Quantum Mechanical Systems with Indefinite Metric. I*†

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The structure of quantum theories with indefinite metric is studied with the aid of several simple models. It is shown that the pseudo-unitary scattering matrices entering such a theory are not inconsistent with physical interpretation provided a suitable invariant projection of physical state is carried out from among all the states. A relativistic quantum theory of interacting fields is outlined and is suggested as a basis for a dynamical theory of elementary particles. It is argued that the formal introduction of an indefinite metric together with supplementary interpretive postulates may help to reinstate the principle of simplicity in a consistent theory of elementary particles.

1. INTRODUCTION

NEGATIVE probabilities and the notion of an indefinite metric have been considered many times previously in quantum theory; and while no explicit contradictions could be pointed out it has often been suggested that the introduction of an indefinite metric would make quantum theory inconsistent. On the other hand, it has also been suggested that if it is ever possible to construct a self-consistent quantum theory using an indefinite metric it should be possible to reformulate the theory without using an indefinite metric; this suggestion, which derives some confirmation from the structure of quantum electrodynamics (where it is possible either to work with the Gupta-Bleuler form involving an indefinite metric or with the Dirac-Schwinger form involving an instantaneous Coulomb interaction), also points to the possibility that the indefinite metric with "local" interactions may actually be a method of presenting a theory with a definite metric but "nonlocal" interactions. Thus, rather than suggest that the indefinite metric is superfluous in this case, it in fact suggests that the use of the indefinite metric may provide the most elegant method of construction of consistent nonlocal relativistic theories.

The question of "local" versus "nonlocal" interactions has also been discussed often and it has sometimes been claimed that nonlocal theories cannot satisfy both relativistic invariance and the dynamical principles leading to specification of Cauchy data for the operator fields entering a Lagrangian theory of quantized fields. This claim is unsound in view of the existence of the Dirac-Schwinger formulation of quantum electrodynamics. Since such a success is not duplicated in any other quantum field theory, one is not able to generalize the dynamical structure of quantum electrodynamics; such a situation is very unsatisfactory.

In recent years methods of dealing with strong interactions not explicitly dependent upon a Lagrangian

formulation have been developed. The success of these methods in regard to quantitative correlation of strong-interaction data have to a certain extent hidden the fact that such a theory is primarily an elegant scheme of "self-consistent field approximation" and the basic questions of elementary particle architecture (as distinct from the calculation of strong-interaction transition amplitudes) are outside the scope of such a theory. The various conventional Lagrangian schemes for relativistic fields have been completely powerless in providing an answer to these questions; the exceptions are certain unorthodox attempts but these do not subscribe to the "orthodox" postulational scheme, noticeably the one relating to the restriction to positive definite probabilities. On the basis of the case of quantum electrodynamics (and to a certain extent, on the basis of some of the simple models discussed below), one is led to assert that for a Lagrangian formulation of theory of interacting fields manifest covariance and positive-definite metric are complementary aspects, and that if we indulge in the luxury of manifest covariance it may not be possible to argue on the basis of physics to rule out negative probabilities. This view is elaborated in the following sections of this paper. In Sec. 2 we discuss the simple example of a two-level system with indefinite metric and several elementary results are deduced. In Sec. 3 this case is generalized to provide a discussion of the neutral scalar theory; this particular case which is completely solvable in closed form exhibits many interesting features. In Sec. 4 we again consider a simple model which cannot be completely solved but a partial diagonalization into "sectors" can be obtained and explicit solutions for several sectors can be obtained. The resemblance of the Lee model to this case is also discussed. A system similar to the ones discussed by Pais and Uhlenbeck is considered in Sec. 5. A model of quantized field theory which involves the indefinite metric for which certain scattering states can be explicitly solved is constructed in Sec. 6. A new formulation of a relativistic theory of interacting quantized fields is outlined in Sec. 7. The theory is compared with some earlier work of Bogoliubov in Sec. 8. Certain questions of principle and outlook are discussed in Sec. 9. The Appendix compares the Gupta-

Bleuler and Dirac-Schwinger formulations of quantum electrodynamics.

2. THE TWO-LEVEL SYSTEM

Let us consider a quantum mechanical system with only two states labeled 1 and 2; the state vector is a 2-component vector and the Hamiltonian operator must now be a 2×2 matrix which would be pseudo-Hermitian if the metric is indefinite. No generality is lost by choosing the metric operator to be diagonal and to have unit square. Then an arbitrary time-dependent Hamiltonian process is expressible in terms of a pseudo-unitary S-matrix transforming initial state vectors into final state vectors. That such a matrix, and more generally a pseudo-unitary matrix connecting the state vectors at two finite times, exists can be deduced easily from the existence of the pseudo-unitary matrices relating infinitesimal time differences; and the latter follows from the existence of the Schrödinger equation with a pseudo-Hermitian matrix. To make the analogy with a real scattering process closer, we may assume that the Hamiltonian matrix H consists of a constant multiple of the unit matrix plus an interaction term which vanishes at both initial and final times. Under these restrictions there exists a pseudo-unitary S matrix which may be expressed in the form:

$$S = \exp(2i\delta), \qquad (1)$$

where the phase shift matrix δ is a pseudo-Hermitian matrix. It is easier to parametrize this matrix and (choosing the basis so that the metric η is identical with the Pauli matrix σ_3) the general form may be written

$$\delta = \begin{pmatrix} a+b & ic+d \\ ic-d & a-b \end{pmatrix}, \qquad (2a)$$

where a, b, c, d are all real parameters. The eigenvalues of δ are given by

$$\delta = a \pm [b^2 - (c^2 + d^2)]^{\frac{1}{2}}, \qquad (2b)$$

and are real as long as $c^2 + d^2 < b^2$ in spite of the fact that the matrix δ was not Hermitian. We may also verify that as long as these roots are real the eigenvectors have nonvanishing norm, but as soon as the eigenvalues become complex (i.e., $c^2 + d^2 > b^2$) the vectors become null vectors.

Consequently, as long as the eigenvalues of δ are real and distinct (and consequently those of S unitary), there exists an eigenstate of the system with a non-vanishing norm and which can be normalized to have $(\psi, \psi) = +1$. Hence in this case physical probabilities are positive definite provided one imposes the supplementary condition that

$$\Lambda \psi = 0, \quad \Lambda = \varphi \varphi^* \eta \qquad (3)$$

where φ is the state with negative norm. Two points are to be noted in this connection:

(i) The important question is not the unitarity of the S-matrix but the unitarity of the eigenvalues. In this case the supplementary condition can always be imposed and the "physical" states[1] so obtained can be chosen to have norm $+1$.

(ii) The interaction Hamiltonian cannot be chosen arbitrarily if one works within the framework of the 2-level amplitude, but must be such as to ensure the unitarity of the eigenvalues of the S matrix.

These considerations which are explicitly demonstrated here for the two-level system can be immediately generalized.[2] In the n-level case we work again with the pseudo-Hermitian phase shift matrix δ. If the eigenvalue λ of δ is complex, the corresponding eigenvector is a null vector (i.e., has zero norm) since the pseudo-Hermiticity requires that

$$\lambda (\psi, \psi) = (\psi, \delta \psi) = (\delta \psi, \psi) = \lambda^*(\psi, \psi),$$

or

$$(\lambda - \lambda^*)(\psi, \psi) = 0.$$

It can also be shown that if the eigenvalue λ is real the corresponding eigenvector can be chosen to have norm ± 1 at least if the characteristic equation has no multiple roots. Consequently the supplementary conditions annihilating all negative norm states can be imposed with consequent assurance that all "physical" probabilities are nonnegative.

Let us now consider the related case of a particular system with an infinite set of states called the Yukawa oscillator; this system is defined by a "pseudo-oscillator" coupled to a static source:

$$H = H_0 + H_1 = -\omega a^\dagger a + g\rho(a^\dagger + a), \qquad (4)$$

where

$$a^\dagger = \eta a^* \eta \quad \text{and} \quad [a, a^\dagger] = -1.$$

The states ψ_n are defined as usual by the eigenvalues of $a^\dagger a$ and have the scalar product

$$(\psi_n, \psi_m) = (-1)^n \delta_{nm}; \quad a^\dagger a \psi_n = -n\psi_n.$$

For Hermiticity of the Hamiltonian it is necessary that if the source strength ρ is real, the coupling constant g is real. The Hamiltonian is pseudo-Hermitian but it can be diagonalized by the introduction of new oscillator variables:

$$b = a - (g/\omega)\rho, \quad b^\dagger = a^\dagger - (g/\omega)\rho.$$

Then

$$H = -\omega b^\dagger b + (g^2/\omega)\rho^2; \qquad (5)$$

the energy eigenvalue is consequently *increased* by the amount $(g^2/\omega)\rho^2$ and the eigenstates are eigenfunctions of $b^\dagger b$ which have the norm $\exp(i\pi b^\dagger b)$. Here again the "physical" probabilities can be chosen to be nonnegative

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[1] It may be necessary to strengthen the supplementary condition by further selecting a subset from among the eigenstates of the S matrix with positive norm; compare reference 7.
[2] See, for example, L. K. Pandit, Suppl. Nuovo cimento 11, 157 (1959).

by the appropriate supplementary condition

$$[1+\exp(i\pi b^\dagger b)]\psi = 0. \tag{6}$$

3. NEUTRAL SCALAR THEORY AND POTENTIALS

The Yukawa oscillator discussed in the last section could be thought of alternatively as a (one-dimensional) particle obeying the pseudo-canonical commutation relation

$$[p,q] = +i$$

(note the opposite sign to the usual case) with a suitable restoring force. We can formally extend this to introduce now a field theory with indefinite metric. Let $\phi(\mathbf{r})$ and $\pi(\mathbf{r})$ be the operators for a neutral scalar field and its canonically conjugate momentum density satisfying the commutation relations:

$$[\pi(\mathbf{r}),\phi(\mathbf{r}')] = +i\delta(\mathbf{r}-\mathbf{r}').$$

A theory of this neutral scalar field interacting with a source density $\rho(\mathbf{r})$ is obtained by choosing the Hamiltonian density:

$$\mathcal{H}(\mathbf{r}) = \pi^2 + \mu^2\phi^2 + (\nabla\phi)^2 + g\rho\phi. \tag{7}$$

The commutation relations have the sign opposite the usual scalar field commutation relations and can be seen to correspond to a theory with indefinite metric. Introducing destruction and creation operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$ for each momentum vector $\mathbf{k}$, one finds that

$$[a(\mathbf{k}),a^\dagger(\mathbf{k}')] = -\delta(\mathbf{k}-\mathbf{k}'),$$

so that the metric operator is

$$\eta = \exp\left[i\pi \int d^3k\, a^\dagger(\mathbf{k})a(\mathbf{k})\right]$$

and the Hamiltonian is

$$H = -\int d^3k(\mu^2+k^2)^{\frac{1}{2}}a^\dagger(\mathbf{k})a(\mathbf{k}) + [g/(16\pi^3)^{\frac{1}{2}}]$$
$$\times \int d^3k(\mu^2+k^2)^{-\frac{1}{2}}[\tilde{\rho}(\mathbf{k})a^\dagger(\mathbf{k})+\tilde{\rho}^*(\mathbf{k})a(\mathbf{k})].$$

By the substitution

$$b(\mathbf{k}) = a(\mathbf{k}) - (16\pi^3)^{-\frac{1}{2}}g(\mu^2+k^2)^{-\frac{3}{4}}\tilde{\rho}(\mathbf{k}),$$
$$b^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k}) - (16\pi^3)^{-\frac{1}{2}}g(\mu^2+k^2)^{-\frac{3}{4}}\tilde{\rho}^*(\mathbf{k}),$$

one obtains the diagonalized Hamiltonian:

$$H = -\int d^3k(\mu^2+k^2)^{\frac{1}{2}}b^\dagger(\mathbf{k})b(\mathbf{k})$$
$$+ (g^2/4\pi)\int\int d^3r\,d^3r'\rho(\mathbf{r})\rho(\mathbf{r}')\frac{\exp(-\mu|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{8}$$

Thus the net effect of the Yukawa coupling in this case is to introduce a *repulsive* interaction between the

sources; if we had used a theory with the usual commutation relations and consequently a positive definite metric, the transformed Hamiltonian would have looked the same except that one would have had an *attractive* Yukawa interaction between sources of the same kind. This change in sign is accompanied by the still indefinite metric $\exp[i\pi\int d^3k\, b^\dagger(\mathbf{k})b(\mathbf{k})]$. Again, because of the simple structure of the theory, one could introduce the supplementary condition:

$$\left\{1+\exp\left[i\pi\int d^3k\, b^\dagger(\mathbf{k})b(\mathbf{k})\right]\right\}\psi = 0. \tag{9}$$

This simple example also illustrates for us that "local" interactions and "nonlocal" interactions are not necessarily physically different. It was also necessary in the case of a repulsive potential to introduce "local" interaction structure only by the artifice of introducing fields with indefinite metric.

In this theory there is no interaction between the "mesons" and the sources as seen explicitly by the transformation. Because of the simple structure of this theory it is possible to introduce any required potential between *static* sources; one need only express the potential as the superposition of Yukawa potentials and use the square root of the "strength" as the coupling constant. For attractive Yukawa components one has fields with "normal" commutation relations, and for repulsive components one has fields with "abnormal" commutation relations.

A particularly interesting example is provided by the exponential potential (either attractive or repulsive); no contact interaction theory with positive definite metric can give such a potential because the asymptotic form is given by the "exchange of the quantum of lowest mass." With indefinite metric this can be considered as a dipole field case; namely, the limiting case of the sources being coupled to a normal field and an abnormal field (possessing indefinite metric) of nearly equal masses and equal strengths.

4. THE LEE OSCILLATOR

We now consider another dynamical system with an infinite number of states which cannot be solved completely, but a partial diagonalization reduces it to the special finite-dimensional case discussed in Sec. 2. For this purpose we introduce the normal oscillator variables $a^\dagger$, a, $c^\dagger$, c and abnormal oscillator variables $b^\dagger$, b which satisfy the relations

$$[a,a^\dagger] = [c,c^\dagger] = +1; \quad [b,b^\dagger] = -1; \tag{10}$$
$$\text{all other commutators} = 0.$$

The total Hamiltonian is

$$H = -mb^\dagger b + \omega c^\dagger c + g(b^\dagger ac + a^\dagger c^\dagger b), \tag{11}$$

where pseudo-Hermiticity requires that

$$m^* = m; \quad \omega^* = \omega; \quad g^* = g.$$

There are two conserved quantities:

$$N_1 = a^\dagger a - b^\dagger b; \quad N_2 = -b^\dagger b + c^\dagger c, \tag{12}$$

which take on nonnegative integral values; one easily verifies that these two operators commute with the Hamiltonian and with each other and can hence be simultaneously diagonalized with the energy. In terms of these operators, we can simplify the Hamiltonian to the form

$$H = mN_2 + \omega' c^\dagger c + g(b^\dagger ac + a^\dagger c^\dagger b),$$

with $\omega' = \omega - m$. A complete basis is constituted by the eigenstates of the number operators labeled by the eigenvalues of the 3 number operators in the form $|n_a, n_b, n_c\rangle$ with the metric $(-1)^{n_b}$.

We now observe that there are precisely $1 + \min(n_1, n_2)$ such states corresponding to the eigenvalues n_1 and n_2 of the number operators: hence for every choice of n_1 and n_2 the eigenstate of the coupled system can only be linear superpositions of a finite number of states and the problem of finding the exact states of the coupled system reduces to an eigenvalue problem for a pseudo-Hermitian finite-dimensional matrix. The associated finite-dimensional subspace will be called the "sector" (n_1, n_2).

The simplest sectors are those where $\min(n_1, n_2) = 0$; these are eigenstates of the exact Hamiltonian and correspond to the states $|n_a, 0, 0\rangle$ and $|0, 0, n_c\rangle$. All these states have positive norm. The next simplest corresponds to $\min(n_1, n_2) = 1$ and the sectors are two-dimensional; and is constituted by the states $|n_a - 1, 1, 0\rangle$ and $|n_a, 0, 1\rangle$ and by the states $|0, 1, n_c - 1\rangle$ and $|1, 0, n_c\rangle$. In either case the pairs of states have opposite norm. The eigenvalue problem reduces to the quadratic equations

$$\det\begin{pmatrix} m - E & +g(n_a)^{\frac{1}{2}} \\ -g(n_a)^{\frac{1}{2}} & \omega - E \end{pmatrix} = 0;$$

$$\det\begin{pmatrix} n_c\omega + m - E & +g(n_c)^{\frac{1}{2}} \\ -g(n_c)^{\frac{1}{2}} & (n_c + 1)\omega - E \end{pmatrix} = 0. \tag{13}$$

The conditions for real roots are

$$(m + \omega)^2 - 4(m\omega + ng^2) \geqslant 0,$$

with $n = n_a$ or $n = n_c$ in the two cases. Recalling that g is real, we have

$$|\omega - m| / |g| \geqslant 2\sqrt{n}, \tag{14}$$

if E is to be real. As long as the inequality is satisfied, the two eigenstates of the coupled system can be chosen to have norms $+1$ and -1, respectively. On the other hand, if the condition is not satisfied, the roots become complex and the eigenstates have vanishing norm. The only admissible states are those of positive-definite norm: and consequently we should impose the supple-

mentary condition that only these states are allowed. It is important to notice from the structure of the condition for real roots that for fixed values of the parameters m, ω, g (with $|g| \neq 0$), for a sufficiently large value of n_a or n_c, this condition is violated. Hence, if we consistently apply the supplementary conditions to select out the "physical states," only a finite number of these "sectors" are allowed; the specific number is dependent on the specific numerical values of the system parameters and is not particularly relevant.

Similar conclusions result from a study of the next set of sectors defined by $\min(n_1, n_2) = 2$. These sectors are three-dimensional and consist of the states $|n_a - 1, 2, 0\rangle$, $|n_a, 1, 1\rangle$, $|n_a + 1, 0, 2\rangle$ and the states $|0, 2, n_c - 1\rangle$, $|1, 1, n_c\rangle$, $|2, 0, n_c + 1\rangle$. The eigenvalue equations for the two cases are cubic equations:

$$\det\begin{bmatrix} 2m - E & +g(2n_a)^{\frac{1}{2}} & 0 \\ -g(2n_a)^{\frac{1}{2}} & m + \omega - E & +g(n_a + 1)^{\frac{1}{2}} \\ 0 & -g(n_a + 1)^{\frac{1}{2}} & 2\omega - E \end{bmatrix} = 0,$$

$$\det\begin{bmatrix} (n_c - 1)\omega + 2m - E & +g(2n_c)^{\frac{1}{2}} & 0 \\ -g(2n_c)^{\frac{1}{2}} & n_c\omega + m - E & +g(n_c + 1)^{\frac{1}{2}} \\ 0 & -g(n_c + 1)^{\frac{1}{2}} & (n_c + 1)\omega - E \end{bmatrix} = 0. \tag{15}$$

The condition of reality for both these equations can be seen to be the same and is equivalent to requiring that the cubic equation

$$x^3 - [\omega'^2 - (3n + 1)g^2]x - (n - 1)\omega'g^2 = 0,$$

must have 3 real roots; namely

$$4[(m - \omega)^2 - (3n + 1)g^2]^3 \geqslant 27(n - 1)(m - \omega)^2 g^4. \tag{16}$$

Again we notice that the sectors contain states of both norms; and the condition for three real roots is again dependent on n although one root is always real. For suitably large values of n the condition is always violated; and since physical states correspond to eigenstates of the coupled Hamiltonian with norm $+1$, these are only a finite fraction of such states. One can proceed in a similar fashion to construct the physical states for the various sectors; by virtue of the oscillator commutation relations it follows that the off-diagonal matrix elements will increase with n and for every nonzero value of g, there are only a certain number of states in the various sectors yielding real eigenvalues for the energy.

As long as the restriction to "physical" states is imposed by the supplementary conditions all physical energy eigenvalues are real, and correspondingly an arbitrary physical state will have its time dependence completely in accordance with the real phase changes corresponding to real energies; and all transition probabilities will be positive and no inconsistencies will arise. If we so desire we may reformulate the theory in terms of a "reduced" Hamiltonian operating on the physical states only; and one can entirely dispense with the indefinite metric and pseudo-unitary transition

matrices. But this reduction is achieved at a price: *We can no longer express the dynamics of the system in terms of a simple set of coupled oscillators.* Note that we cannot talk about arbitrary operators constructed out of the original oscillator variables as if they were *observables*; observables are those operators which have no matrix elements connecting the physical states with the unphysical states. In general, in the original oscillator representation these are also represented by pseudo-Hermitian operators.

5. PAIS-UHLENBECK OSCILLATORS

In connection with the study of a certain class of field theories, Pais and Uhlenbeck[3] were led to consider a system defined by the variational principle

$$\delta \int dt \left\{ -qF\left(\frac{\partial^2}{\partial t^2}\right)q \right\} = 0, \tag{17}$$

where $F(x)$ is a real polynomial of degree N in the variable x with $F(0)=1$. Since the equation of motion

$$F(\partial^2/\partial t^2)q = 0$$

is a linear differential equation in q with constant coefficients, it can be solved explicitly. This system can be cast into a Hamiltonian form by introducing, following Pais and Uhlenbeck, the variables:

$$q_j = \prod_{i=1}^{N}{}' \left(1 + \omega_i^{-2}\frac{\partial^2}{\partial t^2}\right)q, \tag{18}$$

where the constants ω_i are related to the decomposition of $F(x)$ in the form

$$F(x) = \prod_{i=1}^{N}(1 + \omega_i^{-2}x).$$

One can then show that this dynamical system corresponds to the Hamiltonian

$$H = \sum_{j=1}^{N} \{ (p_j^2/4\alpha_j) + (\alpha_j\omega_j^2 q_j^2) \} = \sum_{j=1}^{N} H_j,$$

where

$$\alpha_j = [\omega_j^4 F'(-\omega_j^2)]^{-1}. \tag{19}$$

If $F(x)$ is a polynomial function it follows that the quantities α_j alternate in sign. We call a quantum-mechanical system of this type (assumed to correspond to N real distinct values ω_j for convenience) a Pais-Uhlenbeck oscillator.

Since the contributions from the various H_j alternate in sign an indefinite metric is called for; the commutation relations for the p_j, q_j are

$$[p_j, q_{j'}] = i(-1)^j \delta_{jj'}, \quad [p_j, p_{j'}] = [q_j, q_{j'}] = 0. \tag{20}$$

We now introduce the oscillator variables:

$$a_j = (2\omega_j)^{-\frac{1}{2}}(p_j' + i\omega_j q_j'),$$
$$a_j{}^\dagger = (2\omega_j)^{-\frac{1}{2}}(-1)^{j+1}(p_j' - i\omega_j q_j'),$$

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with

$$p_j' = (|\alpha_j/2|)^{-\frac{1}{2}}p_j, \quad q_j' = (|\alpha_j/2|)^{\frac{1}{2}}q_j.$$

The quantization corresponds to the indefinite metric

$$\eta = \exp[i\pi\sum(j+1)a_j{}^\dagger a_j]. \tag{21}$$

With these commutation rules and the definitions of these oscillator variables in terms of the basic variable q the equation of motion is satisfied. Since the various degrees of freedom are completely independent, the eigenvalue problem is trivial. However, the supplementary condition that eigenstates of negative norms do not enter the set of physical states has to be imposed.

If we impose these conditions, it follows that neither q nor the full set of functions of all p_j and q_j are observables. Since p_j, q_j have matrix elements connecting states with opposite norm if j is even, those operators which correspond to odd functions of the p_j, q_j for even j are not observables. It is to be particularly noticed that neither q nor any of its time derivatives is an observable; the imposition of the supplementary condition thus makes it impossible to consider the basic variable q as an observable. If we so wish, we may rewrite the theory in terms of only the physical states and the indefinite metric then no longer enters the theory. However, this "reduced" Hamiltonian becomes extremely *complicated*; if one attempts to write down an operator equation of motion in terms of q in this "reduced" form, this equation is no longer a differential equation, but an integro-differential equation. Of course the formal simplicity of the primitive theory using the indefinite metric is somewhat misleading because the physical interpretation requires that one goes to the "reduced" form. Here again we see that the indefinite metric leads to formal simplification of the theory and that the "reduced" theory, while it employs only a positive-definite metric, is formally more complicated.

6. A MODEL OF QUANTIZED FIELD THEORY

We shall now construct a solvable model of an interacting quantized field that has some resemblance to a model constructed by Lee.[4] The model is defined by the Hamiltonian,

$$H = \int d^3k \{ a^\dagger(\mathbf{k})a(\mathbf{k})E_1(k) + b^\dagger(\mathbf{k})b(\mathbf{k})E_2(k)$$

$$+ c^\dagger(\mathbf{k})c(\mathbf{k})\omega_1(k) - d^\dagger(\mathbf{k})d(\mathbf{k})\omega_2(k) \}$$

$$+ \frac{1}{(4\pi)^{\frac{1}{2}}} \cdot \int\int d^3p d^3k \left(\frac{M}{E_1(k+p)}\right)^{\frac{1}{2}} \cdot \left(\frac{m}{E_2(p)}\right)^{\frac{1}{2}}$$

$$\times \left\{ a(\mathbf{p}+\mathbf{k})b^\dagger(\mathbf{p})\left(\frac{g_1 c^\dagger(\mathbf{k})}{[2\omega_1(k)]^{\frac{1}{2}}} + g_2\frac{d^\dagger(\mathbf{k})}{[2\omega_2(k)]^{\frac{1}{2}}}\right) \right.$$

$$\left. + a^\dagger(\mathbf{p}+\mathbf{k})b(\mathbf{p})\left(\frac{g_1 c(\mathbf{k})}{[2\omega_1(k)]^{\frac{1}{2}}} + \frac{g_2 d(\mathbf{k})}{[2\omega_2(k)]^{\frac{1}{2}}}\right) \right\}, \tag{22}$$

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[3] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[4] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

where the oscillator variables satisfy the commutation relations:

$$[a(\mathbf{k}),a^\dagger(k')] = [b(\mathbf{k}),b^\dagger(k')] = [c(\mathbf{k}),c^\dagger(k')]$$
$$= -[d(\mathbf{k}),d^\dagger(\mathbf{k})] = \delta(\mathbf{k}-\mathbf{k}'), \quad (23)$$

all other commutators $= 0$, and

$$E_1(k) = (M^2+k^2)^{\frac{1}{2}}; \quad E_2(k) = (m^2+k^2)^{\frac{1}{2}};$$
$$\omega_1(k) = (\mu_1^2+k^2)^{\frac{1}{2}}; \quad \omega_2(k) = (\mu_2^2+k^2)^{\frac{1}{2}}.$$

There are two constants of motion:

$$B = \int d^3k\{a^\dagger(\mathbf{k})a(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})\},$$

$$Q = \int d^3k\{a^\dagger(\mathbf{k})a(\mathbf{k})+c^\dagger(\mathbf{k})c(\mathbf{k})-d^\dagger(\mathbf{k})d(\mathbf{k})\},$$

which both have non-negative integral eigenvalues. A pair of such eigenvalues n_1,n_2 defines a "sector." The "abnormal" commutation rules obeyed by the variables $d(\mathbf{k})$, $d^\dagger(\mathbf{k})$ necessitates an indefinite metric and in the Fock representation in terms of the eigenstates of the number operators n_a, n_b, n_c, n_d, the metric is given by

$$\eta = (-1)^{n_d}.$$

As in the Lee model, here again the no-particle state and the one-particle states corresponding to the quanta b, c, d are stationary states. The first nontrivial state where genuine scattering occurs is for $n_1=n_2=1$. A detailed study of this model has been made by Schnitzer and the present author; here we shall be merely content to write down the T matrix for the sector $(1,1)$ which is obtained using known methods[5]:

$$T(E) = \begin{pmatrix} f_1^2 & +f_1f_2 \\ -f_1f_2 & -f_2^2 \end{pmatrix}\frac{e^{i\delta(E)}\sin\delta(E)}{f_1^2-f_2^2}, \quad (24)$$

where

$$e^{2i\delta(E)} = h(E-i\epsilon)/h(E+i\epsilon),$$

$$f_{1,2} = g_{1,2}[E^4-(m^2-\mu_{1,2})^2]^{-\frac{1}{2}},$$

$$h(z) = M-z+\frac{m}{8\pi}(g_1^2\alpha(\mu_1,z)-g_2^2\alpha(\mu_2,z)), \quad (25)$$

$$\alpha(\mu,z) = \int_{\mu+m}^{\infty} dx\frac{\{(x+m+\mu)(x+m-\mu)\times(x-m+\mu)(x-m-\mu)\}^{\frac{1}{2}}}{x^2(z-x)}.$$

We note that, as was to be expected, the S matrix is not unitary but only pseudo-unitary. However, its eigenvalues are unimodular for all values of the energy and the two (unrenormalized) coupling constants. The particular structure of the T matrix also shows that

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[5] Compare W. Heisenberg, Nuclear Phys. 4, 532 (1957).

only *one* state has any scattering; and this state has positive or negative square according as $f_1^2>f_2^2$ or $f_1^2<f_2^2$. (Null eigenvectors will not occur except for the case $f_1^2=f_2^2$; but in this case we can always redefine the eigenvectors to have positive and negative norms, respectively.) If we now invoke our rule and project out the eigenstates with negative norms, we will have a consistent theory of a two-particle system with only *one* channel and an energy-dependent scattering phase shift.

Let us consider the energy dependence in some detail; according to Eq. (25), for sufficiently high energy, f_1/f_2 tends to g_1/g_2 monotonically. Hence it is necessary to have $g_1^2 \geqslant g_2^2$ if we have to have some scattering in the "physical" channel at all energies. On the other hand, the integral defining (2) is divergent logarithmically, *except* for the special choice $g_1^2=g_2^2$. In this case the integral is convergent and it can be shown that the corresponding field theory is now free of *all* divergences. The cancellation of divergences comes from the fact that the high-energy contributions to the various integrals are suppressed by virtue of the comparable contributions of opposite signs from the two fields entering the original Lagrangian.

For $g_1^2=g_2^2$, the condition $f_1^2>f_2^2$ requires that

$$(m^2-\mu_1^2)^2 > (m^2-\mu_2^2)^2,$$

so that they are all satisfied by a suitable choice of the three masses m, μ_1, μ_2. This simple possibility (as well as the simple solution) are a result of the fact that the "reduced" matrix element (corresponding to a Feynman diagram with all external lines amputated) is the same for either c-field or d-field external lines; and this in turn was made possible by the specific choice of the interaction.

In a more general case, the reduced matrix element will itself depend upon the nature of the coupled quantum and the exact eigenstates of the transition matrix can be found only after a complete solution of the dynamical problem; there will in general be "scattering" in both channels but the procedure for selecting all the physical states remains unchanged. The question of nonunitary eigenvalues for the S matrix cannot now be settled without detailed investigation of the particular dynamical scheme. But as long as the eigenvalues are unimodular and distinct, there exist eigenstates with positive square which can be chosen to be physical states.

7. RELATIVISTIC QUANTUM THEORY OF INTERACTING FIELDS

On the strength of the demonstrations in the last section, we are led to ask if similar ideas can be used to construct a consistent relativistic quantum theory. We shall defer the discussion of such a theory for interacting particles to another paper but briefly consider the question of interacting fields. The present status of

Lagrangian field theories is that all such manifestly covariant theories with interaction involve divergent quantities; and while for "renormalizable" theories a graphical calculus (involving only the physical mass and coupling constant and no divergent quantities explicitly) can be formulated, the connection with the original Lagrangian is made only through meaningless divergent expressions. Also any attempt to compute at least some of the masses and coupling constants in terms of other parameters of the theory seems impossible.

The possibility of obtaining finite expressions in such a theory using cancellation of the high-energy contributions was recognized in the so-called Feynman cutoff; but this was considered as a formal procedure without physical meaning because of the negative probabilities arising out of such cutoffs.[6] But the program outlined in the previous sections shows that the negative probabilities and the resulting pseudo-unitarity of the S matrix do not prevent the selection of a suitable set of "physical" states.

We are thus led to the following program for a relativistic quantum field theory. Construct a simple manifestly covariant Lagrangian density involving an arbitrary number of "normal" fields with local couplings; with every "normal" scalar field associate an "abnormal" field with all quantum numbers the same (but corresponding to a different mass) and couple it in the Lagrangian with the *same* coupling constant. In other words, it is the sum of the normal and abnormal fields which is coupled. For a theory involving only scalar fields the resultant theory involves no infinities and can always be solved more or less using appropriate approximation techniques. The resulting "steady states" of the theory (both one-particle states and scattering states) may then be classified as "physical" or "unphysical" according as whether the square of the vector is positive definite or not. Select out the physical states and in this physical sector, the theory contains only a positive-definite metric. Note that the relativistic invariance is still preserved since the steady states of a relativistically invariant theory themselves form invariant manifolds. The one-particle states being, in general, nondegenerate, one set of them will correspond to physical states for every type of field, but this is no longer true of the two-particle and higher states[7]; nor does one obtain the simple case outlined in the last section since in general the "reduced matrix elements" do depend upon the type of the external lines to which

they are to be attached. Clearly no definite and general answer can be given regarding the nature and composition of the scattering states without a systematic study of this framework.

While the theory involves no infinities, there are still mass and coupling constant renormalizations to be performed in this theory; the important difference is that these renormalizations are now *finite* and are analytic functions of the parameters (masses and coupling constants) entering the primitive Lagrangian. The proof of renormalizability can be carried through exactly as in the conventional theory and one can, in complete analogy,[8] formulate a graphical calculus for the transition amplitudes in the theory. However, the new element entering the present theory is the introduction of the abnormal fields as dynamical fields and the selection of the "physical" states; while an exact separation requires a knowledge of the complete solution and is thus inaccessible in a realistic theory, the separation can be carried through to any order in the finite perturbation theory. Thus we have an iterative process to define the physical states, the observables and the physical transition amplitudes which can be carried through to any desired degree of approximation.

The functional relation between the primitive masses and coupling constants and the corresponding renormalized quantities is considerably more complicated, and the perturbation expansion in terms of the primitive parameters may be much less convergent.[9] But even here, the relations are analytic in the primitive parameters and consequently the physical amplitudes are analytic in the renormalized parameters. The original Lagrangian contains *two* primitive masses (rather than the one primitive mass that is usually written down); such additional constants usually appear already in current calculations, but under the guise of "subtraction constants."

For a theory involving spinor fields, these subtractions are not sufficient and this depends essentially on the fact that the free propagator decreases only as the first power of the momentum. A prescription in complete analogy to the bosons can be worked out for this more general case by requiring the effective propagator to decrease as the fourth power of the momentum and this requires that in place of every spinor field one introduces *two* normal fields and two abnormal fields. The corresponding renormalized theory would contain *two* distinct one-particle states with all quantum numbers identical but with different masses[10]; the proof

[6] See, for example, N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 395.

[7] In order that a particle interpretation may exist in the field theory it seems necessary to further restrict the allowed physical states. There are two kinds of two-particle states with positive norm, but only one of them corresponds to a state of two physical particles in the limit of no interaction; the other corresponds to a state of two unphysical particles. The requirement is the "asymptotic condition" in the present formulation and is required only if we demand a particle interpretation of this field theory.

[8] The rules of the graphical calculus have to be changed in one respect: The relative phase of the emission and absorption factors for abnormal field contains an additional minus sign; in particular internal lines in a diagram have abnormal propagators with the opposite sign from normal propagators.

[9] The author is indebted to I. Bialynicki-Birula for a discussion of this question.

[10] The existence of the electron-muon doublet is perhaps not unconnected with this circumstance; the author wishes to thank P. Cziffra for a discussion of this point.

of renormalizability is more complicated in this case and we shall not discuss it any further here.

Needless to say, this theory involving the indefinite metric is fully relativistic even in the physical sector and there must exist a reformulation of the "reduced" theory. But such a theory would involve nonlocal interactions and would be extremely complicated; and as far as computations are concerned, it is inferior to the manifestly covariant theory involving the indefinite metric.

Having thus related the problem of finding a satisfactory relativistic theory of interacting fields to the construction of manifestly covariant relativistic field theories involving both normal and abnormal fields, the problem of a dynamical theory of elementary particles can now be considered.[11] One is led to consider a theory involving a minimum of fields required to "support" all the conserved additive quantum numbers and a "simple" dynamical scheme. The scheme presented above does not of course given any immediate solution of nonlinear dynamics, but the problem can at least be formulated in a consistent fashion.

8. COMPARISON WITH THEORY OF BOGOLIUBOV

A somewhat similar procedure was outlined by Bogoliubov[12,13] some time ago; to exhibit the difference between this procedure and the one proposed in the present paper let us crystallize our essential computational procedure in the following form (compare Sec. 7). Let $\mathcal{3C}$ be the primitive generalized Hilbert space with indefinite metric (in which the quantized fields are linear operators) underlying the formulation. Denote by Λ the projection operator to the manifold of $\mathcal{3C}$ spanned by those elements of $\mathcal{3C}$ which have positive norm and which are eigenstates of the scattering operator. Then the "physical states" are to be made to corresponds to the elements of the "reduced space" $\mathcal{3C}_0 = \Lambda\mathcal{3C}$. The physical S matrix is

$$S_0 = \Lambda S \Lambda. \qquad (26)$$

We then note that S_0 is unitary in the reduced space $\mathcal{3C}_0$ since

$$S_0{}^\dagger S_0 = S_0 S_0{}^\dagger = \Lambda, \qquad (27)$$

and the space $\mathcal{3C}_0$ is a true Hilbert space with positive definite norm (by construction). And the "observables" $\mathcal{O}$ are pseudo-Hermitian operators which commute with the projection Λ:

$$\Lambda\mathcal{O}(1-\Lambda) = 0; \quad \Lambda^\dagger = \Lambda. \qquad (28)$$

For these operators the restriction to the reduced

space $\mathcal{3C}_0$ gives a Hermitian operator $\mathcal{O}_0$:

$$\mathcal{O}_0{}^\dagger = \mathcal{O}_0; \quad \mathcal{O}_0 = \Lambda\mathcal{O}\Lambda = \mathcal{O}\Lambda. \qquad (29)$$

In view of these the reduced theory has both kinematics and dynamics which can be eventually formulated in the reduced space $\mathcal{3C}_0$ without reference to the space $\mathcal{3C}$ consistent with our view of the use of the indefinite metric as only an auxiliary [compare Eqs. (3), (9), (24), (25) as well as the subsequent paper[14]].

Bogoliubov also starts with a generalized Hilbert space $\mathcal{3C}$ and decomposes it into two orthogonal subspaces $\mathcal{3C}_1$ and $\mathcal{3C}_2$ using a projection operator P such that $\mathcal{3C}_1$ is a true Hilbert space with positive definite metric. If ψ is any vector in $\mathcal{3C}$, the projections of ψ in $\mathcal{3C}_1$ and $\mathcal{3C}_2$ are given by

$$\psi_1 = P\psi; \quad \psi_2 = (1-P)\psi.$$

If S is the generalized (pseudo-unitary) scattering matrix in the space $\mathcal{3C}$ the projection P satisfies, according to Bogoliubov's criterion, the requirement:

$$(1-P)\psi + (1-P)S\psi = 0. \qquad (30)$$

Using the definition of the "physical part,"

$$\psi_{\mathrm{phys}} = P\psi = \psi_1, \qquad (31)$$

we note that the Bogoliubov criterion yields

$$(S\psi)_{\mathrm{phys}} = S_{\mathrm{phys}}\psi_{\mathrm{phys}}, \qquad (32a)$$

with

$$S_{\mathrm{phys}} = P\{S^{-1} + (1-P)\}^{-1}P. \qquad (32b)$$

It is then asserted that "the norm of the physical part of the state vector is conserved and so are the mean values of energy, momenta, etc., calculated (by means of the physical part of the state vector."

However, in contrast to the physical S matrix S_0 (defined above), the matrix S_{phys} is *not unitary;* the conservation of the norm of the physical part was true of only specially selected vectors and in particular, is not true for any vector in $\mathcal{3C}_1$. The reason is not far to seek: one has used a "double standard" working with two distinct definitions for the "norm," one defined over vectors in $\mathcal{3C}$ and the second defined only for vectors in $\mathcal{3C}_1$. Presumably, one attaches probability interpretation to the metric in $\mathcal{3C}_1$ since only this is positive definite but in that case the restricted matrix S_{phys} is not unitary; this lack of unitarity has been explicitly demonstrated in a special solvable case by Glaser[15]; the above discussion shows that this is to be generally expected.

One might enquire under what conditions the matrix S_{phys} is unitary in the subspace $\mathcal{3C}_1$. This is equivalent to the requirement [compare Eq. (27)];

$$S_{\mathrm{phys}}{}^\dagger \cdot S_{\mathrm{phys}} = P,$$

[11] It is thus not surprising to find that several attempts at a manifestly covariant dynamical theory of elementary particles have arrived at an indefinite metric, starting with very different points of view.

[12] N. N. Bogoliubov, *1958 Annual International Conference on High-Energy Physics at CERN* (CERN Scientific Information Service, Geneva, 1958), p. 129.

[13] See, also, reference 5.

[14] H. J. Schnitzer and E. C. G. Sudarshan, following paper [Phys. Rev. **123**, 2193 (1961)].

[15] V. Glaser, reference 12, p. 130.

but from the explicit expression for S_{phys} and the requirement that the metric in $\mathcal{K}_1$ is positive definite it follows that this will only be satisfied when

$$PS(1-P)=0,$$

i.e., the projection P commutes with the primitive S matrix (defined over $\mathcal{K}$). But in that case the "unphysical parts" vanish and Bogoliubov's criterion is irrelevant.

In conclusion, it must be stressed that the restriction of the primitive generalized Hilbert space to the "physical space" should be done in such a manner that the vectors in this space can be chosen arbitrarily and that all conditions imposed on the scattering matrix can be automatically satisfied; this requirement, essential for the physical relevance of the theory, is satisfied in our theory but not in Bogoliubov's and hence leads us to reject the latter.

9. DISCUSSION

The investigations of the previous sections dealt with widely different systems which involve an indefinite metric; most of these are admittedly simple models, the outstanding exception being the quantized radiation field (briefly discussed in the Appendix). There are some common features of these theories:

(i) The "physical" states must include only eigenstates of the true Hamiltonian which can be normalized to $+1$.

(ii) Only those operators are observables which have no matrix elements connecting physical and unphysical states.

(iii) A "reduced" theory involving only physical states can be constructed which involves only a Hilbert space (with positive definite metric) and no supplementary conditions; but the Hamiltonian of such a theory may be much more complicated than the formal structure of primitive Hamiltonian.

These features demand that not necessarily all heuristic notions associated with theories with indefinite metric may be justified; in particular the nature of asymptotic *physical* states cannot be decided *a priori*.

It has become fashionable to consider that relativistic quantum theory of elementary particles should be formulated in terms of "local fields," and to assert that Lagrangian theories are inconsistent since all attempts at constructing consistent theories with simple interaction structures had failed. All attempts at calculations with "local Lagrangians" (in theories with a positive definite metric) automatically yielded infinities; while manifestly covariant nonlocal theories were "white elephants" with which one could do nothing (not even obtain an infinity). In any case it is difficult to proceed to construct a theory unless it has a simple structure; for simple covariant interaction schemes one could obtain finite results in perturbation theory if one

introduced an indefinite metric, but this appeared to make the theory contain "negative probabilities" and doubts about its consistency are often expressed.

If it were not for preserving relativistic invariance a consistent theory could be constructed using suitable form factors. The theory need then involve no indefinite metric and Lagrangian theories need not be ruled out. There is still the problem of the actual details of computation of physical predictions; but more important is the fact that there is too much ambiguity in choosing the form factors. On the other hand, in general, such a theory will not be relativistically invariant; the special case of the Dirac-Schwinger form of electrodynamics suggests that there may be other field theories with nonlocal interactions with the form factors suitably chosen. However such theories are formally greatly complicated and in any case there is no method of constructing other such theories.

The appeal to "simple" interaction structures is most useful in a theory of elementary particles which aims at predicting at least some of the masses and coupling constants. The study in the previous sections suggest that the use of contact interactions along with the indefinite metric, involving as it does only a few constants, is an alternative "primitive" theory involving supplementary conditions, the corresponding "reduced" theory being a consistent theory with a positive-definite metric but involving a complicated interaction structure. If nontrivial theories of this type exist, the corresponding "reduced" theory also will be relativistically invariant; there is nothing that is known at present which shows that such a possibility is unlikely. The requirement of manifest covariance and the requirement of restriction to positive definite metric may thus be complementary[16]; and this feature may be typical of all relativistic field theories. This gives additional justification for attempts to construct theories of elementary particles with regularized propagators as has been done by Heisenberg and collaborators[17] and by Nambu.[17]

Irrespective of any indefinite metric, in a local Lagrangian theory one has differential equations satisfied by the field operators, which (by nature of the Lorentz space) are hyperbolic equations and hence they have the propagation character and characteristic cones typical of hyperbolic equations. Hence in any such theory "local commutation rules" may hold, though one cannot interpret this condition to say that "physical measurements" at two space-like points commute; in general the field operators are *not measurable* (even spread out over small regions) as was seen in the special case of the Pais-Uhlenbeck oscillator. This also implies

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[16] The author is indebted to J. Schwinger for several comments on this point of view.

[17] *Proceedings of the 1960 Annual Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), Session P3.

that the success of the covariant self-consistent method[18] of treating strong interactions which makes use of analytic properties of transition amplitudes suggested by local commutation rules is not inconsistent with the existence of a Lagrangian theory of strong interactions. In fact, some of the approximations employed in these calculations are most easily understood within a framework involving an indefinite metric. We are thus led to suggest that the principle of simplicity can be reinstated in a consistent relativistic field theory of elementary particles by the formal introduction of an indefinite metric.

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APPENDIX. INDEFINITE METRIC IN QUANTUM ELECTRODYNAMICS

The various results found in the text have their counterpart in the treatment of the radiation field in interaction. The conventional treatment involves an indefinite metric together with a supplementary condition; but the theory can be reformulated in terms of physical states and observables with an apparent increase in the complexity of the formal structure.

The Gupta-Bleuler formalism of the radiation field coupled to a classical charge current density starts with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu A_\nu \partial^\mu A_\nu + j^\mu A_\mu,$$

and the commutation relations

$$[A^\mu(x), A^\nu(x')] = -ig^{\mu\nu} D(x-x'),$$

where $g^{\mu\nu}$ is the (space-like) metric tensor and $D(x,x')$ is the odd invariant function. These commutation relations are similar to the corresponding commutation relations for relativistic scalar fields (of zero mass) except that for $\mu=\nu=0$ the right-hand side has the opposite sign. One can construct an explicit representation of the equal time field operators by going to the momentum space and introducing four kinds of creation and annihilation operators (labeled by a momentum index) associated with the four values of the index μ, which satisfy the commutation relations

$$[a_\mu(\mathbf{k}), a_\nu{}^\dagger(\mathbf{k}')] = g_{\mu\nu}\delta(\mathbf{k}-\mathbf{k}'), \quad [a_\mu, a_\nu] = [a_\mu{}^\dagger, a_\nu{}^\dagger] = 0.$$

The opposite sign of the right-hand side for $\mu=\nu=0$ necessitates an indefinite metric

$$\eta = \exp\left[i\pi \int d^3k\, a_0{}^\dagger(\mathbf{k}) a_0(\mathbf{k}) \right].$$

To make the theory completely equivalent to the Maxwell equations it is necessary to impose the

subsidiary condition,

$$\partial_\mu A^{\mu(+)}(x) \mid \cdots \rangle = 0,$$

for any physical state. By virtue of the equations of motion, one verifies that this supplementary condition is consistent with the equations of motion provided that the current j^μ is locally conserved. The Lorentz invariance and consistency of this formulation has been discussed by various authors.

We now note two special features of this theory: firstly, the supplementary condition does not eliminate all eigenstates of the total Hamiltonian with negative norm or zero norm. In particular, for a noninteracting radiation field it allows states with odd "time-like" photons provided they are accompanied by "longitudinal" photons degenerate in momentum with them. These states have vanishing expectation values for the energy or momentum of the field. A closer analysis shows that the amplitudes of such states is completely nonmeasurable. In other words, these components of an admissible state vector are completely irrelevant as far as all physical predictions are concerned. It is then desirable to choose this field to be unquantized and absent. Secondly, the observables of the radiation field are not arbitrary functionals of the field operators, but only such functionals as commute with the supplementary condition. Thus the simple formal structure is again misleading as not all quantities that enter are physically measurable. We also note the related fact that not all interactions of the quantized field with other dynamical entities is allowed, but only those which are "gauge invariant." Within such a framework this theory with indefinite metric has the same degree of consistency as, say, a scalar field with a positive-definite metric.

The considerations of the previous sections suggest that there should exist a "reduced" theory involving only physical states, a definite metric, and no supplementary conditions but which may look more complicated. This is accomplished by the Dirac-Schwinger formulation which starts with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu) F_{\mu\nu} - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu.$$

There are ten equations of motion for the ten components $F_{\mu\nu}$, A_μ; separating out the defining equations and constraints, one finds the true equations of motion:

$$\partial^0 \mathbf{A}^T = \mathbf{E}^T, \quad \partial_0 \mathbf{E}^T = -\nabla^2 \mathbf{A}^T,$$

where $\mathbf{A}^T$, $\mathbf{E}^T$ are the transverse (i.e., divergence-free) vector potential $\mathbf{A}$ and the electric field $\mathbf{E}$, respectively. The scalar potential A^0 and the longitudinal part of the vector potential A^L are not completely defined but are only related by the requirement that

$$\nabla \cdot \mathbf{E}^L = j^0; \quad \mathbf{E}^L = -\partial^0 \mathbf{A}^L - \nabla A^0.$$

The transverse vector fields contain only two components (instead of three), each of which are linearly

[18] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289 (unpublished).

independent. (This is seen most clearly in the momentum representation where the transversality condition becomes an algebraic constraint.) In terms of these the "reduced" Hamiltonian may be written in the form

$$H = \int d^3x \{ \tfrac{1}{2} [\mathbf{E}^T(\mathbf{x}) \overset{\leftrightarrow}{\partial^0} \mathbf{A}^T(\mathbf{x}) + \mathbf{E}^T(\mathbf{x}) \cdot \mathbf{E}^T(\mathbf{x})$$

$$- \mathbf{A}^T(\mathbf{x}) \cdot \nabla^2 \mathbf{A}^T(\mathbf{x})] + \mathbf{j}(\mathbf{x}) \cdot \mathbf{A}^T(\mathbf{x}) \}$$

$$+ \int \int d^3x d^3x' \times \frac{1}{2} \frac{j^0(\mathbf{x}) j^0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$

Note the nonlocal interaction corresponding to the (instantaneous, nonretarded) Coulomb interaction between the sources. The apparent noncovariance of the theory is irrelevant and it can be shown that this theory is relativistically invariant. The explicit representation of the field operators can be obtained by proceeding to the momentum representation in which the operators

satisfy the commutation relations:

$$[\bar{E}_j(\mathbf{k}), \tilde{A}_{j'}(\mathbf{k}')] = - i \delta(\mathbf{k} - \mathbf{k}') \left(\delta_{jj'} - \frac{k_j k_{j'}}{\mathbf{k}^2} \right).$$

It is now straightforward to introduce the creation and destruction operators for "photons" with arbitrary momentum $\mathbf{k}$ and left- or right-circular polarizations.

The theory thus formulated does not contain either supplementary conditions or an indefinite metric; all the states entering in the formalism are physical states and all Hermitian operators are observables. However, this "reduced" theory is formally much more complicated; it is particularly interesting to note that there is now an instantaneous "action at a distance" which is consistent with relativistic invariance; and the true observables of the electromagnetic field, namely, the transverse field operators (and their functionals), are not "localizable" since transversality is a nonlocal condition.

Quantum Mechanical Systems with Indefinite Metric. II*

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Several simple models, similar to that of Lee, involving indefinite metric are studied in this paper. In this connection, a dispersion-theoretic treatment is applied to a simple "equal-mass" model. It is shown that, at least for these models, the scattering amplitude is analytic in the upper-half energy plane provided time-reversal invariance holds; the rules of the dispersion-theoretic formulation in the case of an indefinite metric theory are given. The solution is reinterpreted as the exact solution of a slightly different model, which can also be obtained by Hamiltonian techniques; further techniques are generalized to include recoil in a relativistic no-pair model. Certain basic questions of interpretation are discussed in some detail in the concluding section.

1. INTRODUCTION

IN the preceding paper[1] it had been suggested that in a truly dynamical theory of quantized fields the principle of simplicity could be reinstated and a consistent theory formulated by the formal introduction of an indefinite metric. The systems discussed in that section were very simple and the important problem of interacting particles and the structure of the scattering amplitudes was not discussed in detail. Nor was it shown how the interpretive postulate restricting "physical" states to the subspace spanned by the eigenstates of the S matrix with positive definite norm could be reconciled with certain intuitive notions regarding asymptotic bare particle amplitudes, par-

ticularly in view of some recent discussions in the literature[2] regarding the lack of a consistent physical interpretation for such theories. This paper attempts to remedy these shortcomings and, in this sense, is to be considered as a sequel to the preceding paper. We choose for discussion certain models patterned after a simple example considered by Lee.[3] In the course of this analysis we formulate the rules for applying dispersion-theoretic techniques to a theory involving an indefinite metric. We also analyze, in the framework of this model, the construction of physical particle variables and physical configuration amplitudes.

In Sec. 2 we develop the dispersion-theoretic techniques to solve for the scattering amplitude in theories with an indefinite metric; and these are

[1] E. C. G. Sudarshan, preceding paper [Phys. Rev. **123**, 2183 (1961)].

[2] G. Kallen and W. Pauli, Kgl. Danske Videnskab Selskab, Mat-fys. Medd. **30**, No. 7 (1955); G. Barton, Nuovo cimento **17**, 864 (1960).
[3] T. D. Lee, Phys. Rev. **95**, 1329 (1954).