Exact Electric Analogy to the Vernotte Hypothesis

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The Fourier equation of heat conduction predicts the paradoxical result that the effect of a thermal impulse in an infinite medium will be felt instantaneously in all parts of the medium. In other words, a thermal impulse is propagated at infinite velocity. The result is paradoxical because it is incompatible with a dynamic interpretation of the mechanism of heat transfer in solids. In order to avoid this apparent paradox, Vernotte has proposed a modification of the Fourier hypothesis. This modification results in the transformation of the equation of heat conduction from a parabolic to a hyperbolic differential equation predicting finite velocity of propagation of thermal impulses. Vernotte's proposal is shown here to have an exact electric analogy. The proposal is equivalent to postulating the existence of a heat transfer quantity which is analogous to the electric quantity inductance. The transformed equation of heat transfer is therefore analogous to the differential equation of telegraphy.

INTRODUCTION

IT it well known that the Fourier equation of heat conduction predicts infinite velocity of propagation of thermal impulses. This has usually been reconciled by reference to the statistical nature of the solutions. This paper shows that a recent revision of the Fourier hypothesis which eliminates this apparent paradox completes the analogy between electricity and heat transfer.

DISCUSSION

The general partial differential equation of second order,

$$A'\varphi_{xx}+2B'\varphi_{xt}+C'\varphi_{tt}=D', \tag{1}$$

where the subscripts indicate partial derivatives, has physical characteristics whose differential equation is

$$dt/dx=[B'\pm(B'^2-A'C')^{\frac{1}{2}}]/A', \tag{2}$$

where A', B', C', and D' are, in general, functions of φ_x, φ_t, x, and t. When one of the independent variables represents distance (x) and the other time (t), Eq. (2) has the units of a reciprocal velocity. The velocity in question can be shown to be the velocity of propagation of an impulse along the characteristics. Designating this velocity as v_0, it is found for the Fourier equation of heat conduction in one dimension,

$$\frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}, \tag{3}$$

that v_0 is infinite. (Here T=temperature, k=thermal conductivity, ρ=density, and c=specific heat.) In other words, the effect of a thermal impulse in an infinite medium is felt instantaneously in all parts of the medium.

This result is well known and has usually been explained by attributing a statistical nature to the propagation of thermal impulses. But a physical interpretation requires the mechanism of heat transfer to be of a dynamical nature, thus excluding an infinite velocity.

The most recent attempt to eliminate this apparent paradox is due to Vernotte.[1] He argues that the Fourier hypothesis incorrectly predicts that the establishment of a temperature gradient occurs instantaneously. He concludes that the gradient should depend on the rate of change of heat flux and assumes a linear relationship between the two. The resulting differential equation of heat conduction predicts finite velocity of propagation of thermal impulses.

This revision of Fourier's hypothesis has completed the analogy between electricity and heat transfer. Previously no heat transfer quantity had been proposed which would be similar to the electrical quantity inductance. But Vernotte's hypothesis completes the analogy by introducing a new quantity which makes the differential equation of heat conduction of exactly the same form as the differential equation of telegraphy. To describe this in mathematical terms, it will first be shown that when inductance is neglected the equations of telegraphy are of exactly the same form as the Fourier equations of heat conduction.

Considering a noninductive transmission line along which no leakage occurs, Ohm's hypothesis can be written

$$-\partial V/\partial x=Ri, \tag{4}$$

in terms of properties per unit length of the conductor. (R is the uniformly distributed electrical resistance, V is the electrical potential and i the electric current.) This is an exactly analogous form to the Fourier hypothesis. The expression of conservation of charge for the line is

$$-\frac{\partial i}{\partial x}=C\frac{\partial V}{\partial t}, \tag{5}$$

where C is the uniformly distributed electrical capacitance. This is exactly analogous to the expression of conservation of heat energy. Eliminating i between these two equations, the resulting differential equation of telegraphy is

$$\frac{1}{RC}\frac{\partial^2 V}{\partial x^2}=\frac{\partial V}{\partial t}. \tag{6}$$

[1] P. Vernotte, Compt. rend. **246**, 3154 (1958).

This is an exactly analogous form to the Fourier equation of heat conduction. The equation predicts infinite velocity of propagation of electrical impulses. Since this prediction is obviously in error the discrepancy must lie in Ohm's hypothesis where the inductance of the line has been neglected. If its effect is included, Eq. (4) is modified as follows:

$$-\frac{\partial V}{\partial x}=Ri+L\frac{\partial i}{\partial t}, \tag{7}$$

where L is the uniformly distributed electrical inductance. Since sources and sinks have been excluded, Eq. (5) remains unchanged and elimination of i between Eqs. (5) and (7) results in the correct differential equation of telegraphy;

$$\frac{\partial^2 V}{\partial x^2}-LC\frac{\partial^2 V}{\partial t^2}-RC\frac{\partial V}{\partial t}=0. \tag{8}$$

Equation (7) is exactly analogous to Vernotte's hypothesis. Considering, as usual, T, $\dot{Q}$ (heat flux), $1/k$, and ρc to be analogous to V, i, R, and C, respectively, and introducing a new constant τ, analogous to L/R and having the units of time, the heat transfer equation analogous to Eq. (7) would be

$$-\frac{\partial T}{\partial x}=\dot{Q}+\tau\frac{\partial \dot{Q}}{\partial t}. \tag{9}$$

This is the revised form of the Fourier hypothesis presented by Vernotte. When $\dot{Q}$ is eliminated between Eq. (9) and the expression of conservation of heat energy analogous to Eq. (5), a new differential equation of heat conduction arises;

$$\frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2}-\tau\frac{\partial^2 T}{\partial t^2}-\frac{\partial T}{\partial t}=0. \tag{10}$$

This is exactly analogous to Eq. (8). It is easily shown that i, as well as V, satisfies Eq. (8) and, analogously, that $\dot{Q}$ satisfies Eq. (10). Substituting the values of A', B', and C' in Eq. (2), it is found that

$$V_0=(k/\tau\rho c)^{\frac{1}{2}}, \tag{11}$$

and the velocity of propagation of thermal impulses for this case is therefore finite. That this velocity is the maximum velocity of propagation is most easily shown by the following example. When the surface temperature of a semi-infinite slab is allowed to vary sinusoidally, a solution of Eq. (10) for the temperature distribution in the slab is

$$\theta=\theta_m \exp\left(-\frac{\xi}{\sqrt{2}}x\right)\cos\left(\omega t-\frac{\eta}{\sqrt{2}}x\right), \tag{12}$$

where $\theta=T-T_r$ (T_r being a reference temperature), $\theta_m-T_m-T_r$ (T_m being the maximum temperature), ω is the angular frequency of temperature variation, and

$$\xi=\left\{\left[\frac{\omega^4}{v_0^4}+\left(\frac{\omega\rho c}{k}\right)^2\right]^{\frac{1}{2}}-\frac{\omega^2}{v_0^2}\right\}^{\frac{1}{2}}, \tag{13}$$

$$\eta=\left\{\left[\frac{\omega^4}{v_0^4}+\left(\frac{\omega\rho c}{k}\right)^2\right]^{\frac{1}{2}}+\frac{\omega^2}{v_0^2}\right\}^{\frac{1}{2}}. \tag{14}$$

Equation (12) represents a temperature wave moving into the slab with the velocity

$$v=\sqrt{2}\frac{\omega}{\eta}=\left\{2\bigg/\left[\frac{1}{v_0^4}+\left(\frac{\rho c}{k\omega}\right)^2\right]^{\frac{1}{2}}+\frac{1}{v_0^2}\right\}^{\frac{1}{2}}. \tag{15}$$

Since all the properties of the material have been assumed to be independent of temperature, then for a slab of given material, v is a function of ω only. It therefore assumes its minimum value ($v=0$) when $\omega=0$, which corresponds to no temperature change at the surface, and its maximum value ($v=v_0$) when $\omega=\infty$, which corresponds to an impulse.

The equations presented so far are applicable only to one-dimensional heat conduction without internal heat generation. However, they are easily extended to the three-dimensional case and to include heat sources and sinks. When this is done, the general differential equation of heat conduction becomes

$$\frac{k}{\rho c}\nabla^2 T=\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}-\dot{Q}'''-\tau\frac{\partial \dot{Q}'''}{\partial t}, \tag{16}$$

where $\dot{Q}'''$ is the internal heat generation per unit volume. When heat losses (or gains) are proportional to the temperature such that $\dot{Q}'''$ can be written

$$\dot{Q}'''=-\alpha T, \tag{17}$$

then Eq. (16) becomes

$$\nabla^2 T=\frac{\rho c\tau}{k}\frac{\partial^2 T}{\partial t^2}+\left\{\frac{\rho c}{k}+\frac{\tau}{k}\alpha\right\}\frac{\partial T}{\partial t}+\frac{\alpha}{k}T, \tag{18}$$

which is an exactly analogous form to the general differential equation of telegraphy with α analogous to the leakage conductance.

Although Vernotte's proposed revision of Fourier's hypothesis adequately circumvents the paradox of infinite velocity, no apparent physical justification can be offered for the addition of the second term on the right side of Eq. (9). Each of the heat transfer quantities k and ρc can be shown to have physical justification by arguments similar to those which justify the existence of the electrical quantities R and C. But no argument which justifies the existence of L can be used as a basis in explaining the existence of the quantity τ. Whether or not τ actually exists and is of appreciable magnitude can be determined only by future experiment and further research into the mechanism of heat transfer in solids.