^*\exp[-i(q_x x+q_y y)]\}e^{-\eta_1 z}dxdydz\\&=2\alpha^2 f(\theta)\frac{eV_1 S}{\Omega}q^2(b_q+b_{-q}^*)I,\end{aligned} \tag{2.6}$$

where

$$q_x=k_x'-k_x;\quad q_y=k_y'-k_y, \tag{2.7}$$

and the spin state of $\mathbf{k}$ is the same as that of $\mathbf{k}'$. S is the surface area of interface and

$$I=\int_0^\infty\sin(k_z'z+\delta')\sin(k_z z+\delta)e^{-\eta_1 z}dz. \tag{2.8}$$

It is convenient to use the notation of second quantization to describe the perturbation. In this representation it becomes

$$\sum_{\mathbf{k},\mathbf{k}'}q^2\frac{2SeV_1}{\Omega}\alpha^2 f(\theta)I(b_q+b_{-q}^*)a_{\mathbf{k}'}^*a_{\mathbf{k}}, \tag{2.9}$$

where $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ are the Fermi creation and destruction

[11] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949).

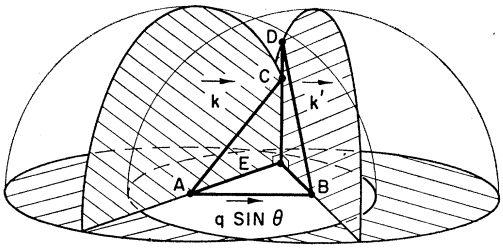

FIG. 1. Construction showing how an electron $\mathbf{k}$ scatters to $\mathbf{k}'$ off the surface disturbance conserving momentum in the plane of the surface EAB but not perpendicular to it.

operators which satisfy the usual anticommutation rules.

The mean value of the number operator $a_k{}^*a_k$ over an ensemble of systems is

$$\langle a_k{}^*a_k\rangle_{av}=\frac{1}{\exp[(E_k-E_f)/kT]+1}. \quad (2.10)$$

CALCULATION OF HEAT FLOW ACROSS THE INTERFACE

The flow of heat per unit time, $\dot{Q}$, across the interface may now be calculated using time-dependent perturbation theory.

$$\dot{Q}=\sum_q\sum_{k,k'}\{|V_{ij}|^2-|V_{ji}|^2\}\hbar\omega_q\frac{4\sin^2(t\Delta\omega/2)}{t\hbar^2(\Delta\omega)^2}, \quad (3.1)$$

where V_{ij} is the matrix element corresponding to the process in which an electron is scattered and a phonon destroyed. V_{ji} corresponds to the inverse process.

$$\hbar\Delta_\omega=E_k+\hbar\omega_q-E_{k'}. \quad (3.2)$$

Utilizing expression (2.9) and (2.7) and transforming the sums to integrals, one obtains

$$\dot{Q}=\frac{V}{(2\pi)^3}\int_q d^3q\,\frac{2\Omega}{(2\pi)^3}\int_k d^3k\,\frac{\Omega}{2\pi S}\int dk_z'\times\{|V_{ij}|^2-|V_{ji}|^2\}\hbar\omega_q\frac{4\sin^2(t\Delta\omega/2)}{t\hbar^2(\Delta\omega)^2}. \quad (3.3)$$

The integral over d^3k and dk_z' can be simplified by noting that the energy of an electron near the Fermi surface is much greater than $\hbar\omega_q$. Therefore, in a scattering process the electron propagation vector $\mathbf{k}$ changes its direction, but its length remains essentially the same, i.e., $|\mathbf{k}|=|\mathbf{k}'|$. The way in which the integrals may be done may be visualized most easily by the following construction illustrated in Fig. 1.

From point A as origin on the surface describe a hemisphere of radius k. Move from A to B a distance, $q\sin\theta$. From B describe a second hemisphere of radius k', where $k'=k$. Drop a perpendicular to the surface to pass through hemispheres k and k' at points C and D, respectively. Now an electron represented by the point C will scatter to the point D satisfying the conditions (2.7) and $\Delta\omega=0$. We may therefore perform the integration

$$\int d^3k\int dk_z \quad \text{as} \quad \int d|\mathbf{k}|\int\left(\frac{dk'}{d(\Delta\omega)}\right)d(\Delta\omega)\int dS_\omega',$$

where $\int dS_\omega$ is an integral over the surface of the k' hemisphere at constant $\Delta\omega$.

The frequency integration can now be done, and after some rearrangement one obtains

$$\dot{Q}=\frac{8S(eV_1)^2m}{(2\pi)^6\rho_1\hbar}\int d^3q\,\frac{q^2\alpha^4f^2(\theta)}{\omega_q}\int_0^\infty\frac{dE_k}{E_k}\int dS_\omega' I^2\times\{\mathfrak{N}_q a_k{}^*a_k a_{k'}a_{k'}{}^*-(\mathfrak{N}_q+1)a_k a_k{}^*a_{k'}{}^*a_{k'}\}. \quad (3.4)$$

The interesting properties of the heat flow come from considering the properties of $\int I^2dS_\omega'$. The form of (2.10) assures us that only those electrons near the Fermi surface contribute to $\dot{Q}$. So that at low temperatures $|\mathbf{q}|\ll|\mathbf{k}|$ and thus the sines in the integral I usually vary much more rapidly than the exponential term. Consequently, the small phase factors δ and δ' can be ignored.

The integral I then becomes

$$I=\frac{1}{2}\left\{\frac{\eta_1}{(k_z'+k_z)^2+\eta_1^2}-\frac{\eta_1}{(k_z'-k_z)^2+\eta_1^2}\right\}. \quad (3.5)$$

In order to make the problem more tractable, we propose to approximate I^2 in the following way.

Case (a). If $|k_z'-k_z|<\eta_1$ and $|k_z'+k_z|>\eta_1$, we set

$$I^2=1/(4\eta_1^2). \quad (3.6)$$

Case (b). If $|k_z'-k_z|<\eta_1$ and $|k_z'+k_z|<\eta_1$, we take

$$I^2=1/(4\eta_1^2). \quad (3.7)$$

If none of the above conditions are satisfied, we set

$$I^2=0. \quad (3.8)$$

We can then find the areas on the k' sphere within which either case (a) or case (b) conditions are satisfied. After a little trigonometry, one finds the value of these integrals to be approximately

Case (a)

$$\int I^2dS_\omega'=\frac{k^2}{2\eta_1^2}\tan^{-1}\left(\frac{\eta_1}{q\sin\theta}\right), \quad (3.9)$$

whereas in

Case (b)

$$\int I^2dS_\omega=\eta_1/8q\sin\theta. \quad (3.10)$$

These areas are shown shaded in Fig. 2. Recalling that

$$\eta_1=q(\sin^2\theta-\alpha^2)^{\frac{1}{2}},$$

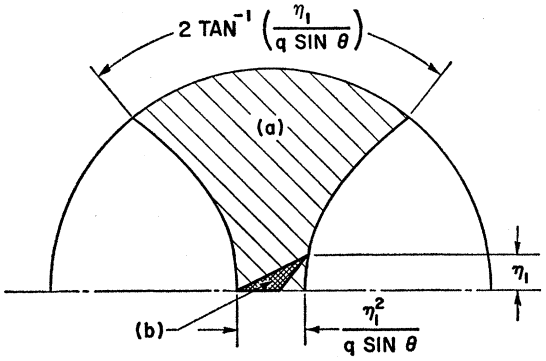

FIG. 2. The regions of the Fermi hemisphere which contribute to the heat flow across an interface due to processes of type (a) and (b) described in the text.

we see that the integral in the two cases depends in a different way upon the phonon momentum $\mathbf{q}$. This gives rise to a different temperature dependence for the two contributions because the Fermi momentum k_f is essentially independent of temperature while the mean value of $|\mathbf{q}|$ is proportional to it. We note, too, that the electrons which contribute to the heat flow come from different parts of the Fermi sphere. This we will take up later in considering the effect of band structure.

Utilizing the anticommutation properties of the $a_{\mathbf{k}}$, expressions (2.10) and (1.19), and taking the electrons to be at a temperature T, and the phonons at a temperature $T+\Delta T$, the curly bracket of (3.4) simplifies to give

$$(\mathfrak{N}_{\mathbf{q}}+1)a_{\mathbf{k}}a_{\mathbf{k}}{}^{*}a_{\mathbf{k}'}{}^{*}a_{\mathbf{k}'}\frac{\hbar\omega_{\mathbf{q}}}{kT}\frac{\Delta T}{T}. \tag{3.11}$$

The integral of $E_{\mathbf{k}}$ is now trivial and one obtains after some rearrangement for case (a)

$$\dot{Q}_a=\frac{8S(eV_1)^2m^2k^4}{(2\pi)^5\rho_1\hbar^6C_1{}^3}\mathcal{J}_4\left(\frac{\Theta}{T}\right)F(\theta)T^3\Delta T, \tag{3.12}$$

$$\mathcal{J}_4\left(\frac{\Theta}{T}\right)=\int_0^{\Theta/T}\frac{z^4dz}{(e^z-1)(1-e^{-z})}, \tag{3.13}$$

and Θ is the Debye temperature of the liquid helium $\approx 20°$K. At low temperatures, $\mathcal{J}_4(\Theta/T)\approx 26.0$.

$$F(\theta)=\int_{\theta_0}^{\frac{1}{2}\pi}\frac{\alpha^4f^2(\theta)\sin\theta\tan^{-1}\{[(\sin^2\theta-\alpha^2)^{\frac{1}{2}}/\sin\theta]\}}{(\sin^2\theta-\alpha^2)}d\theta. \tag{3.14}$$

One obtains for case (b)

$$\dot{Q}_b=\frac{S(eV_1)^2mk^6}{(2\pi)^5\rho_1E_f\hbar^6C_1{}^5}\mathcal{J}_6\left(\frac{\Theta}{T}\right)G(\theta)T^5\Delta T, \tag{3.15}$$

where

$$\mathcal{J}_6\left(\frac{\Theta}{T}\right)=\int_0^{\Theta/T}\frac{z^6dz}{(e^z-1)(1-e^{-z})}, \tag{3.16}$$

$$\mathcal{J}_6(\infty)=732.3,$$

and

$$G(\theta)=\int_{\theta_0}^{\frac{1}{2}\pi}\alpha^4f^2(\theta)(\sin^2\theta-\alpha^2)d\theta. \tag{3.17}$$

Putting in reasonable numerical values, one obtains approximately

$$\dot{Q}=2.2\times10^9F(\theta)T^3\Delta T +5\times10^7G(\theta)T^5\Delta T \text{ erg sec}^{-1}\text{ cm}^{-2}. \tag{3.18}$$

A numerical evaluation of $F(\theta)$ has been made for a solid with Poisson's ratio$=0.33$ and

$$\rho_2/\rho_1=77 \quad \text{and} \quad \alpha=0.2.$$

This gives $\dot{Q}=5.4\times10^3T^3\Delta T$ erg sec^{-1} cm^{-2}, with a much smaller contribution from the second term. The major contribution to $F(\theta)$ comes from the region close to the critical angles.

This value is considerably smaller than the value of approximately $1.5\times10^6T^3\Delta T$ obtained by Challis[10] for lead. This is not entirely surprising because in our calculation of the electron-phonon interaction we have ignored the fact that in all metals the potential in which the electrons move is not constant, but a strong function of position. Consequently, we obtained in our derivation an interaction with only the longitudinal component of the surface disturbance. If we take into account the periodicity of the ionic potential, we would obtain in addition an interaction with the transverse components. In this particular problem, any such transverse component would be very important for one can easily show that while the dilation of the solid is small, the actual displacement at the surface is quite large.

The dilation is small because of the peculiar phase relationship between the transverse and normal displacements in the surface disturbance, indeed, almost complete cancellation (to order α^2) occurs in the contribution of each of these to the dilation. This has the effect of reducing the cross section by α^4. Consequently, one would require a precise knowledge of the phonon-electron interaction for a particular metal to obtain a precise numerical check. It should be noted too, that the usual value of the electron-phonon interaction obtainable for example from the high-temperature resistivity may not be used in this case, for the frequency-wave number relationship is entirely different for the surface disturbance and for the body-phonon, i.e., one is off the "mass shell."

Furthermore, one should note that in addition to the matrix elements considered in (2.9) the electrons will also be perturbed by the periodic displacement of the

solid surface. This occurs even in our simple model, for, in the surface, the potential which is assumed constant in the body of the metal, drops to zero. Consequently in the surface region the actual potential depends upon the displacement at that point and the gradient of V_0. This gives a perturbation

$$eV_1=\mathbf{s}\cdot\text{grad}(eV_0). \tag{3.20}$$

Using our assumption that V_0 is constant within the metal, grad(eV_0) must be normal to the surface and thus the only contribution to (3.20) comes from s_z

$$\langle\mathbf{k}_1\cdots\mathbf{k}'\cdots\mathbf{k}_n|\,\mathbf{s}\cdot\text{grad}(eV_0)\,|\mathbf{k}_1\cdots\mathbf{k}\cdots\mathbf{k}_n\rangle$$
$$=\frac{2S}{\Omega}q_z(b_q+b_q^*)\int\sin(q_zz+\epsilon_0)\sin(k_z'z+\delta')$$
$$\times\sin(kz+\delta)[\text{grad}eV_0(z)]dz, \tag{3.21}$$

where

$$k_x'=q_x+k_x \text{ and } k_x \text{ and } k_y'=q_y+k_y. \tag{3.22}$$

If we make the reasonable assumption that $eV_0(z)$ decreases rapidly at $z=0$, then the terms in the integral other than the gradient may be considered as constant and be given their value at $z=0$. For values of $1/k_z$ comparable to the lattice spacing, $\sin\delta$ will be somewhat less than unity, Γ, while for the phonons $\sin\epsilon_0$ will be much less than unity [Eq. (1.17)]. This perturbation then becomes

$$\sum_{\mathbf{q},\mathbf{k},\mathbf{k}'}\frac{2S\Gamma\Gamma'}{\Omega}(eV_0)q\cos\theta\sin\epsilon_0(b_\mathbf{q}+b_{-\mathbf{q}}^*)a_{\mathbf{k}'}^*a_\mathbf{k}. \tag{3.23}$$

This will give a transfer of heat across the boundary,

$$\dot{Q}=\frac{8S(eV_0)^2m^2k^4}{(2\pi)^4\rho_1\hbar^6C_1^3}\mathcal{J}_4\left(\frac{\Theta}{T}\right)H(\theta)T^3\Delta T, \tag{3.24}$$

where

$$H(\theta)=\int_0^{\frac{1}{2}\pi}\Gamma^2(\Gamma')^2(\sin\epsilon_0)^2\cos^2\theta\sin\theta d\theta. \tag{3.25}$$

This transfer of heat will exist for angles of incidence both less than and greater than the critical angle. One should note, too, that because of the simplifying assumption used in obtaining (3.23), all parts of the Fermi surface contribute equally to (3.24) provided (3.22) can be satisfied for these electrons.

Putting in numerical values, one obtains approximately

$$\dot{Q}=2\times10^4T^3\Delta T \text{ erg sec}^{-1}\text{ cm}^{-2}. \tag{3.26}$$

This is somewhat larger than the contribution from (3.18) but in the absence of a detailed calculation of the phonon-electron interaction for a particular metal, it would be rash to conclude that this surface modulation would always dominate.

INTERACTION IN THE SUPERCONDUCTING STATE

In the superconducting state, three types of interaction are possible. First, the totally reflected phonon may create a quasi-particle out of the ground state. To do this the phonon must have an energy of at least the energy gap, $3.5kT_c$. Consequently, the number of such phonons will fall with temperature approximately as $\exp(-3.5T_c/T)$ and will become vanishingly small at temperatures well below the superconducting transition temperature.

Second, a thermally excited quasi-particle may be scattered by the phonon and thereby absorb energy from the fluid. The number of these quasi-particles will depend also upon the temperature as $\exp(-3.5T_c/T)$ and hence for $T\ll T_c$ will vanish.

Third, the whole superconducting state may be excited to one of its collective modes, for example, by the absorption of momentum from the phonons giving the electrons a net drift. The absence of an appreciable contribution of these modes to the specific heat indicates that there can be relatively few of them. Moreover, the matrix elements for such excitations must be exceedingly small. For if this were not so, this mechanism (the emission or absorption of a phonon) would provide a means for a persistent current to decay. The tremendous stability (or metastability) of such currents shows how unlikely these processes are. The smallness of the matrix elements is probably due to the destructive interference between the different parts of the element because of the phase coherence between electrons in the superconducting state.

From the above we conclude that as the temperature falls appreciably below the superconducting transition temperature, all three processes become negligibly small and hence the contribution of the conduction electrons to the heat flow across the boundary vanishes. This then gives a greater Kapitza resistance for a metal in the superconducting state than for one in the normal state.

EFFECTS OF BAND STRUCTURE, MAGNETIC FIELD, AND PHONON-DRAG

If the major contribution to the heat transfer comes from processes involving the totally reflected phonons, then one can see from Fig. 2 that the electrons which contribute come from a fairly narrow band on the Fermi surface. Any band structure which causes a deformation of the Fermi surface in this region would alter the magnitude of the Kapitza resistance. While it would be difficult to measure in detail the shape of the Fermi surface in this way because of the inherent experimental difficulties and the additional angular dependence of the phonon-electron interaction, it might be used as a check of any shape determined in any of the more usual ways. If the major contribution to the heat transfer comes from the modulation of the surface, however, the band structure will be much less impor-

tant. This could be used to determine which of the two mechanisms is dominant.

If the former interaction is the dominant one, the Kapitza resistance should be altered by a magnetic field applied parallel to the surface. For, in this case the electron orbits will be curved and the integral (2.8) will be a function of the field. If the latter type of interaction is dominant, no such field dependence should occur, because the integral (3.23) is not sensitive to the curvature of the electron orbits. Challis' preliminary results suggest that it is, indeed, the former type of interaction which is dominant, i.e., the interaction with the totally reflected phonons.

If this turns out to be the case, then one might expect in addition to the above, de Haas-van Alphen type of oscillations to occur in the Kapitza resistance due to the variation with magnetic field of the density of states in those regions of the Fermi surface which contribute to the heat flow. This would probably be difficult to observe except for certain specific orientations of some metals and very large magnetic fields.

Another consequence of the above theory is the possibility of observing a phonon-drag phenomenon in the fluid due to an electron current in the metal. This is due to the momentum transfer which occurs via the surface disturbance [Eqs. (2.7) and (3.22)]. Conversely, if there is a drift of the normal fluid past the metallic surface, it will induce an electrical current in the metal. This is a rather unusual situation in that the phonons can be observed in a region where the electrons are not, and vice versa.

The above theory has been applied to the specific case of the helium-metal interface, however, it requires little modification to apply it to any solid-metal interface. For similar reasons to those given above, one should expect to find an appreciable difference in the thermal contact resistance between two dissimilar metals when one is in the normal state as compared to it being in the superconducting state. A column composed of many such pairs, e.g., Cu-Pb-Cu-Pb-Cu, should exhibit a greater difference in thermal conductivity in the two states than that of the superconducting metal alone. The application of this as a thermal valve is obvious.

DISCUSSION

The treatment of the interaction of the totally reflected phonons with the conduction electrons has been shown to give an appreciable contribution to the heat flow across a helium/metal interface. Numerical agreement with the measurements on lead, however, is not good. The reason for this is believed to be due partly to the rather naive electron-phonon interaction used in this paper, which has been discussed earlier. There is another possibility for improving the agreement and that is the one pointed out by Challis and Wilks[12] in connection with the Khalatnikov mechanism. Close to the metal interface the van der Waals attractive force is sufficient to solidify a thin layer of helium. This solid layer will act as an acoustic match between the liquid and the metal because of its intermediate acoustic impedance. Consequently, the amplitude of the disturbance in the solid will be correspondingly greater than in the absence of such a layer and will increase the contribution to the heat flow of both the transmitted phonons and the conduction electrons. As yet no detailed calculation has been made of the magnitude of this effect; however, it is clear that it will depend upon the wavelength of the phonons and consequently will change to some extent the temperature dependence calculated in this paper as well as the magnitude.

In conclusion, we may point out that there is a more general way of calculating the Kapitza resistance than that used here. The propagation vectors of the phonons which we have used have been *real*. However, we have shown that it is the scattering of these phonons by the electrons which give a contribution to the heat flow. The scattering attenuates the phonon waves and consequently one may describe the phonons in terms of complex propagation vectors. In such a representation, the interaction with the electrons would be taken into account automatically. In this case the heat flow across the boundary could be calculated as was done by Khalatnikov[6] or Little[9] and would depend upon the densities, acoustic velocities, *and* the attenuation coefficients of the longitudinal and transverse phonons in the two media.

[12] L. J. Challis and J. Wilks, Physica **24**, S 145 (1958).