PHYSICAL REVIEW VOLUME 123, NUMBER 1 JULY 1, 1961

Lagrangian Formalism in Relativistic Dynamics

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(Received April 28, 1960)

A covariant Lagrangian formalism is put forward with an explicit variation of the proper time in the action functional. This approach conforms with the geometrical interpretation in space-time. A general equation of motion is derived, which is not identical with the Euler-Lagrange equation. Momentum and mass are unambiguously defined through the requirement of translational invariance. The rest mass is constant in the special case of electromagnetic field only. A conservation law for the combination of the momentum and the energy momentum-tensor of the free field is derived. No satisfactory Hamiltonian formalism can be established within the framework of the formalism.

1. INTRODUCTION

DIFFERENT methods have been proposed to derive the equation of motion of a mass point in a given external field of a particular transformation property. A method invented by Dirac[1] in connection with the electromagnetic field consists of calculating the divergence of energy-momentum tensor of the field within a small tube surrounding the world-line of the moving particle. This treatment has been later extended by Bhabha, Harish-Chandra,[2] and Havas[3] to fields other than the electromagnetic. Another approach, first used by Infeld and Wallace,[4] applies a small gravitational field in which the equation of motion follows by virtue of the nonlinearity of the gravitational equations. The derivation of the equation of motion from the variation of an invariant Lagrangian has been used occasionally, but no general method has been established which could be used independently from the transformation properties of the field. The present note aims at pointing out the possibility of such an approach.

If the behavior of a system—field plus particle—can be described with the aid of a Lagrangian formalism, one expects to find an action functional composed of three parts,[5]

$$S=S^{\text{particle}}+S^{\text{interaction}}+S^{\text{field}}$$

$$=\int_{\sigma_1}^{\sigma_2} L^{\text{p}}(\dot{\xi})d\tau +\int_{\Sigma_1}^{\Sigma_2}\int_{\sigma_1}^{\sigma_2} L^{\text{i}}(\phi(x),\phi_{,\alpha}(x),\dot{\xi})\delta(x-\xi)d\tau dx +\int_{\Sigma_1}^{\Sigma_2} \mathfrak{L}^{\text{f}}(\phi(x),\phi_{,\alpha}(x))dx, \quad (1)$$

the first part containing the quantities characteristic of the particle, the third the field variables, while the interaction term contains both. The variation of S^{particle} and S^{field} yield the equation of motion of a free particle and of a free field, respectively; variation of $S^{\text{interaction}}$ with respect to particle coordinates gives the equation of motion in the field, and with respect to field coordinates, the inhomogeneous terms in the field equations. The advantage of such a procedure is obvious: It helps to define conservation laws unambiguously; the resulting equation of motion may be regarded as based on a simpler physical principle; the procedure may pave the way towards the transition from classical to quantum mechanics. From this point of view it is essential, though not obvious, that one and the same interaction term should be used for both of these equations.

The above program, as it is well known, can be carried through without any difficulty in the case of the electromagnetic field.[6] The Euler-Lagrange equations of the interaction Lagrangian give the correct equation of motion of a particle in an electromagnetic field. This method does not work, however, if the field possesses transformation properties other than a vector. In this event, the Euler-Lagrange equations of a reasonable interaction Lagrangian are not equivalent to the equation of motion. Nevertheless, it has been shown by Szamosi,[7] for instance, that the equation of motion can be derived from an interaction Lagrangian, if two different Lagrangians are chosen for the field equations and the equation of motion of the particle. It has been demonstrated also by Szamosi[8] that if the proper time τ is replaced by a new independent parameter $s=\tau/M$, where M is the (generally not constant) rest mass of the particle, the procedure becomes more natural.

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[2] H. J. Bhabha and Harish-Chandra, Proc. Roy. Soc. (London) **A183**, 134 (1944); **A185**, 250 (1946); and Harish-Chandra, Proc. Roy. Soc. (London) **A185**, 269 (1946).
[3] P. Havas, Phys. Rev. **87**, 309 (1952); **91**, 997 (1953); **93**, 882 (1954); and **113**, 732 (1959).
[4] L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).
[5] We use the following notations: x_μ stands for the field coordinates, ξ_μ for the coordinates of the particle. x, ξ (without index) denote the set of four coordinates, dx is the four-dimensional volume-element divided by ic. τ is the proper-time. The fourth component of a vector is imaginary; no metric is introduced. Σ_1, Σ_2 signify two space-like surfaces; σ_1, σ_2 two arbitrary points on them. L is the Lagrangian and $\mathfrak{L}$ is the Lagrangian density. ϕ represents the set of field variables, $\phi_{(\sigma)}$ one component of it, the bracketed suffix may correspond to indexes $1, 2\cdots n$ or may be omitted, depending on the transformation properties of the field.
[6] See, e.g., L. Landau and E. Lifschitz, *Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), translated by M. Hamermesh, p. 70.
[7] G. Szamosi, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1958), Vol. 30.
[8] G. Szamosi, private communication (to be published).

The present approach is based on attaching a geometrical meaning to the variational principle. The equation

$$\delta(S^{\mathrm{p}}+S^{\mathrm{i}})=0, \quad (2)$$

will be regarded as prescribing the extremal of the weighted path between points σ_1 and σ_2 in space-time. This is an extension of the requirement that a free particle moves along a geodesic: indeed, if $L^{\mathrm{particle}}=m$ (constant), then (2) gives the equation of motion for a free particle. This interpretation requires, however, that τ be not regarded as an independent parameter; indeed, it must be varied together with the coordinates. Consequently, the Euler-Lagrange equations will not be the differential equations of the variational principle. The modified differential equation which we propose to put identical with the equation of motion will be derived in Sec. 2. In Sec. 3, momentum and mass will be defined in the new formalism. In Sec. 4, the existence of a conservation law for the momentum will be demonstrated. In Sec. 5, it will be pointed out that it is not possible, as a rule, to construct a Hamiltonian formalism in this treatment. In Sec. 6, the equation of motion obtained will be applied to fields of various transformation properties; it will be shown that the equations of motion derived from a reasonable interaction Lagrangian are equivalent to the equations of motion which are believed to be correct. The special role of the electromagnetic field will be indicated.

2. THE EQUATION OF MOTION

In this section we drive the generalized equation of motion of a point particle from a Lagrangian,

$$L=-mc^2+\int L^{\mathrm{i}}(\xi,\phi(x),\phi_{,\alpha}(x))\delta(x-\xi)dx$$
$$=L(\xi,\dot{\xi};m). \quad (3)$$

In writing down the Lagrangian in this form, we assumed (i) that the free-particle Lagrangian is $-mc^2$, where m is the constant rest mass, which is equivalent to the requirement that the free particle should move along a geodetic; (ii) that the interaction Lagrangian depends only on the first-order time derivative in the particle coordinate and the first-order partial derivatives of the field coordinates only. Inclusion of higher derivatives is not difficult in principle, but it does not correspond to any known physical system. m is defined as the mass measured in the particle's rest frame, infinitely far apart from any other particle with which it may interact. The negative sign is chosen in order to obtain a maximum for the varied integral in the case of an extremum.

In accordance with the philosophy of the previous section, we decompose the variational integral in the following way:

$$\delta S=\delta\int_{\sigma_1}^{\sigma_2} Ld\tau=\int_{\sigma_1}^{\sigma_2}\delta Ld\tau+\int_{\sigma_1}^{\sigma_2}L\delta d\tau. \quad (4)$$

The first term can be expressed[9] clearly as

$$\int\delta Ld\tau=\int\left(\frac{\partial L}{\partial\xi_\mu}\delta\xi_\mu+\frac{\partial L}{\partial\dot{\xi}_\mu}\delta\dot{\xi}_\mu\right)d\tau. \quad (5)$$

It must be noted that the δ and the $d/d\tau$ operation do not commute now. Instead, the following relation holds:

$$\delta\dot{\xi}_\mu=(\delta d\xi_\mu/d\tau)-[d\xi_\mu/(d\tau)^2]\delta d\tau. \quad (6)$$

Employing the identity

$$\delta d\tau=\delta(-d\xi_\mu d\xi_\mu)^{\frac{1}{2}}=-\dot{\xi}_\mu\delta d\xi_\mu, \quad (7)$$

we obtain, by combining the previous equations,

$$\int\delta Ld\tau=\int\frac{\partial L}{\partial\xi_\mu}\delta\xi_\mu d\tau+\int\frac{\partial L}{\partial\dot{\xi}_\mu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\delta d\xi_\nu. \quad (8)$$

The second integral may be transformed considering that $\delta d\xi_\mu=d\delta\xi_\mu$ and applying partial integration. Finally, one obtains

$$\int\delta Ld\tau=\frac{\partial L}{\partial\dot{\xi}_\mu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\delta\xi_\nu\Big|_{\sigma_1}^{\sigma_2}$$
$$+\int\left\{\frac{\partial L}{\partial\xi_\mu}-\frac{d}{d\tau}\left[\frac{\partial L}{\partial\dot{\xi}_\nu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\right]\right\}\delta\xi_\nu d\tau. \quad (9)$$

The second term in (4) yields, in virtue of (7),

$$\int L\delta d\tau=-\int L\dot{\xi}_\mu\delta d\xi_\mu$$
$$=-L\dot{\xi}_\mu\delta\xi_\mu\Big|_{\sigma_1}^{\sigma_2}+\int\frac{d}{d\tau}(L\dot{\xi}_\mu)\delta\xi_\mu d\tau. \quad (10)$$

Finally, combining (9) with (10), we obtain

$$\delta S=\left[\frac{\partial L}{\partial\dot{\xi}_\mu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)-L\dot{\xi}_\nu\right]\delta\xi_\nu\Big|_{\sigma_1}^{\sigma_2}$$
$$+\int\left\{\frac{d}{d\tau}\left[L\dot{\xi}_\mu-\frac{\partial L}{\partial\dot{\xi}_\nu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\right]+\frac{\partial L}{\partial\xi_\mu}\right\}\delta\xi_\mu d\tau. \quad (11)$$

If we put $\delta\xi_\mu(\sigma_1)=\delta\xi_\mu(\sigma_2)=0$ at the end points, (11) leads to the following differential equation:

$$\frac{d}{d\tau}\left\{\left(\frac{\partial L}{\partial\dot{\xi}_\mu}\dot{\xi}_\mu-L\right)\dot{\xi}_\nu+\frac{\partial L}{\partial\dot{\xi}_\nu}\right\}=\frac{\partial L}{\partial\xi_\nu}. \quad (12)$$

[9] From here on we use units $c=1$. Limits of the integration will be generally omitted.

This equation will be regarded as the equation of motion of the particle described by the Lagrangian L. It differs from the Euler-Lagrange equation in the bracketed term on the left-hand side. This term disappears only if τ is not varied in the variation process.

3. MOMENTUM AND MASS

We proceed to establish the connection between momentum, mass, and the Lagrangian in this formalism. It should be recalled that the only unambiguous definition of momentum in classical mechanics can be achieved through exploiting the translation invariance of the Lagrangian.[10] This method ensures at the same time that the momentum defined in that fashion will go over into the proper quantum mechanical quantity, described by the momentum-operator $-i\hbar\partial/\partial x_\mu$. To this end we consider, as usual, a variation,

$$\delta\xi_\mu=\epsilon_\mu=\text{const}, \tag{13}$$

of the actual path. In this case, (11) reduces to

$$\delta S=[P_\mu(\sigma_2)-P_\mu(\sigma_1)]\epsilon_\mu, \tag{14}$$

where

$$P_\mu=(\partial L/\partial\dot{\xi}_\mu)+[(\partial L/\partial\dot{\xi}_\nu)\dot{\xi}_\nu-L]\dot{\xi}_\mu. \tag{15}$$

The condition that S should remain invariant under the translation is that

$$P_\mu(\sigma_1)=P_\mu(\sigma_2). \tag{16}$$

Equation (15) defines the momentum, while (16) expresses the conservation of momentum for an isolated system. Generally, the system is not closed and the conservation of momentum assumes more complicated form. This will be discussed in the next section.

In a similar fashion, we shall define mass as the quantity conjugate to the translation of the proper time. To do this, we rewrite (11), separating out the terms which depend on $\delta\xi_\mu$ through $\delta\tau$ only in the integrated-out part of the right-hand side.

$$\delta S=\left[\left(L-\frac{\partial L}{\partial\dot{\xi}_\mu}\dot{\xi}_\mu\right)\delta\tau+\frac{\partial L}{\partial\dot{\xi}_\mu}\delta\xi_\mu\right]\Bigg|_{\sigma_1}^{\sigma_2}+\int\left\{\frac{d}{d\tau}\left[L\dot{\xi}_\mu-\frac{\partial L}{\partial\dot{\xi}_\nu}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\right]+\frac{\partial L}{\partial\xi_\mu}\right\}\delta\xi_\mu d\tau. \tag{17}$$

We consider a variation of the actual path again, such that

$$\delta\tau=-\epsilon=\text{const}, \tag{18}$$

and we concentrate on the change of S which is brought about on account of this variation only, $\delta' S$. Then we obtain

$$\delta' S=[M(\sigma_2)-M(\sigma_1)]\epsilon, \tag{19}$$

where

$$M=(\partial L/\partial\dot{\xi}_\mu)\dot{\xi}_\mu-L. \tag{20}$$

Equation (20) defines the variable rest mass which is generally a function of ξ and $\dot{\xi}$. In view of our definition of the Lagrangian (3),

$$M=m+\Delta M, \tag{21}$$

and ΔM disappears in the absence of interaction.

Finally, we may consider the remaining part of δS which is due to the direct variation of ξ_μ. Through this g_μ may be defined as

$$g_\mu=\partial L/\partial\dot{\xi}_\mu, \tag{22}$$

which is the direct contribution of the field to the momentum.

An alternative formulation of the foregoing results can be achieved if S is regarded as function of the coordinates. This can be realized if the $S=\int Ld\tau$ value is attributed to every point in space, the integration being carried out along the actual trajectory connecting the point of interest with an arbitrary field point. Then

$$S=S(\tau(x),x), \tag{23}$$

and the following three derivatives are equivalent to our previous definitions:

$$dS/dx_\mu=P_\mu;\quad \partial S/\partial\tau=-M;\quad \partial S/\partial x_\mu=g_\mu. \tag{24}$$

The equation of motion (12) can be written down in a simple form with the aid of (15), (20), and (22).

$$dP_\mu/d\tau=\partial L/\partial\xi_\mu, \tag{25}$$

or defining the kinetic momentum, with

$$p_\mu=M\dot{\xi}_\mu,\quad P_\mu=p_\mu+g_\mu, \tag{26}$$

in an alternative form:

$$dp_\mu/d\tau=F_\mu,\quad F_\mu\equiv\partial L/\partial\xi_\mu-(d/d\tau)(\partial L/\partial\dot{\xi}_\mu). \tag{27}$$

(27) may be regarded as the definition of the force, and it is the most appropriate formulation for comparison with the standard forms of the equation of motion.

In concluding, it is worthwhile to emphasize that generally there are two distinct contributions to the momentum of a particle by the field: through the mass and through g_μ. Only in special cases does one of these disappear. The scalar field has no field-momentum, while the vector field leaves $m=M$ invariant.

4. CONSERVATION OF MOMENTUM

In this section, the conservation law obeyed by the momentum in a system which is not closed, that is, where the field does not vanish on the space-like surfaces Σ_1, Σ_2, is considered. The derivation follows the standard line; the general formulation of the conservation law is, however, new.

We consider the total Lagrangian density, $\mathfrak{L}^{\text{int}}+\mathfrak{L}^{\text{field}}$

[10] E. Noether: Nachr. kgl. Ges. Wiss. Göttingen. (1918), *235*; and J. Rzewuski, *Field Theory* (Panstwowe Wydawuictwo Naukowe, Warsaw, 1958), Vol. I, p. 131.

discarding the free-particle contribution, which is constant.

$$\mathcal{L}=\mathcal{L}^{f}(\phi(x),\phi_{,\alpha}(x))+\int L^{i}(\xi,\phi(x),\phi_{,\alpha}(x))\delta(x-\xi)d\tau$$
$$=\mathcal{L}(\xi,\dot{\xi},x,\phi(x),\phi_{,\alpha}(x)). \tag{28}$$

The total variation of the action integral can be written down as[11]

$$\delta S=\int_{\Sigma_1}^{\Sigma_2}\left(\frac{\delta\mathcal{L}}{\delta\xi_\mu}\delta\xi_\mu+\frac{d\mathcal{L}}{dx_\mu}\delta x_\mu+\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma)}}\delta_0\phi_{(\sigma)}+\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma),\mu}}\delta_0\phi_{(\sigma),\mu}\right)dx. \tag{29}$$

δ_0 refers to a variation which is due to the change of the function form of $\phi(x)$ and not to the variation of x. We apply a rigid translation of the actual field and of the actual trajectory together. In this case

$$\delta\xi_\mu=\delta x_\mu=\epsilon_\mu,\quad \delta_0\phi_{(\sigma)}=-\phi_{(\sigma),\mu}\delta x_\mu,\quad \delta_0\phi_{(\sigma),\nu}=-\phi_{(\sigma),\mu\nu}\delta x_\mu. \tag{30}$$

The latter relations hold, because the total variation of ϕ should vanish; therefore, the two variations compensate each other. (29) now becomes

$$\delta S=\int\left(\frac{d\mathcal{L}}{dx_\mu}-\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma)}}\phi_{(\sigma),\mu}-\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma),\nu}}\phi_{(\sigma),\mu\nu}+\frac{\delta\mathcal{L}}{\delta\xi_\mu}\right)\epsilon_\mu dx$$
$$=\int\left\{\frac{d}{dx_\nu}\left[\mathcal{L}\delta_{\mu\nu}-\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma),\nu}}\phi_{(\sigma),\mu}\right]+\left[\frac{d}{dx_\nu}\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma),\nu}}-\frac{\partial\mathcal{L}}{\partial\phi_{(\sigma)}}\right]\phi_{(\sigma),\mu}+\frac{\delta\mathcal{L}}{\delta\xi_\mu}\right\}\epsilon_\mu dx. \tag{31}$$

The second bracketed part in this expression disappears by virtue of the field equations.[12] If the canonical energy-momentum tensor of the free field,

$$T_{\mu\nu}{}^{0}=-\frac{\partial\mathcal{L}^{f}}{\partial\phi_{(\sigma),\nu}}\phi_{(\sigma),\mu}+\mathcal{L}^{f}\delta_{\mu\nu}, \tag{32}$$

is introduced and the explicit form of $\mathcal{L}^{i}$ is considered, one is left with the transformation of the expression

$$\int\left(\frac{d}{dx_\mu}+\frac{\delta}{\delta\xi_\mu}-\frac{d}{dx_\mu}\phi_{(\sigma),\nu}\frac{\partial}{\partial\phi_{(\sigma),\mu}}\right)\epsilon_\mu\times\int L(\dot{\xi},\phi(x),\phi_{,\alpha}(x))\delta(x-\xi)d\tau dx. \tag{33}$$

For a moment we omit the last term in the bracket, which is present only in the case of some kind of derivative coupling. Then (33) can be further transformed to yield

$$\int\int\left[\frac{\partial}{\partial\xi_\mu}L(\dot{\xi},\phi(x))\right]\delta(x-\xi)d\tau dx+\int\int L(\dot{\xi},\phi(x))\delta(x-\xi)\delta d\tau$$
$$+\int\int L(\dot{\xi},\phi(x))\left[\frac{\partial}{\partial\xi_\mu}\delta(x-\xi)+\frac{\partial}{\partial x_\mu}\delta(x-\xi)\right]\epsilon_\mu d\tau+\int\frac{\partial}{\partial\xi_\mu}L(\dot{\xi},\xi)\epsilon_\mu d\tau. \tag{34}$$

The third integral here identically vanishes. The first and second can be transcribed, employing (6), (7), and (15) and applying partial integration to

$$\epsilon_\mu(P_\mu(\sigma_2)-P_\mu(\sigma_1))-\int_{\sigma_1}^{\sigma_2}\dot{P}_\mu\epsilon_\mu d\tau, \tag{35}$$

which also vanishes, in virtue of the constancy of ϵ_μ. Finally, only the last integral remains in (34), which, combined with (31), (32), and (33) gives, if use is made of the equation of motion (25), the law of conservation of momentum

$$-\int_{\Sigma_1}^{\Sigma_2}\frac{dT_{\mu\nu}{}^{0}}{dx_\mu}dx=P(\sigma_2)-P(\sigma_1). \tag{36}$$

We comment on this result, as follows:

(a) The conservation law holds for the canonical momentum; no conservation law exists for the kinetic momentum.

(b) Only the free field energy-momentum tensor shows up in the conservation law. This is, indeed, what one expects on the basis of a naive physical picture that describes the process as the flow of momentum from the field to the particle.

(c) If the interaction Lagrangian contains derivatives of the potential also, the omitted term in (33) must be taken into account, and the simple law of conservation (36) does not hold. In this event, one must work with the total energy-momentum tensor, which depends on the sources as well. Conservation

[11] One must distinguish between partial, total, and functional derivatives here. As a rule, a functional derivative is needed everywhere, but if the Lagrangian density depends only explicitly on a variable, it may be replaced by a partial derivative; in the event that it depends through an undefined functional form, it may be replaced by a total derivative; otherwise, as in the case of ξ_μ, one must retain $\delta\mathcal{L}/\delta\xi_\mu$ on account of the variation of $d\tau$.

[12] See, e.g., J. Rzewuski, reference 8, p. 99.

laws expressed through this tensor are neither physically illuminating nor practically useful. Moreover, we did not succeed in deriving a conservation law in a simple form for this case. One is inclined to believe that the reason for the complication is that, in physical cases, derivative coupling exists only when the particle has some kind of internal structure.[13] Since the present considerations concern structureless particles only, it is not surprising, perhaps, that no consequent description can be achieved.

Equation (36) can be put in an alternative form by introducing the energy-momentum tensor of the particle:

$$T_{\mu\nu}{}^{\mathrm{p}}=\int \dot{\xi}_\mu(\tau(\xi))P_\nu(\tau(\xi))\delta(x-\xi)d\tau. \tag{37}$$

From this, one may deduce

$$\begin{aligned}\int \frac{dT_{\mu\nu}{}^{\mathrm{p}}}{dx_\mu}dx &= \int\int P_\nu \dot{\xi}_\mu \frac{d}{dx_\mu}\delta(x-\xi)d\tau dx \\ &= \int \frac{d}{d\xi_\mu}(P_\nu \dot{\xi}_\mu)d\tau \\ &= -\int \dot{\xi}_\mu \frac{d}{d\tau}(P_\nu \dot{\xi}_\mu)d\tau \\ &= -\int \left(\dot{\xi}_\mu \dot{P}_\nu \dot{\xi}_\mu + \dot{\xi}_\mu P_\nu \frac{d\dot{\xi}_\mu}{d\tau}\right)d\tau \\ &= P_\nu(\sigma_2)-P_\nu(\sigma_1). \end{aligned} \tag{38}$$

Comparing (36) with (38), we may reformulate the conservation law,

$$(d/dx_\mu)(T_{\mu\nu}{}^0+T_{\mu\nu}{}^{\mathrm{p}})=0. \tag{39}$$

Note that the energy-momentum tensor of the particle generally is not symmetric.

5. ON THE HAMILTONIAN FORMALISM

In this section, the possibility of a Hamiltonian formalism will be touched on, and it will be indicated that no satisfactory Hamiltonian formalism can be worked out within the framework of the present theory. The root of the difficulties is that

$$(\partial L/\partial \dot{\xi}_\mu)\equiv g_\mu \neq P_\mu \tag{40}$$

in our formalism, while the canonical equations are equivalent to the equation of motion only under the fulfillment of the above condition. In the attempt to find a partly acceptable Hamiltonian, one may start by considering various characteristics of the conventional Hamiltonian and singling out one of them, postulating it as a defining property. Then it can be shown that the remaining conditions are not satisfied. Any of the following properties may be regarded as the basic ones: (i) The Hamiltonian should describe the Hamilton-Jacobi equation. (ii) It should lead to the canonical equations. (iii) Its numerical value should be equal to the rest energy of the particle.

We consider here in some detail the possibility of defining a Hamiltonian through (i) and (ii), as these results are of certain interest.

Condition (i) defines the Hamiltonian by

$$\partial S/\partial\tau+H(\xi,P)=0. \tag{41}$$

Then, since

$$L=dS/d\tau=\partial S/\partial\tau+\dot{\xi}_\mu(\partial S/\partial\xi_\mu), \tag{42}$$

one obtains

$$H=P_\mu\dot{\xi}_\mu-L. \tag{43}$$

This conforms with the standard form of the Hamiltonian. It is easy to show, however, that on account of (40) it does not satisfy the canonical equations. Indeed, one receives by differentiation

$$\begin{aligned}\partial H/\partial\xi_\mu &= P_\nu(\partial\dot{\xi}_\nu/\partial\xi_\mu)-(\partial L/\partial\xi_\mu)_P \\ &= -\dot{P}_\mu+(\partial\dot{\xi}_\nu/\partial\xi_\mu)[P_\mu-(\partial L/\partial\dot{\xi}_\mu)],\end{aligned} \tag{44}$$

and

$$\partial H/\partial P_\mu=\dot{\xi}_\mu+(\partial\dot{\xi}_\nu/\partial P_\mu)[P_\nu-(\partial L/\partial\dot{\xi}_\nu)]. \tag{45}$$

The Hamiltonian (43) has, however, the interesting property that because of (15)

$$\dot{\xi}_\mu P_\mu=L, \tag{46}$$

and thus

$$H(\xi,P)\equiv 0;\quad S=\int^{\xi} P_\mu d\xi_\mu;\quad \partial S/\partial\tau=0. \tag{47}$$

One may exploit these relations in the transition to quantum mechanics (compare with reference 7).

If condition (ii) is regarded as definition, then the canonical equations serve as differential equations defining the unknown Hamiltonian. The first canonical equation,

$$\frac{\partial H}{\partial\xi_\mu}=-\dot{P}_\mu=-\left(\frac{\partial L}{\partial\xi_\mu}\right)_P+g_\nu\frac{\partial\dot{\xi}_\nu}{\partial\xi_\mu}, \tag{48}$$

may be integrated to yield

$$H=-L+g_\mu\dot{\xi}_\mu-\int(\partial g_\nu/\partial\xi_\mu)\dot{\xi}_\nu d\xi_\mu+\alpha(P), \tag{49}$$

where $\alpha(P)$ is an arbitrary function of the momentum. Substituting (49) into the second canonical equation, one obtains

$$\frac{\partial H}{\partial P_\mu}=-\frac{\partial L}{\partial P_\mu}+\frac{\partial g_\nu}{\partial P_\mu}\dot{\xi}_\nu+g_\nu\frac{\partial\dot{\xi}_\nu}{\partial P_\mu}-\frac{\partial}{\partial P_\mu}\int\frac{\partial g_\nu}{\partial\xi_\lambda}\dot{\xi}_\lambda d\xi_\nu+\frac{\partial\alpha}{\partial P_\mu}. \tag{50}$$

[13] This is an empirical fact. In principle, one may have derivative coupling with structureless particles, of course.

Since

$$(\partial L/\partial P_\mu)=(\partial L/\partial\dot{\xi}_\nu)(\partial\dot{\xi}_\nu/\partial P_\mu)=g_\nu(\partial\dot{\xi}_\nu/\partial P_\mu), \quad (51)$$

(50) reduces to

$$\frac{\partial}{\partial P_\mu}\alpha(P)=\left(\delta_{\mu\nu}-\frac{\partial g_\nu}{\partial P_\mu}\right)\dot{\xi}_\nu+\frac{\partial}{\partial P_\mu}\int\frac{\partial g_\nu}{\partial \xi_\lambda}\dot{\xi}_\nu d\xi_\lambda. \quad (52)$$

One can demonstrate that (52) cannot be generally satisfied. In particular cases, however, one may obtain a solution. Such is the case of the electromagnetic field, when $\alpha=\frac{1}{2}m^{-1}P_\mu P_\mu$ and $H=\frac{1}{2}m^{-1}(P_\mu-e\phi_\mu)(P_\mu-e\phi_\mu)$. Alternatively, if one is concerned with a scalar field, $g_\mu=0$ and (52) is self-contradictory, as the right-hand side depends on ξ, while the left-hand side does not.

Finally we note that the definition

$$H=M=(\partial L/\partial\dot{\xi}_\mu)\dot{\xi}_\mu-L, \quad (53)$$

according to (iii), is of no use for us; as one can easily see, it does not satisfy either the Hamilton-Jacobi equation or the canonical equations.

6. APPLICATIONS

In this section, we survey the application of the formalism to concrete physical fields. The motion of a particle in a scalar, vector (electromagnetic), and tensor (weak gravitational) field will be considered. The guiding principle in constructing the interaction Lagrangian is the "principle of simplicity,"[14] which requires that it should be the simplest invariant expression which can be constructed from the field quantities and the four-velocity of the particle. One can easily convince oneself, on the other hand, that the Lagrangian built up in this fashion leads to the correct (inhomogeneous) field equations.

A. Scalar Field

In the case of a scalar field with scalar potential ϕ and interaction constant g, we choose the Lagrangian[15] as

$$L=-mc^2-g\phi. \quad (54)$$

Then we obtain with the aid of (15), (20), (22), and (27),

$$\begin{aligned}&P_\mu=p_\mu=[m+(g/c^2)\phi]\dot{x}_\mu;\quad g_\mu=0;\\&M=m+(g/c^2)\phi;\quad F_\mu=-g\partial\phi/\partial x_\mu.\end{aligned} \quad (55)$$

We note that the rest mass is not constant. This is in accordance with the results of Marx and Szamosi.[7,16] Comparison with the results of Havas[3] and Bhabha and Harish-Chandra[2] based on the Dirac method is hindered, since in the result of these workers the "radiation reaction" terms are intertwined with the direct effect of the field.

The equation of motion in a scalar field takes up finally the form

$$\frac{d}{d\tau}\left[\left(m+\frac{g}{c^2}\phi\right)\dot{x}_\mu\right]=-g\frac{\partial\phi}{\partial x_\mu}. \quad (56)$$

B. Vector Field

The Lagrangian in the vector field, which may be associated with the electromagnetic field (whether the field-mass is zero or not is irrelevant from our point of view), with the vector potential ϕ_μ and interaction constant e, takes the form

$$L=-mc^2+e\dot{x}_\mu\phi_\mu. \quad (57)$$

From this, one derives the relations

$$\begin{aligned}&P_\mu=m\dot{x}_\mu+e\phi_\mu;\quad M=m;\quad g_\mu=e\phi_\mu;\\&F_\mu=eF_{\mu\nu}\dot{x}_\mu;\quad F_{\mu\nu}\equiv\phi_{\mu,\nu}-\phi_{\nu,\mu},\end{aligned} \quad (58)$$

which conform with the standard forms of momentum, mass, and force in the electromagnetic field. It is worthwhile to point out the root of the rather peculiar feature of the vector field, that it conserves the rest mass. A glance at (20) shows the reason for this: As the interaction Lagrangian is a first-order homogeneous expression in $\dot{\xi}_\mu$, the two terms $\partial L/\partial\dot{\xi}_\mu$ and L compensate each other. At the same time, this consideration indicates that no other field may have this property.

C. Tensor Field

The Lagrangian of a tensor field (tensor potential $\phi_{\mu\nu}$, interaction constant f) can be chosen as

$$L=-mc^2-\tfrac{1}{2}f\dot{x}_\mu\phi_{\mu\nu}\dot{x}_\nu. \quad (59)$$

Clearly, the tensor potential $\phi_{\mu\nu}$ should be symmetric; if it is not, the antisymmetric part plays no role at all.

This Lagrangian leads to the following expressions for the momentum, etc.:

$$\begin{aligned}&P_\mu=[m-(f/2c^2)\dot{x}_\alpha\phi_{\alpha\beta}\dot{x}_\beta]\dot{x}_\mu-f\phi_{\mu\nu}\dot{x}_\alpha,\\&M=m-(f/2c^2)\dot{x}_\alpha\phi_{\alpha\beta}\dot{x}_\beta,\\&g_\mu=-f\phi_{\mu\alpha}\dot{x}_\alpha,\\&F_\mu=f(\dot{x}F_{\alpha\mu\beta}\dot{x}+\ddot{x}_\alpha\phi_{\alpha\mu}),\\&F_{\alpha\mu\beta}=-\tfrac{1}{2}(\phi_{\alpha\beta,\mu}+\phi_{\alpha\mu,\beta}+\phi_{\mu\beta,\alpha}).\end{aligned} \quad (60)$$

The equation of motion may be written in the following form

$$\begin{aligned}(d/d\tau)\{[m-(f/2c^2)\dot{x}_\alpha\phi_{\alpha\beta}\dot{x}_\beta]\dot{x}_\mu-f\phi_{\mu\alpha}\dot{x}_\alpha\}&\\=-\tfrac{1}{2}f(\partial\phi_{\alpha\beta}/\partial x_\mu)\dot{x}_\alpha\dot{x}_\beta.&\end{aligned} \quad (61)$$

If f is associated with the constant rest mass, (61) can be recognized as the equation of motion of a particle in a weak gravitational field. Indeed, if one starts with

[14] See, e.g., P. Roman, *Theory of Elementary Particles* (North-Holland Publishing Company, Amsterdam, 1960), p. 101.

[15] We write out c in this section and discard the distinction between x and ξ.

[16] G. Marx and G. Szamosi, Acta Physica. Acad. Sci. Hung. 4, 219 (1954).

the general gravitational equation of motion

$$g_{\mu\nu}(du^{\nu}/ds)=-\Gamma_{\mu,\alpha\beta}u^{\alpha}u^{\beta},\qquad \Gamma_{\mu,\alpha\beta}\equiv\tfrac{1}{2}(g_{\mu\alpha,\beta}+g_{\mu\beta,\alpha}-g_{\alpha\beta,\mu}), \tag{62}$$

and uses the weak field assumption

$$g_{\mu\nu}=\delta_{\mu\nu}+\phi_{\mu\nu}, \tag{63}$$

neglecting second and higher powers of $\phi_{\mu\nu}$, the identity of (61) and (62) is easily established. The only nontrivial step in doing this is distinguishing between ds ($u^{\nu}=dx^{\nu}/ds$) on the one hand, and $d\tau$ ($\dot{x}_{\nu}=dx_{\nu}/d\tau$) on the other:

$$\begin{aligned} d\tau^2&=-dx_{\mu}dx_{\mu},\\ ds^2&=-g_{\mu\nu}dx^{\mu}dx^{\nu}=-(\delta_{\mu\nu}+\phi_{\mu\nu})dx^{\mu}dx^{\nu}\\ &=d\tau^2(1-\dot{x}_{\mu}\phi_{\mu\nu}\dot{x}_{\nu}),\\ \frac{d}{ds}&=\frac{d\tau}{ds}\frac{d}{d\tau}=(1+\tfrac{1}{2}\dot{x}_{\alpha}\phi_{\alpha\beta}\dot{x}_{\beta})\frac{d}{d\tau}. \end{aligned} \tag{64}$$

Equations (62), (63), and (64)[17] can be now combined to yield (61). We note that the mechanism which brings about the variation of the rest mass can be clearly observed in this case: The physically existing and measurable proper time is s; τ is a physically meaningless artifice. The observer moving along with the test particle will find a changing rest mass only, if he artificially sets his watch to measure τ; otherwise, he will experience no variation. Whether in the case of other fields the variability of the rest mass can be traced back to some similar reason cannot be predicted in the absence of corresponding nonlinear theories. But speculations along this line[8] indicate such possibilities.

[17] Since in the Lorentz frame we do not distinguish between covariant and contravariant indexes, these are not correct tensor equations.

7. CONCLUSIONS

In this paper, a consequent Lagrangian formalism has been worked out in covariant form. The equation of motion in an arbitrary field may be derived in this way from first principles. The main points in this formalism have been the following:

(a) The equation of motion is not the Euler-Lagrange equation for the Lagrangian.

(b) The rest mass generally is not constant; it is constant in a vector field only.

(c) The canonical momentum generally differs from the kinetic one; they coincide in a scalar field only.

(d) The canonical momentum and the energy-momentum tensor of the *free* field combine into a conserved system in the case of no derivative coupling; otherwise they do not.

(e) No satisfactory companion Hamiltonian formalism can be built up; generally there is no function which satisfies the canonical equations.

ACKNOWLEDGMENTS

The author is indebted to Professor G. Szamosi for many stimulating discussions and for making him acquainted with his results prior to publication. He is grateful to Dr. R. Englman for further discussions on the subject.