ϕ'. Also, because the cone is narrow, $|q'| \approx q_z'$,

$$\dot{Q}_R = \frac{c_1}{(2\pi)^3} \int\int\int \sum_{\mathbf{k}} \gamma_{\mathbf{k}}^2 \hbar\omega_q [\mathfrak{N}_q(T_1) - \mathfrak{N}_q(T_2)] \times f^2(\theta') \cos\theta \sin\theta' d\theta' d\phi' dq_z, \quad (2.8)$$

$$\dot{Q}_R = \frac{c_1}{(2\pi)^3} \int \sum_{\mathbf{k}} \gamma_k^2 \hbar\omega_q [\mathfrak{N}_q(T_1) - \mathfrak{N}_q(T_2)] \cos\theta q_z^2 dq_z \times \int f^2(\theta') \sin\theta' d\theta' \int d\phi', \quad (2.9)$$

$$q_z = (q^2 - k^2)^{\frac{1}{2}} = q[1-(k/q)^2]^{\frac{1}{2}} = q\cos\theta, \quad (2.10)$$

$$\dot{Q}_R = \frac{c_1}{(2\pi)^3} \int \sum_{\mathbf{k}} \gamma_k^2 \hbar\omega_q [\mathfrak{N}_q(T_1) - \mathfrak{N}_q(T_2)] \times \left[1 - \left(\frac{k}{q}\right)^2\right]^{\frac{3}{2}} q^2 dq \int f^2(\theta') \sin\theta' d\theta' \int d\phi'. \quad (2.11)$$

If the surface were smooth, all γ_k's would be zero except for γ_0. The difference between the rough and the smooth surface lies only in the factor

$$\gamma_{\mathbf{k}}^2[1-(k/q)^2]^{\frac{3}{2}},$$

which is necessarily less than γ_k^2 for any finite value of k. Consequently,

$$\sum_{\mathbf{k}} \gamma_{\mathbf{k}}^2[1-(k/q)^2] \leqslant \sum_{\mathbf{k}} \gamma_{\mathbf{k}}^2 \leqslant 1. \quad (2.12)$$

The equality sign occurs for a truly smooth surface. Hence

$$\dot{Q}_R < \dot{Q}_s.$$

Thus we have shown that the heat flow across the solid-helium interface is necessarily smaller for a rough surface than for a smooth surface. This result is not to be confused with the increased heat flow which occurs when a surface is roughened due to the increase of the macroscopic area of the surface. The distinction between macroscopic roughening and microscopic roughening being given by the mean free path of the phonon. This effect has been discussed previously.[2]

ACKNOWLEDGMENT

I wish to thank Professor Felix Bloch for two stimulating discussions: one which raised the question of the roughness of the surface, and one which contained most of the basis of the proof that it was not important for an understanding of the Kapitza resistance.

PHYSICAL REVIEW VOLUME 123, NUMBER 6 SEPTEMBER 15, 1961

Generalized Bardeen-Cooper-Schrieffer States and the Proposed Low-Temperature Phase of Liquid He^3

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(Received May 15, 1961)

Particle interactions in a Fermi gas may be such as to attract pairs near the Fermi surface more strongly in $l=1, 2, 3$ or higher states than in the simple spherically symmetrical s state. In that case the Bardeen-Cooper-Schrieffer condensed state must be generalized, and the resulting state is an anisotropic superfluid. We have studied the properties of this type of state in considerable detail, especially for $l=1$ and 2. We have derived expressions for the energy, the moment of inertia, the magnetic susceptibility and the specific heat. We also derive the density correlation function and the density-current density correlation; in some cases the latter implies that the liquid has net surface currents and a net orbital angular momentum. The ground state for $l=2$ is different from those previously considered, and has cubic symmetry and no net angular momentum. A general method for replacing the possibly rather complicated potential by a simple scattering matrix is given. A brief discussion of possible collective effects is included. We apply our results to liquid He^3; after correction for scattering by a method due to Suhl, it is found that the predicted transition should take place below 0.02°K. Other possible applications are suggested.

I. INTRODUCTION

SINCE the publication of the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity,[1] attempts have been made to extend their method to describe possible condensations of other interacting fermion systems, particularly liquid helium-3. It has been recently observed by several authors[2–4] that the problem of determining the ground state of a fermion

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] J. C. Fisher (private communication); D. J. Thouless, Ann. Phys. **10**, 553 (1960).

[3] K. A. Brueckner, T. Soda, P. W. Anderson, and P. Morel, Phys. Rev. **118**, 1442 (1960).

[4] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 43 (1960).

assembly with attractive forces has solutions which correspond to condensing the fermion pairs into $l=1, 2, 3\cdots$ states rather than into the spherically symmetric s state of BCS theory. Brueckner *et al.*[3] first pointed out that the forces in liquid helium-3 are favorable to a condensation into an $l=2$ state at a temperature of the order of 0.07°K. This situation is a consequence of the strongly repulsive short-range part of the He^3-He^3 interaction potential which dominates the first few terms of its expansion in terms of spherical harmonics, thus preventing a condensation in the $l=0$ or 1 configurations. In higher order, however, (beginning with $l=2$) the long-range attractive part contributes more than the repulsive core so that the balance is attractive and a condensation is possible. It is not inconceivable that a similar situation could occur in other cases of interacting fermion systems, such as electrons in transition metals.

It must be pointed out that the existence of such condensed states has not been observed yet in any physical system. Measurements[5] of the specific heat of liquid helium-3 at temperatures as low as 0.008°K failed to show any transition to an ordered state; this experimental result does not, however, preclude a condensed state of the type we are discussing here, for the critical temperature T_c derived for an ideal fluid (without scattering) is certainly well above the true transition temperature for the real fluid. Indeed, Suhl argues that a transition to a condensed state cannot take place unless the energy broadening due to scattering is smaller than kT_c.[6] One can see from self-diffusion measurements[5] that the amount of particle-particle scattering is still quite large at 0.03°K and that the resulting energy broadening will be small enough to allow a condensation, only at a significantly lower temperature, probably of the order of 0.01° to 0.02°K. It is thus possible that the stable state of liquid helium-3 be a condensed state at the absolute zero and at temperatures close to zero, even though helium-3 is still a Fermi liquid in the range of temperatures which are attainable now. In any case, these nonisotropic condensed states appear interesting enough to justify as complete an investigation as possible of their properties and of the conditions for condensation.

We first introduce a general theory of ideal many-fermion systems specifically designed to include possible condensation of the particles into nonspherically symmetrical pair states, following a straightforward generalization of the method of Anderson[7] (Sec. II); then, we renormalize the interaction potential in order to eliminate formally divergent matrix elements (resulting from the strong repulsive core of the He^3-He^3 potential) and reduce it to an "effective potential" acting only near the Fermi surface (Sec. III). The problem of finding the ground state is then reduced to solving a BCS-type nonlinear equation: Sec. IV is devoted to the treatment of this equation and the derivation of the properties of the ground state, particularly the ground state energy (condensation energy). We present in Sec. V a novel approach to the derivation of the thermodynamics of the condensed system, based on Anderson's formalism; the critical temperature T_c, the specific heat, and the paramagnetic susceptibility of the condensed fluid are evaluated. The two following sections are devoted to the study of the flow properties of this model; in Sec. VI we show that a small fraction of the particles take part in a spontaneous circulation in the ground state at 0°K, because the average correlation between the density of particles and the current-density fails to vanish on account of the anisotropy of the ground state; in Sec. VII, we show that the condensed fluid is superfluid as could be expected from the analogy with the superconducting property of the condensed electron gas in metals, and in accordance with the finding of Glassgold and Sessler.[8] Finally, we discuss the alterations of the properties derived in the previous sections for an ideal fluid, brought about by scattering and in general departure of the real fluid from the ideal model; we emphasize particularly in this last section (VIII) the significant lowering of the transition temperature by scattering in the case of liquid helium-3.

II. PRESENTATION OF THE FORMALISM

We write the Hamiltonian of our system of interacting fermions in second quantization and expand the wave function in term of plane waves:

$$H=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k}}c_{\mathbf{k},\sigma}{}^{*}c_{\mathbf{k},\sigma}-\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\sum_{\sigma,\sigma'}V_{\mathbf{k}\mathbf{k}'}c_{\mathbf{k}'\sigma}{}^{*}c_{\mathbf{q}-\mathbf{k}',\sigma'}{}^{*}c_{\mathbf{q}-\mathbf{k},\sigma'}c_{\mathbf{k}\sigma}. \quad (2.1)$$

Here $c_{\mathbf{k},\sigma}{}^{*}$ and $c_{\mathbf{k},\sigma}$ are, respectively, the creation and annihilation operators of the fermion field, $\epsilon_{\mathbf{k}}$ the energy appropriate to single-particle excitations of the system, and $V_{\mathbf{k}\mathbf{k}'}$ the interaction potential matrix element. The most basic approach to treat the Hamiltonian (2.1) would be to take $\epsilon_{\mathbf{k}}$ as the "bare" particle kinetic energy and $V_{\mathbf{k}\mathbf{k}'}$ equal to the Fourier transform of the free space He^3-He^3 interaction potential, and to derive all many-body effects, including a possible condensation, from there. We shall not, however, follow this course since we are only interested in finding the energy difference (or condensation energy) between the condensed state and the Fermi states of a system of "dressed" particles. Anderson[7] and others[9] have shown that the usual many-body effects, leading to a re-

[5] A. C. Anderson, H. R. Hart, Jr., and J. C. Wheatley, Phys. Rev. Letters **5**, 133 (1960); A. C. Anderson, G. L. Salinger, W. A. Steyert, and J. C. Wheatley, *ibid.* **6**, 331 (1961).
[6] H. Suhl, Bull. Am. Phys. Soc. **6**, 119 (1961).
[7] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[8] A. E. Glassgold and A. M. Sessler, Nuovo cimento **19**, 723 (1961).
[9] E.g., N. N. Bogolyubov, Uspekhi Fiz. Nauk **67**, 549 (1959); Y. Nambu, Phys. Rev. **117**, 648 (1960).

normalization of the Hamiltonian in terms of quasiparticles or "dressed" particles, are probably substantially unaffected by the transition into a BCS-type condensed state, when the condensation involves only a small fraction of the total number of particles; the many-body corrections, such as the exchange energy, the screening of the interaction potential, etc., are practically the same in the normal and condensed states. Consequently, it is adequate to replace the basic Hamiltonian (2.1) by the renormalized Hamiltonian which is the result of a complete treatment of the usual many-body effects in the normal state, that is to say to include effective-mass corrections into $\epsilon_{\mathbf{k}}$ and to use a screened potential instead of the free-space interaction potential.

Furthermore, Anderson has shown that the reduced Hamiltonian H_{red} which includes only the term $\mathbf{q}=0$ of the full Hamiltonian (2.1), leads to the correct zero-order equations of motion for the excitations of the system, in the generalized random-phase approximation. It is thus satisfactory to use this reduced Hamiltonian instead of (2.1) for the purpose of finding the ground state of the system. In the ground state, all particles are associated in pairs with zero total momentum and either zero total spin (antiparallel pairing) or a total spin equal to 1 (parallel pairing). In the former case, we can define the BCS operators:

$$b_{\mathbf{k}}^*=c_{\mathbf{k}\uparrow}^*c_{-\mathbf{k}\downarrow}^*,$$
$$b_{\mathbf{k}}=c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow},$$
$$n_{\mathbf{k}}=c_{\mathbf{k}\uparrow}^*c_{\mathbf{k}\uparrow}=c_{-\mathbf{k}\downarrow}^*c_{-\mathbf{k}\downarrow}.$$

We find, in this case, that the odd part of $V_{\mathbf{k}\mathbf{k}'}$ can be canceled by symmetrizing the interaction terms of the reduced Hamiltonian. Since we shall later use the expansion of $V_{\mathbf{k}\mathbf{k}'}$ in term of spherical harmonics, it is pertinent to remark that only even terms of this expansion remain after the symmetrization: even l terms are associated with antiparallel pairing.

In the case of parallel pairing, we can instead define the operators:

$$b_{\mathbf{k},\sigma}^*=c_{\mathbf{k},\sigma}^*c_{-\mathbf{k},\sigma}^*,$$
$$b_{\mathbf{k},\sigma}=c_{-\mathbf{k},\sigma}c_{\mathbf{k},\sigma},$$
$$n_{\mathbf{k},\sigma}=c_{\mathbf{k},\sigma}^*c_{\mathbf{k},\sigma}=c_{-\mathbf{k},\sigma}^*c_{-\mathbf{k},\sigma}.$$

Now, only odd terms are left upon symmetrization of the interaction, meaning that the odd l terms of the expansion of $V_{\mathbf{k}\mathbf{k}'}$ in terms of spherical harmonics are associated with parallel pairing. A slight complication arises in this case because the two spin populations are left uncoupled by the interaction: One can then write two "half-Hamiltonians," one for each spin, and consider, in principle, the case of unequal populations of the two spin states. It can be shown, however, that this situation could not occur in the weak-coupling limit,[10] so that the average values of the above operators

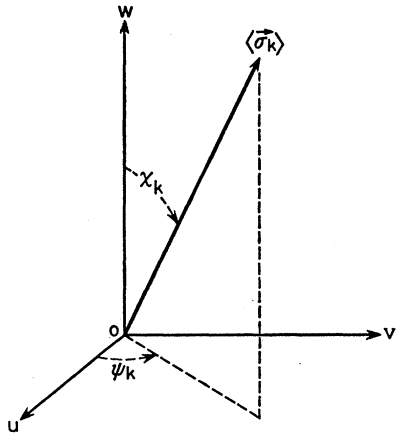

FIG. 1. Representation of the mean value of the "pseudo-spin" operator σ_k in the occupation space of the individual particle state k.

are independent of the spin; this allows us to forget the distinction between the two spin states in this case and to write the reduced Hamiltonian both in the cases of parallel and antiparallel pairing as

$$H_{\text{red}}=2\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}n_{\mathbf{k}}-2\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}b_{\mathbf{k}'}^*b_{\mathbf{k}}. \tag{2.2}$$

Following Anderson's method, we express the three operators $b_{\mathbf{k}}^*$, $b_{\mathbf{k}}$ and $n_{\mathbf{k}}$ in terms of one vectorial operator $\boldsymbol{\sigma}_{\mathbf{k}}$, formally identical, in the occupation space of the pair state $\mathbf{k}\uparrow$, $-\mathbf{k}\downarrow$ (antiparallel pairing) or $\mathbf{k}\uparrow$, $-\mathbf{k}\uparrow$ (parallel pairing), to the spin $\frac{1}{2}$ operator in spin space:

$$b_{\mathbf{k}}^*=\tfrac{1}{2}(\sigma_{\mathbf{k}u}-i\sigma_{\mathbf{k}v}),$$
$$b_{\mathbf{k}}=\tfrac{1}{2}(\sigma_{\mathbf{k}u}+i\sigma_{\mathbf{k}v}),$$
$$1-n_{\mathbf{k}}-n_{-\mathbf{k}}=\sigma_{\mathbf{k}w}.$$

We now carry these expressions into (2.2) and also slightly alter the definition of $\epsilon_{\mathbf{k}}$ to the effect that, from now on, this energy will be measured relative to the Fermi level. Omitting an irrelevant constant, we find

$$H_{\text{red}}=-\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}\sigma_{\mathbf{k}w}-\tfrac{1}{2}\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}[\sigma_{\mathbf{k}u}\sigma_{\mathbf{k}'u}+\sigma_{\mathbf{k}v}\sigma_{\mathbf{k}'v}]. \tag{2.3}$$

The analogy with the problem of ferromagnetism leads us to use the so-called "semiclassical method" for the purpose of finding the ground state of the system; this method consists in replacing the spin operators (here the operators $\boldsymbol{\sigma}_{\mathbf{k}}$) by their mean values, which can be treated as ordinary vectors of moduli equal to 1, in their respective spin (occupation) spaces $(Ouvw)_{\mathbf{k}}$.[11] Let then $\chi_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$ be the usual latitude and longitude angles defining the direction of the vector $\langle\boldsymbol{\sigma}_{\mathbf{k}}\rangle$ (see Fig. 1). In term of these new variables, the mean value

[10] Indeed, if the interaction has a reasonable strength, the binding energy gained by concentrating more particles in one spin state than in the other is more than offset by the resulting increase of the kinetic energy.

[11] G. Heller and H. A. Kramers, Proc. Acad. Sci. Amsterdam **37**, 378 (1934); M. J. Klein and R. S. Smith, Phys. Rev. **80**, 1111 (1951); P. W. Anderson, *ibid.* **86**, 694 (1952).

E of the reduced Hamiltonian is

$$E=-\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}\cos\chi_{\mathbf{k}}-\tfrac{1}{2}\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i(\psi_{\mathbf{k}'}-\psi_{\mathbf{k}})}. \quad (2.4)$$

The ground state at $T=0°\mathrm{K}$ is found by minimizing E with respect to $\chi_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$. The former operation leads to the condition:

$$\tan\chi_{\mathbf{k}}e^{i\psi_{\mathbf{k}}}=\frac{1}{\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}. \quad (2.5)$$

Minimizing with respect to $\psi_{\mathbf{k}}$ does not bring a more restrictive condition, for the derivative of E with respect to $\psi_{\mathbf{k}}$ vanishes as a consequence of condition (2.5). This of course was to be expected since adding an arbitrary constant to all $\psi_{\mathbf{k}}$ is irrelevant to the physical situation.

Finally, using relations (2.4) and (2.5), we find the energy in the ground state:

$$E_g=-\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}(\cos\chi_{\mathbf{k}}+\tfrac{1}{2}\sin\chi_{\mathbf{k}}\tan\chi_{\mathbf{k}}). \quad (2.6)$$

The condensation energy is then the difference between E_g and the energy corresponding to the Fermi state ($\cos\chi_{\mathbf{k}}=-1$ inside the Fermi sphere and $+1$ outside):

$$W_0=\sum_{\mathbf{k}}[|\epsilon_{\mathbf{k}}|-\epsilon_{\mathbf{k}}(\cos\chi_{\mathbf{k}}+\tfrac{1}{2}\sin\chi_{\mathbf{k}}\tan\chi_{\mathbf{k}})]. \quad (2.7)$$

Although this energy W_0 is usually quite small, it is indeed the main contribution to the binding energy gained by the pairing of the particles. The contribution of the interaction terms included in H_{red} to the usual correlation energy is altogether negligible (to order $1/N$). On the other hand, the usual correlation energy results from terms which are not included in the reduced Hamiltonian (scattering, exchange, exchange-scattering matrix elements, . . .) and therefore, is not significantly altered by the condensation.[12]

III. EFFECTIVE POTENTIAL

Since we are interested here in the weak coupling limit, we consider only states of the system not too different from the Fermi state: We expect, in other words, the distribution of individual particle states in the ground state of the system to be essentially identical to the Fermi distribution outside a fairly thin shell about the Fermi surface. Restricting our interest to these quasi-Fermi distributions, we assume that $|\sin\chi_{\mathbf{k}}|$ is significantly different from zero only inside the shell $|\epsilon_{\mathbf{k}}|<\xi$ in momentum space, ξ being much smaller than the Fermi energy. It is clear that most of the binding energy results from the contribution to (2.7) of transitions inside this shell. This region near the Fermi surface is also the domain where Eq. (2.5) is nonlinear, for $\sin\chi_{\mathbf{k}}$ is almost equal either to $+$ or $-\tan\chi_{\mathbf{k}}$ outside the shell (that is to say, $\chi_{\mathbf{k}}\to\pi$ below the Fermi level and $\chi_{\mathbf{k}}\to 0$ above the Fermi level, if the actual distribution of individual particle states approaches the Fermi distribution). Let us then split the summation on the right-hand side of (2.5) into summation inside and outside the shell:

$$\tan\chi_{\mathbf{k}}e^{i\psi_{\mathbf{k}}}=\frac{1}{\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}'}^{\mathrm{in}}V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}+\sum_{\mathbf{k}'}^{\mathrm{out}}V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}, \quad (3.1)$$

and replace $\sin\chi_{\mathbf{k}'}$ in the second term on the right-hand side by

$$\sin\chi_{\mathbf{k}}=\frac{\epsilon_{\mathbf{k}}}{|\epsilon_{\mathbf{k}}|}\tan\chi_{\mathbf{k}}+O(|\sin^3\chi_{\mathbf{k}}|). \quad (3.2)$$

We obtain thereby, as far as the summation outside the shell is concerned, a linear integral equation which can be solved by iteration. Substituting expression (3.1) for $\tan\chi_{\mathbf{k}}$ in (3.2), we find:

$$\begin{aligned}\tan\chi_{\mathbf{k}}e^{i\psi_{\mathbf{k}}}&=\frac{1}{\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}'}^{\mathrm{in}}V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}+\frac{1}{\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}'}^{\mathrm{out}}V_{\mathbf{k}\mathbf{k}'}\tan\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}\frac{\epsilon_{\mathbf{k}'}}{|\epsilon_{\mathbf{k}'}|}\\&=\sum_{\mathbf{k}'}^{\mathrm{in}}\frac{V_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}}{\epsilon_{\mathbf{k}}}+\sum_{\mathbf{k}''}^{\mathrm{in}}\sum_{\mathbf{k}'}^{\mathrm{out}}\frac{V_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}'\mathbf{k}''}\sin\chi_{\mathbf{k}''}e^{i\psi_{\mathbf{k}''}}}{\epsilon_{\mathbf{k}}|\epsilon_{\mathbf{k}'}|}\\&\quad+\sum_{\mathbf{k}'''}^{\mathrm{in}}\sum_{\mathbf{k}''}^{\mathrm{out}}\sum_{\mathbf{k}'}^{\mathrm{out}}\frac{V_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}'\mathbf{k}''}V_{\mathbf{k}''\mathbf{k}'''}\sin\chi_{\mathbf{k}'''}e^{i\psi_{\mathbf{k}'''}}}{\epsilon_{\mathbf{k}}|\epsilon_{\mathbf{k}'}||\epsilon_{\mathbf{k}''}|}+\cdots.\end{aligned} \quad (3.3)$$

By relabeling the summation indexes, this equation can be written in the more convenient form:

$$\tan\chi_{\mathbf{k}}e^{i\psi_{\mathbf{k}}}=\frac{1}{\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}'}^{\mathrm{in}}U_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}, \quad (3.4)$$

where the "effective potential" $U_{\mathbf{k}\mathbf{k}'}$ is defined by

$$\begin{aligned}U_{\mathbf{k}\mathbf{k}'}&=V_{\mathbf{k}\mathbf{k}'}+\sum_{\mathbf{k}''}^{\mathrm{out}}V_{\mathbf{k}\mathbf{k}''}\frac{1}{|\epsilon_{\mathbf{k}''}|}V_{\mathbf{k}''\mathbf{k}'}\\&\quad+\sum_{\mathbf{k}''}^{\mathrm{out}}\sum_{\mathbf{k}'''}^{\mathrm{out}}V_{\mathbf{k}\mathbf{k}''}\frac{1}{|\epsilon_{\mathbf{k}''}|}V_{\mathbf{k}''\mathbf{k}'''}\frac{1}{|\epsilon_{\mathbf{k}'''}|}V_{\mathbf{k}'''\mathbf{k}'}-\cdots.\end{aligned} \quad (3.5)$$

Note that Eq. (3.4) is quite similar to (2.5) although much simplified since the summation is to be extended only on a small region of momentum space; on the other hand, the original potential V is replaced by the rather awkward expression (3.5). In this respect, we shall find convenient to use instead of this definition of $U_{\mathbf{k}\mathbf{k}'}$, the equivalent integral equation:

$$U_{\mathbf{k}\mathbf{k}'}=V_{\mathbf{k}\mathbf{k}'}+\sum_{\mathbf{k}''}^{\mathrm{out}}V_{\mathbf{k}\mathbf{k}''}\frac{1}{|\epsilon_{\mathbf{k}''}|}U_{\mathbf{k}''\mathbf{k}'}. \quad (3.6)$$

[12] See reference 7, Sec. V.

We notice furthermore that the terms of the second order in $\sin\chi_{\mathbf{k}}$ cancel from the expression (2.6) for the ground-state energy, in the limit of small $\sin\chi_{\mathbf{k}}$ (that is to say, outside the shell):

$$E_g=-\sum_{\mathbf{k}}^{\text{in}}\epsilon_{\mathbf{k}}(\cos\chi_{\mathbf{k}}+\tfrac{1}{2}\sin\chi_{\mathbf{k}}\tan\chi_{\mathbf{k}})$$
$$-\sum_{k}^{\text{out}}[|\epsilon_{\mathbf{k}}|+O(|\sin\chi_{\mathbf{k}}|^4)]. \quad (3.7)$$

The last term on the right-hand side is of the order of Δ^3/ξ^3 (where Δ is a measure of the condensation energy near the Fermi level) and must be neglected in the weak-coupling limit. In this "effective potential" scheme, therefore, the binding energy W_0 results exclusively from the contribution of transitions inside the shell. We justify thereby the "cutoff" hypothesis of BCS since we have obtained an equivalent expression of the ground-state energy by replacing the actual potential $V_{\mathbf{k}\mathbf{k}'}$ by the effective potential $U_{\mathbf{k}\mathbf{k}'}$ defined by (3.6) if both $\mathbf{k}$ and $\mathbf{k}'$ are inside the shell $|\epsilon_{\mathbf{k}}|<\xi$ and equal to zero otherwise.

This procedure was a necessary step in the helium-3 problem, because of the presence of a practically divergent repulsive core, which is to a great extent renormalized out in the integral Eq. (3.6). It is very close to the Bethe-Goldstone scheme and Eq. (3.6) can indeed be re-expressed in position space and integrated like a Bethe-Goldstone equation.[13] It is, in addition, extremely convenient in any weak-coupling case, since it allows the replacing of the energy-dependent $V_{\mathbf{k}\mathbf{k}'}$ by the substantially energy-independent $U_{\mathbf{k}\mathbf{k}'}$ thereby leading in general to a soluble integral equation (since this effective potential is factorizable).

To elaborate on this last point, let us expand Eq. (3.6) in terms of spherical harmonics, with the help of the "addition theorem" for Legendre polynomials.[14] The expansion of $V_{\mathbf{k}\mathbf{k}'}$ is

$$V_{\mathbf{k}\mathbf{k}'}=\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(r)e^{i\mathbf{k}\cdot\mathbf{r}}d\tau$$
$$=\sum_{l=0}^{\infty}P_l(\hat{k}\cdot\hat{k}')\frac{2\pi^2(2l+1)}{(kk')^{\frac{1}{2}}}\int_0 J_{l+\frac{1}{2}}(k'r)V(r)J_{l+\frac{1}{2}}(kr)rdr$$
$$=\sum_{l=0}^{\infty}\frac{2l+1}{2}V_l(k,k')P_l(\hat{k}\cdot\hat{k}'), \quad (3.8)$$

where we denote by $\hat{k}$ the unit vector parallel to $\mathbf{k}$. Here $P_l(u)$ is the (un-normalized) Legendre polynomial of order l and $J_{l+\frac{1}{2}}$ is the Bessel function of order $l+\frac{1}{2}$. In the limit of small r, $J_{l+\frac{1}{2}}(kr)$ is approximately proportional to $(kr)^{l+\frac{1}{2}}$; consequently the short-range part of the potential $V(r)$ (inside a range of the order of k_F^{-1}) contributes little to the high-order terms of expansion (3.8). This property is particularly significant in the case of a long-range attractive potential with a repulsive core; although the core may be strongly repulsive, the long-range part eventually dominates the higher order terms of (3.8) so that V_l is positive (corresponding to an attractive interaction) for large enough l. Similarly, the expansion of $U_{\mathbf{k}\mathbf{k}'}$ is

$$U_{\mathbf{k}\mathbf{k}'}=\sum_{l=0}^{\infty}\frac{2l+1}{2}U_l(k,k')P_l(\hat{k}\cdot\hat{k}'). \quad (3.9)$$

These coefficients at large values of k are of course important only because the equations we are to solve are for pairs of particles *in the presence of the Fermi sea;* the repulsive centripetal barrier, which for ordinary wave equations always ensures that a spherical potential binds $l=0$ states most strongly, is here at least partially overcome by the zero-point kinetic energy of the degenerate gas.

Carrying these expressions into (3.6) and replacing the summation by integration over the corresponding domain of momentum space,[15] we obtain

$$U_l(k,k')=V_l(k,k')$$
$$+\frac{1}{4\pi^2}\int_{|\epsilon_q|>\xi}V_l(k,q)\frac{1}{|\epsilon_q|}U_l(q,k')q^2dq. \quad (3.10)$$

We can now demonstrate that the divergence due to the repulsive core has been eliminated from the effective potential. For the sake of the argument, let us take $V(r)$ as a square potential of fixed range and arbitrary large strength A. Then $V_l(k,k')$ is proportional to A and (3.10) can only be satisfied if $U_l(k,k')$ is independent of A, hence finite. Thus divergent terms cancel each other on the right-hand side of Eq. (3.6) or (3.10), so that $U_l(k,k')$ is finite, even though $V_l(k,k')$ may be formally infinite.

Furthermore, we note that we only need to solve Eq. (3.10) in a very small range of values of k and k': our basic assumption, relevant to the weak-coupling case, is indeed that the thickness 2ξ of the domain of summation is much smaller than the Fermi energy so that the relative variations of the moduli k and k' are quite small; it is then an excellent approximation to replace the functions $U_l(k,k')$ in expansion (3.9) by

[13] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1957).

[14] If θ, φ and θ', φ' are the usual latitude and longitude angles which determine the directions of the unit vectors u and u' with respect to a fixed frame $Oxyz$, we have:

$$P_l(\hat{u}\cdot\hat{u}')=P_l(\cos\theta)P_l(\cos\theta')+\sum_{m=1}^{m=+l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')$$
$$\times[e^{im(\varphi'-\varphi)}+\text{c.c.}]=\frac{4\pi}{2l+1}\sum_{m=-l}^{m=+l}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi').$$

[15] We assume the system is large enough so that $\Sigma_{\mathbf{k}}$ is approximately equal to $\int d\tau_{\mathbf{k}}/8\pi^3$. More precisely, we shall replace $\Sigma_{\mathbf{k}}^{\text{in}}$ by $(N_0/2\pi)\int_{\epsilon=0}^{\xi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\sin\theta d\theta d\varphi d\epsilon$ and $\Sigma_{\mathbf{k}}^{\text{out}}$ by $(N_0/2\pi)\times\int_{\epsilon=\xi}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\sin\theta d\theta d\varphi d\epsilon$. N_0 is one-half the individual particle density of states at the Fermi level.

their mean values on the Fermi surface: $U_l(k_F,k_F)$ or U_l for short. The U_l are not well (intrinsically) defined parameters of the system since their values depend logarithmically upon the arbitrary parameter ξ (we shall show, however, that the energy of the system and the other physical quantities of interest are independent of ξ). In order to derive explicitly this dependence, let us assume then that the function $U_l(q,k_F)$ of *one variable* q is increased by an infinitesimal amount $\delta U_l(q,k_F)$, proportional to $U_l(q,k_F)$:

$$U_l(q,k_F)+\delta U_l(q,k_F)=U_l(q,k_F)[1+\delta x].$$

This function is the solution of Eq. (3.10) corresponding to a slightly different value $\xi+\delta\xi$ of the parameter:

$$(1+\delta x)U_l=V_l+\frac{1+\delta x}{4\pi^2}\int_{|\epsilon_q|>\xi+\delta\xi}V_l(k_F,q)\frac{1}{|\epsilon_q|}\times U_l(q,k_F)q^2dq. \quad (3.11)$$

Subtracting (3.10) from (3.11) we obtain in first order:

$$\delta U_l=-N_0U_l^2d\xi/\xi \quad (3.12)$$

or:

$$1/N_0U_l=\ln\xi+\text{constant}, \quad (3.13)$$

where N_0 is one-half the density of individual-particle states at the Fermi level.

IV. GROUND STATE AT $T=0°K$

We can more conveniently rewrite the Eq. (3.4) in the form

$$\tan\chi_{\mathbf{k}}e^{i\psi_{\mathbf{k}}}=C(\theta,\varphi)/\epsilon_{\mathbf{k}}, \quad (4.1)$$

thereby separating the angular dependence from the energy dependence. $C(\theta,\varphi)$ is defined by:

$$C(\theta,\varphi)=\sum_{lm}2\pi U_lY_{lm}(\theta,\varphi)\sum_{\mathbf{k}'}^{in}\frac{Y_{lm}^*(\theta',\varphi')C(\theta',\varphi')}{[\epsilon_{\mathbf{k}'}^2+|C(\theta',\varphi')|^2]^{\frac{1}{2}}}. \quad (4.2)$$

This is a nonlinear integral equation, the nonlinearity resulting from the presence of the term $|C(\theta,\varphi)|^2$ in the denominator on the right-hand side. Since this term is small in the weak-coupling limit, one expects that the mixing of different l spherical harmonics will be small and that therefore Eq. (4.2) has a series of almost pure l-type solutions like

$$C(\theta,\varphi)=\sum_m\Delta_mY_{lm}(\theta,\varphi). \quad (4.3)$$

These considerations are borne out by a more refined evaluation of the amount of coupling between different terms of the expansion (3.9) of the effective potential (see Appendix A). One finds that the error involved in solving the simplified equation:

$$C(\theta,\varphi)=2\pi U_l\sum_mY_{lm}(\theta,\varphi)\sum_{\mathbf{k}'}^{\text{in}}\frac{Y_{lm}^*(\theta',\varphi')C(\theta',\varphi')}{[\epsilon_{\mathbf{k}'}^2+|C(\theta',\varphi')|^2]^{\frac{1}{2}}}, \quad (4.4)$$

instead of (4.2), is quite small; the amount of other spherical harmonics in a predominantly l-type solution of (4.2) never exceeds few percent and moreover, the error in the ground state energy involved in neglecting this slight mixing is very small indeed (0.1%). We shall therefore simplify the problem of finding the ground state by taking into account only the most attractive term of the expansion of the effective potential (which is likely to give the most favorable condensed state).

Similarly, one could also argue that different Y_{lm} (for a given l) are slightly coupled in Eq. (4.4) and one sees easily that this equation has indeed simple solutions like

$$C(\theta,\varphi)=\Delta_mY_{lm}(\theta,\varphi). \quad (4.5)$$

For such solutions, (4.4) reduces exactly to the BCS-type integral equation:

$$1=2\pi U_l\sum_{\mathbf{k}'}^{\text{in}}\frac{|Y_{lm}(\theta',\varphi')|^2}{[\epsilon_{\mathbf{k}'}^2+\Delta_m^2|Y_{lm}(\theta',\varphi')|^2]^{\frac{1}{2}}}. \quad (4.6)$$

On the other hand, the perfect decoupling of different m in (4.6) is only the result of the heuristic "Ansatz" (4.5) and there can be no doubt that (4.4) has also mixed solutions like (4.3): for example, the solutions derived from (4.5) by a rotation of the coordinate system. We have therefore performed a complete analysis of this problem for the case $l=2$ (condensed state of helium-3) in the Appendix B. After removing the irrelevant degrees of freedom (three for rotational invariance and one for gauge invariance[16]), we have found that Eq. (4.4) for $l=2$ has at least two independent mixed solutions in addition to the three simple solutions Δ_0Y_{20}, Δ_1Y_{21}, and Δ_2Y_{22}.

At this point, numerical computations are necessary to proceed and choose the most favorable configuration among these solutions. This computation is straightforward enough for the simple solutions (4.5); after integrating (4.6) over the energy and discarding terms of the order of Δ_m^2/ξ^2 (negligible in the weak coupling case), we obtain

$$1=N_0U_l\left(\ln\frac{2\xi}{\Delta_m}-\int\int|Y_{lm}|^2\ln|Y_{lm}|d\Omega\right), \quad (4.7)$$

or, more conveniently,

$$\Delta_m=2\Gamma\xi\exp(-1/N_0U_l), \quad (4.8)$$

where Δ_m plays the part of the energy gap derived by BCS and where the constant Γ is defined by

$$\ln\Gamma=-\int\int|Y_{lm}(\theta,\varphi)|^2\ln|Y_{lm}(\theta,\varphi)|d\Omega. \quad (4.9)$$

[16] We have seen in Sec. II that the physical situation is not changed by adding the same arbitrary phase angle to all $\psi_{\mathbf{k}}$. Such a gauge transformation is equivalent to multiplying $C(\theta,\varphi)$ by an arbitrary phase factor.

TABLE I. Values of $\ln\Gamma$ for different solutions of Eq. (4.2). The binding energy is proportional to Γ^2 for a given U_l.

	$l=0$	$l=1$	$l=2$	$l=3$
$m=0$	1.263	1.048	1.020	1.010
$m=1$		1.201	1.131	1.090
$m=2$			1.131	1.123
$m=3$				1.075
Ground-state configuration (4.12)			1.154	

We have listed the values of $\ln\Gamma$ for the simple solutions Y_{lm} up to $l=3$ in Table I. Note that widely different configurations yield close values of Γ and, consequently, correspond to almost equal condensation energies [according to expression (4.13) for the condensation energy]. We wish to get at the same convenient relation (4.8) for mixed solutions also, and therefore we define

$$C(\theta,\varphi)=\Delta f(\theta,\varphi), \tag{4.10}$$

where $f(\theta,\varphi)$ is the relevant combination of spherical harmonics normalized to unity. Δ is again given by (4.8), with Γ defined in general by

$$\ln\Gamma=-\int\int |f(\theta,\varphi)|^2 \ln|f(\theta,\varphi)|\, d\Omega. \tag{4.11}$$

We find that the p-type ground state is given by the simple solution Y_{11}; the d-type ground state configuration, on the other hand, is not a simple solution but rather a mixture of Y_{20} and Y_{22} spherical harmonics, in accordance with a suggestion of Thouless.[17] More precisely, the most favorable d-type solution is

$$f(\theta,\varphi)=\left[\frac{1}{\sqrt{2}}Y_{20}+\tfrac{1}{2}(Y_{22}-Y_{2,-2})\right]. \tag{4.12}$$

This configuration does indeed yield an energy about 5% lower than the energy corresponding to either simple solutions Y_{21} or Y_{22} (see Table I).

Ground-State Energy

Carrying the expression (4.10) for $C(\theta,\varphi)$ into (4.1) and (3.7), we obtain in a straightforward fashion the following convenient expression for the condensation energy W_0:

$$\begin{aligned} W_0&=\frac{N_0}{2\pi}\int d\Omega\int_0^{\xi} d\epsilon\left[\epsilon-\frac{\epsilon^2+\frac{1}{2}|C(\theta,\varphi)|^2}{[\epsilon^2+|C(\theta,\varphi)|^2]^{\frac{1}{2}}}\right],\\ &=-\frac{N_0}{8\pi}\int |C(\theta,\varphi)|^2 d\Omega=-\frac{N_0\Delta^2}{8\pi}. \end{aligned} \tag{4.13}$$

Although the expression (4.8) for Δ formally involves the "cutoff" parameter ξ, we see from (3.13) that the dependence upon ξ actually cancels out; Δ and consequently the ground state energy are indeed independent of the "cutoff" scheme we have introduced to compute them.

Equation (4.13) shows that Δ must be maximized in order to maximize the ground-state energy. Equations (4.11), (4.10), and (4.8), in turn, show that, in order to do so, we must maximize

$$-\langle f^2(\theta,\varphi)|\ln|f^2(\theta,\varphi)|\rangle_{\rm av}$$

relative to $\langle|f^2|\rangle_{\rm av}$ itself. Since the above function is convex downward, this means that we must minimize the total variation of $|f|^2$; essentially, we must make the "gap" spread as uniformly as possible over the Fermi sphere. This concept is very useful in visualizing what is likely to be a good solution and what its properties are. First, of course, $m=0$ and other real solutions are very bad because clearly we can always find another real spherical harmonic which is large when this is small and vice versa, and make then a complex solution which is more favorable because $|f_1^2|+|f_2|^2$ is more uniform than $|f_1^2|$. That is important in our later discussion of currents. Second, clearly the solution should have as few zeros of as low order as possible. That is why (4.12), with only point zeros, is preferable to the simple (4.5) in the case $l=2$.

Individual Particle Excitation Spectrum

As BCS pointed out, the condensation perturbs strongly the distribution of individual particle states near the Fermi level because of the appearance of an extra "condensation energy" term in the expression for the individual-particle excitation energy:

$$E_k=[\epsilon_k^2+|C(\theta,\varphi)|^2]^{\frac{1}{2}}. \tag{4.14}$$

For $l=0$ this extra term is a constant ϵ_0; the corresponding spectrum has therefore a gap of width $2\epsilon_0$. In the non-spherically symmetrical condensed states, however, $C(\theta,\varphi)$ can vanish for some directions, so that the energy spectrum does not exhibit a true gap, but only a sharp reduction of the density of states near the Fermi level (in fact, the density of states vanishes at the Fermi level). Now, the Jacobian for expressing the density of states in terms of the variables E, θ, φ instead of k, θ, φ is

$$\frac{D(k,\theta,\varphi)}{D(E,\theta,\varphi)}=\frac{m|E|}{\hbar^2 k_F[E^2-|C(\theta,\varphi)|^2]^{\frac{1}{2}}}, \tag{4.15}$$

and therefore

$$N(E)=\frac{N_0|E|}{4\pi}\int\int_{\Sigma}\frac{d\Omega}{[E^2-|C(\theta,\varphi)|^2]^{\frac{1}{2}}}. \tag{4.16}$$

The integration here is extended over the region Σ of the sphere, where $|C(\theta,\varphi)|$ is smaller than E. It is clear from (4.16) that the excitation energy spectrum

[17] D. Thouless (private communication); see also D. Thouless, Ann. Phys. **10**, 553 (1960).

is altered only near the Fermi level since the right-hand side reduces to the normal value N_0 if E is large compared to Δ. Furthermore, one notices that the integral on the right-hand side is finite, except maybe when E is equal to a relative maximum of $|C(\theta,\varphi)|$; the integral is still finite if the maximum is reached at discrete points but is divergent if the maximum is reached on a line. We see then that this maximum is analogous to the gap ϵ_0 of BCS, as far as the energy spectrum is concerned. We shall find examples of both situations below:

$l=1$: $|C(\theta,\varphi)|=\epsilon_0\sin\theta$.

In this special case, (4.16) can be integrated exactly and one finds:

$$N(E)=\frac{N_0|E|}{2\epsilon_0}\ln\left|\frac{E-\epsilon_0}{E+\epsilon_0}\right|.$$

The density of states $N(E)$ vanishes like $N_0(E/\epsilon_0)^2$ in the small excitation energy limit, and becomes infinite at $E=\epsilon_0$.

$l=2$: $|C(\theta,\varphi)|=\frac{1}{2}\epsilon_0[(3\cos^2\theta-1)^2+3\sin^2 2\varphi\sin^4\theta]^{\frac{1}{2}}$

In the small energy limit, the domain of integration Σ splits into 8 subdomains Σ_1 in the neighborhood of the nodes of the function $|C(\theta,\varphi)|$: $\theta=\theta_1$ or $\pi-\theta_1$, $\varphi=n\pi/2$ ($\cos\theta_1=1/\sqrt{3}$). Each subdomain contributes an equal amount to the right side of (4.16) on account of the symmetry of $|C(\theta,\varphi)|$, so that this expression can be written:

$$N(E)=8\frac{N_0|E|}{4\pi}\int\int_{\Sigma_1}\frac{d\Omega}{[E^2-|C|^2]^{\frac{1}{2}}},\qquad(4.17)$$

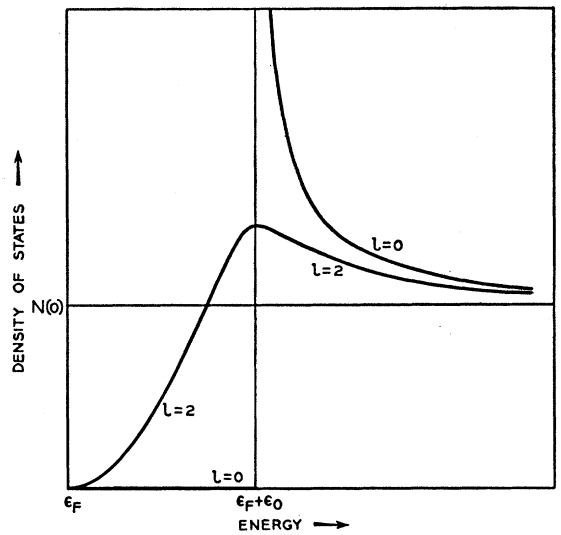

FIG. 2. Comparison of the individual-particle excitation energy spectra for normal fluid (constant density of states equal to N_0), an s-type condensed fluid ($l=0$) and a d-type condensed fluid ($l=2$).

where $|C|$ is approximately proportional to the distance from the node. Transforming (4.17) to a new system of coordinates θ', φ' with the z' axis passing through the node, we obtain therefore in this limit:

$$N(E)=8\frac{N_0|E|}{4\pi}\int\int_{\Sigma_1}\frac{d\Omega'}{[E^2-2\epsilon_0^2\sin^2\theta']^{\frac{1}{2}}}=2N_0\left(\frac{E}{\epsilon_0}\right)^2.\qquad(4.18)$$

On the other hand, we have seen in Appendix B that $|C(\theta,\varphi)|$ reaches the maximum value ϵ_0 at the six points $\theta=0$, $\theta=\pi$, and $\theta=\pi/2$, $\varphi=(2n+1)\pi/4$. The integral on the right-hand side of (4.16) is therefore always finite and reaches a maximum (about $1.43N_0$) at $E=\epsilon_0$ (see Fig. 2).

V. THERMODYNAMICS OF THE SYSTEM

Let us now go back to our "occupation space" formalism of Sec. II, and notice that the total energy (2.4) of the system can be written

$$E=-\sum_{\mathbf{k}}\mathbf{E}_{\mathbf{k}}\cdot\boldsymbol{\sigma}_{\mathbf{k}},\qquad(5.1)$$

where $\boldsymbol{\sigma}_{\mathbf{k}}$ is (in the ground state) a unit vector in the occupation space of the pair $\mathbf{k}$, $-\mathbf{k}$, and $\mathbf{E}_{\mathbf{k}}$ a pseudo-field defined by

$$\begin{aligned}E_{\mathbf{k}u}&=\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\langle\sigma_{\mathbf{k}'u}\rangle,\\ E_{\mathbf{k}v}&=\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\langle\sigma_{\mathbf{k}'v}\rangle,\\ E_{\mathbf{k}w}&=\epsilon_{\mathbf{k}}.\end{aligned}\qquad(5.2)$$

We can still use here the effective potential $U_{\mathbf{k}\mathbf{k}'}$ instead of $V_{\mathbf{k}\mathbf{k}'}$, for the derivation which led to Eq. (3.4) at the absolute zero, is very nearly exact at temperatures finite but much smaller than the cutoff energy ξ. Since we are only interested in the range of temperature below the critical temperature T_c, this condition is fulfilled and we can write instead of (5.2):

$$\begin{aligned}E_{\mathbf{k}u}&=\sum_{\mathbf{k}'}^{\text{in}}U_{\mathbf{k}\mathbf{k}'}\langle\sigma_{\mathbf{k}'u}\rangle,\\ E_{\mathbf{k}v}&=\sum_{\mathbf{k}'}^{\text{in}}U_{\mathbf{k}\mathbf{k}'}\langle\sigma_{\mathbf{k}'v}\rangle,\\ E_{\mathbf{k}w}&=\epsilon_{\mathbf{k}}.\end{aligned}\qquad(5.3)$$

Since the potential $U_{\mathbf{k}\mathbf{k}'}$ is separable, $E_{\mathbf{k}}$ is indeed the product of a $\mathbf{k}$-dependent factor by a (self-consistent) sum over all states $\mathbf{k}'$ inside the cutoff. It is all right then to write the total energy in the form (5.1), for each term of the sum on the right-hand side depends only upon its subscript $\mathbf{k}$ (there is no hidden dependence upon the other $\boldsymbol{\sigma}_{\mathbf{k}'}$, except through the self-consistent field $\mathbf{E}_{\mathbf{k}}$).

We shall consider first for guidance the case of the Fermi distribution, for which the pseudo-field $\mathbf{E}_\mathbf{k}$ reduces to the w component $\epsilon_\mathbf{k}$, and $\boldsymbol{\sigma}_\mathbf{k}$ is equal to $+1$ and parallel to the w axis (corresponding to full occupancy of the pair $\mathbf{k}, -\mathbf{k}$ below the Fermi level and both $\mathbf{k}$ and $-\mathbf{k}$ empty above the Fermi level). There are altogether four possible occupation number combinations of the individual-particle states $\mathbf{k}, -\mathbf{k}$: the ground state, two single-particle excited states (one occupied, one empty) and one excited pair state (two holes below the Fermi level, or two excited particles above the Fermi level). The same considerations apply to the condensed state where the pseudo-field is a vector of modulus $\mathbf{E}_\mathbf{k}$ [given by (4.14)] not parallel to the w axis in general. The reduced Hamiltonian (2.2) simply does not act on the two single-particle states, so that the pseudo-field $\mathbf{E}_\mathbf{k}$ is ineffective and they simply act like two extra states of the pseudo-spin with energy 0. The occupation vector $\boldsymbol{\sigma}_\mathbf{k}$ must be parallel to the direction of the pseudo-field $\mathbf{E}_\mathbf{k}$ and may take any of the four values given in Table II. Note that this procedure is strictly equivalent to the treatment of Bogoliubov *et al.*,[18] and is simply a more graphic method to picture the independent "quasi-particles" introduced by these authors. It is now

TABLE II. States of the pseudo-spin vector.

	$\boldsymbol{\sigma}_\mathbf{k}$ (parallel to $\mathbf{E}_\mathbf{k}$)	Energy
Ground pair	$+1$	$-E_\mathbf{k}$
Single-particle excitation: $\mathbf{k}$	0	0
Single-particle excitation: $-\mathbf{k}$	0	0
Excited pair	-1	$+E_\mathbf{k}$

straightforward to compute the "sum-over-states" Z and the related thermodynamic quantities for each subsystem $\mathbf{k}, -\mathbf{k}$ and consequently the over-all thermodynamic behavior of the system.

Critical Temperature

The thermodynamic average of $\sigma_\mathbf{k}$ at a finite temperature T is

$$\langle\sigma_\mathbf{k}\rangle_{\mathrm{av}}=\frac{e^{\beta E_\mathbf{k}}-e^{-\beta E_\mathbf{k}}}{e^{\beta E_\mathbf{k}}+e^{-\beta E_\mathbf{k}}+2}=\tanh(\beta E_\mathbf{k}/2), \quad (5.4)$$

where β is $1/kT$. Consequently, Eq. (3.4) must be replaced by

$$\tan\chi_\mathbf{k}e^{i\psi_\mathbf{k}}=\frac{1}{\epsilon_\mathbf{k}}\sum_{\mathbf{k}'}^{\mathrm{in}}U_{\mathbf{k}\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{i\psi_{\mathbf{k}'}}\tanh(\beta E_{\mathbf{k}'}/2). \quad (5.5)$$

The critical temperature T_c is the temperature above which no solution $C(\theta,\varphi)$ of this equation exists. At T_c, the solutions $C(\theta,\varphi)$ vanish exactly and (5.5) becomes a linear equation for which the superposition principle is valid: All simple solutions Y_{lm} (for a given l and various m) and combinations of simple solutions are equivalent energy-wise. Equation (5.5) does indeed split into $2l+1$ equivalent equations:

$$1=2\pi U_l\sum_{\mathbf{k}'}^{\mathrm{in}}\frac{|Y_{lm}(\theta',\varphi')|^2}{\epsilon_{\mathbf{k}'}}\tanh(\beta_c\epsilon_{\mathbf{k}'}/2). \quad (5.6)$$

After performing the angular integration, we find the BCS equation

$$1=N_0U_l\int_0^\xi\tanh(\beta_c\epsilon/2)\frac{d\epsilon}{\epsilon}, \quad (5.7a)$$

or

$$kT_c=1.14\xi\exp(-1/N_0U_l). \quad (5.7b)$$

Again, it is straightforward to verify that the formal dependence of T_c upon the cutoff parameter ξ actually cancels out by virtue of (3.13). It is interesting to compare this critical temperature with the "gap" parameter ϵ_0 defined in the previous section. We have listed the values of the dimensionless ratio $2\epsilon_0/kT_c$ for a few different angular configurations (Table III).

TABLE III. Critical temperatures and specific heats of some generalized BCS states.

Configuration $C(\theta,\varphi)$	$2\epsilon_0/kT_c$	$\left(\frac{C-\gamma T_c}{\gamma T_c}\right)_{T=T_c}$
$l=0$	3.50	1.55
$l=1$, $m=1$	4.03	1.29
$l=2$ $m=1$, or 2	4.20	1.085
$l=2$ state (4.12)	4.93	
$l=3$, $m=2$	4.26	1.055

Specific Heat

The total energy of the system at finite temperature is evidently

$$E=\sum_\mathbf{k}\frac{E_\mathbf{k}e^{-\beta E_\mathbf{k}}-E_\mathbf{k}e^{\beta E_\mathbf{k}}}{e^{\beta E_\mathbf{k}}+e^{-\beta E_\mathbf{k}}+2}=-\sum_\mathbf{k}E_\mathbf{k}\tanh(\beta E_\mathbf{k}/2), \quad (5.8)$$

and the specific heat C is therefore

$$C=\frac{dE}{dT}=2k\beta^2\sum_\mathbf{k}\frac{e^{\beta E_\mathbf{k}}}{(1+e^{\beta E_\mathbf{k}})^2}\left(E_\mathbf{k}^2+\beta E_\mathbf{k}\frac{dE_\mathbf{k}}{d\beta}\right). \quad (5.9)$$

In the low-temperature limit, the modulus Δ of $C(\theta,\varphi)$ becomes approximately constant so that both $d(E_k^2)/dT$ and (*a fortiori*) $T[d(E_k^2)/dT]$ vanish; the expression for the specific heat reduces then to

$$C_{T\to0}=2k\beta^2\sum_k\frac{E_\mathbf{k}^2e^{\beta E_\mathbf{k}}}{[1+e^{\beta E_\mathbf{k}}]^2}. \quad (5.10)$$

It is straightforward to see that (5.10) gives the ex-

[18] N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Academy of Sciences of U.S.S.R., Moscow, 1958).

pected result for the normal Fermi fluid, if E_k is replaced by ϵ_k[19]

$$C_n=4J_2k^2N_0T=\gamma T. \tag{5.11}$$

For a condensed system, however, the excitation energy spectrum is strongly perturbed near the Fermi level and a quite different temperature dependence is expected. One finds, respectively:

p-Type Condensed State ($l=1$, $m=1$):

$$C=\frac{J_4}{J_2}\left(\frac{kT_c}{\epsilon_0}\right)^2(\gamma T_c)\left(\frac{T}{T_c}\right)^3=3.5(\gamma T_c)\left(\frac{T}{T_c}\right)^3.$$

d-Type Condensed State [*State* (4.12)]:

Since the statistical factor on the right-hand side of (5.9) or (5.10) is strongly peaked at $E_k=0$, the domain of integration practically splits into the 8 subdomains Σ_1 near the nodes of $|C(\theta,\varphi)|$. One can use the same technique as in the last paragraph of Sec. IV and one finds:

$$C=\frac{2J_4}{J_2}\left(\frac{kT_c}{\epsilon_0}\right)^2(\gamma T_c)\left(\frac{T}{T_c}\right)^3=4.6(\gamma T_c)\left(\frac{T}{T_c}\right)^3. \tag{5.12}$$

It must be emphasized that the above results have been derived by taking into account only the contribution of individual-particle excitation modes of the system and neglecting all collective excitations. This procedure is rigorous in the case of superconducting electrons only, because the long-range Coulomb repulsive force prevents in this case the existence of low-energy collective excitation modes lying in the gap. It is not so for a system of neutral particles (with a short-range interaction) and it has indeed been shown by Anderson[7] that a neutral Fermi gas condensed into a s-type state has low-energy excitation modes corresponding to long-wavelength (small wave vector $\mathbf{q}$) longitudinal waves:

$$\omega=\frac{1}{\sqrt{3}}\frac{k_Fq}{m}. \tag{5.13}$$

On the other hand, the spectrum of collective excitation modes in the case of helium-3 (anisotropic d-type condensed state) is certainly more complicated than the simple case of the isotropic neutral Fermi gas. In addition to longitudinal waves, there will be transverse or "rotational" waves corresponding to a gradual rotation, in space, of the anisotropy axis of the condensed-state configuration. These rotational modes are rather analogous to spin-waves in the usual ferromagnets and one expects therefore that their frequency may be proportional to the square of their wave vector $\mathbf{q}$; furthermore, one can estimate the energy necessary to cause this gradual rotation by noticing that all condensation energy will be lost if a 2π rotation takes place within a distance of the order of the cohesion length r_c (see next section). The energy spectrum of the rotational modes will therefore be very approximately

$$\omega\simeq\Delta r_c^2q^2. \tag{5.14}$$

It is straightforward to show that the longitudinal modes (5.13) give a small contribution to the specific heat:

$$C_{\text{long}}=0.04(\gamma T_c)(T/T_c)^3, \tag{5.15}$$

of the order of 1% of the contribution (5.12) of individual particle excitations. On the other hand, the total energy of the rotational modes at the temperature T is the statistical average:

$$E_{\text{rot}}=\int_0^\infty\frac{\omega}{e^{\beta\omega}-1}\frac{dn}{d\omega}d\omega, \tag{5.16}$$

where n is the number of modes per unit volume with an energy smaller than ω: $n=q^3/6\pi^2$. We obtain from relation (5.14):

$$E_{\text{rot}}\simeq\frac{1}{4\pi^2\Delta^{\frac{1}{2}}r_c^3}\int_0^\infty\frac{\omega^{\frac{1}{2}}d\omega}{e^{\beta\omega}-1}=\frac{1.8(kT)^{\frac{5}{2}}}{4\pi^2\Delta^{\frac{1}{2}}r_c^3},$$

and therefore:

$$C_{\text{rot}}=\frac{d(E_{\text{rot}})}{dT}\simeq\frac{4.5k}{4\pi^2r_c^3}\left(\frac{kT}{\Delta}\right)^{\frac{3}{2}}. \tag{5.17}$$

This contribution to the specific heat vanishes like $T^{\frac{3}{2}}$ only and will therefore eventually dominate in the low-temperature limit; comparing the expressions (5.12) and (5.17), and noting that the cohesion length is of the order of $k_F^{-1}\epsilon_F/\epsilon_0$, one finds in the case of helium-3:

$$\frac{C_{\text{rot}}}{C}\simeq0.012\left(\frac{\epsilon_0}{\epsilon_F}\right)^2\left(\frac{T}{T_c}\right)^{-\frac{3}{2}}=6\times10^{-5}\left(\frac{T}{T_c}\right)^{-\frac{3}{2}}. \tag{5.18}$$

This ratio is of the order of unity only when the temperature T is of the order of $10^{-3}T_c$ and, consequently, the contribution to the specific heat from these rotational collective modes becomes important in a range of temperature well below the transition temperature and also well below the temperature which could conceivably be attained by the present day cryogenic techniques. These estimates of the contribution of collective excitation modes of the system justify then our claim that they are negligible in the interesting range of temperatures.

It is also interesting to evaluate the specific heat near the critical temperature T_c. In this limit, one can easily derive the expression for $d(E_k^2)/d\beta$ from Eq. (5.5) if one assumes that the configuration $f(\theta,\varphi)$ of the

[19] We have used here the definite integrals:

$$J_n=\int_0^\infty\frac{t^ne^t dt}{(1+e^t)^2}=\int_0^\infty\frac{nt^{n-1}dt}{1+e^t}.$$

The first four J_n are: $J_1=0.693$; $J_2=1.64$; $J_3=5.4$, and $J_4=23$.

stable state does not change; that is to say,

$$\frac{d(E_k^2)}{d\beta}=|f(\theta,\varphi)|^2\frac{d(\Delta^2)}{d\beta}. \tag{5.19}$$

Using the result derived by BCS in the $l=0$ case, one finds

$$\left[\frac{d(\Delta^2)}{d\beta}\right]_{T=T_c}=\frac{10.2}{\beta_c^3\int|Y_{lm}|^2|f|^2d\Omega}. \tag{5.20}$$

We have listed the values of the dimensionless ratio $(C-\gamma T_c)/\gamma T_c$ which measures the relative magnitude of the specific heat discontinuity at the critical temperature, for the simple configurations $\Delta_m Y_{lm}(\theta,\varphi)$ up to $l=3$ (Table III). We would expect the specific heat discontinuity to be of the same order for the ground-state configuration (4.12) as for the Y_{22} configuration; this, however, is only of academic interest since it appears that the actual transition of liquid helium-3 into a condensed state certainly cannot take place at the computed critical temperature T_c (see Sec. VIII).

Paramagnetic Susceptibility

Since the condensation into an even l configuration is based on the formation of pairs of particles with antiparallel spins, one expects that the freedom of the individual-particle spins and, therefore, the over-all paramagnetic susceptibility are reduced in the condensed state. At the absolute zero, the spins have no freedom at all since all particles must be paired and the susceptibility must vanish. On the other hand, condensation into an odd-l configuration leaves the spin-up and spin-down populations uncoupled (exactly like the Fermi state) and therefore the susceptibility of a p-type of f-type condensed fluid is essentially the same as the susceptibility of a normal fluid.[20] Let us then restrict our attention to the case of antiparallel pairing. The excitations of the subsystem $\mathbf{k}\uparrow$, $-\mathbf{k}\downarrow$ are still those of Table II but, in this case, the degeneracy of the two single-particle excitations is lifted by the paramagnetic interaction, their energies are now $+\mu H$ and $-\mu H$, respectively (μ is the nuclear magnetron in the case of helium-3). The average paramagnetic moment at the temperature T is then

$$M_{\rm av}=\mu\sum_{\mathbf{k}}\frac{e^{\beta\mu H}-e^{-\beta\mu H}}{e^{\beta E_k}+e^{-\beta E_k}+e^{\beta\mu H}+e^{-\beta\mu H}}$$

$$=\mu\sum_{\mathbf{k}}\frac{\sinh(\beta\mu H)}{\cosh(\beta E_k)+\cosh(\beta\mu H)}. \tag{5.21}$$

[20] The susceptibility may be slightly increased for an odd l condensed state, if the coupling is strong enough to yield a sizeable increase of the condensation energy when the two-spin populations become unequal.

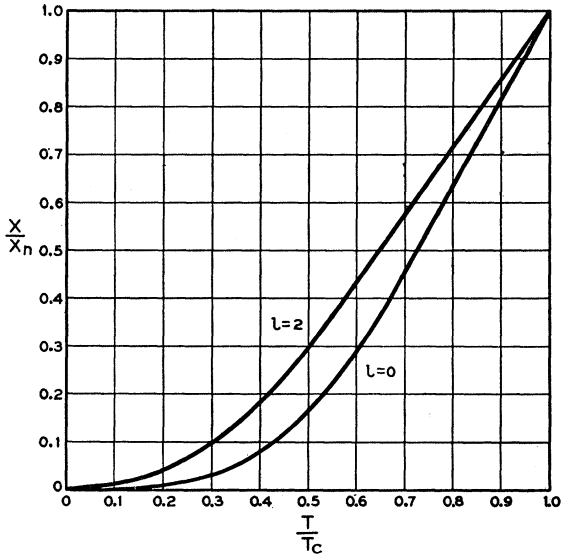

FIG. 3. Comparison of the variation of the paramagnetic susceptibility versus temperature for an s-type condensed fluid ($l=0$) and a d-type condensed fluid ($l=2$).

This expression is identical to the expression derived by Nosanov and Vasudevan[21] by Bogoliubov's method. In the weak-field limit, (5.21) reduces to the expression found by Yosida for the special case $l=0$[22]:

$$M_{\rm av}=\chi H=2\beta\mu^2 H\sum_{\mathbf{k}}\frac{e^{\beta E_k}}{(e^{\beta E_k}+1)^2}. \tag{5.22}$$

It must be emphasized that, in view of the smallness of the nuclear magneton, one is practically always in the weak field limit: Fields of the order of 10^5 oersteds must still be considered as "weak" at 0.01°K. Again, the summation on the right-hand side of (5.22) involves a statistical factor strongly peaked at $E_k=0$ and therefore the integration can be performed by the same technique as for the specific heat (5.10). One finds that the paramagnetic susceptibility vanishes in the low-temperature limit [like $1.08(T/T_c)^2$] as expected (see Fig. 3).

VI. DENSITY AND CURRENT-DENSITY CORRELATION IN THE GROUND STATE

One expects, in the case of the weakly interacting condensed fluid we are considering, that the condensation is a rather long-range cooperative phenomenon, involving the coherent ordering of the momenta of a large number of particles; one expects, in other words, that the condensed state is characterized by a special long-range correlation between the particles. Furthermore, the average particle current around a given particle does not need to vanish everywhere as in a

[21] L. H. Nosanov and R. Vasudevan, Phys. Rev. Letters **6**, 1 (1961).

[22] K. Yosida, Phys. Rev. **110**, 769 (1958).

normal fluid or an isotropic condensed fluid, since the usual symmetry argument does not apply in the case of the anisotropic configurations under discussion. It is therefore instructive to evaluate the average particle density and the average current density versus the distance from a given particle, that is to say, the mean values of the operators $n(\mathbf{x})n(\mathbf{x}')$ and $n(\mathbf{x})\mathbf{j}(\mathbf{x}')$ in the ground state; n and $\mathbf{j}$ are the usual density and current density operators in second quantization:

$$n=\psi^*\psi,$$
$$\mathbf{j}=-(i/2)\hbar[\psi^*\nabla\psi-(\nabla\psi^*)\psi]. \tag{6.1}$$

The ground state is the vector

$$\Psi_g=\prod_{\mathbf{k}}[\cos(\chi_{\mathbf{k}}/2)e^{i\psi_{\mathbf{k}}/2}+\sin(\chi_{\mathbf{k}}/2)e^{-i\psi_{\mathbf{k}}/2}b_{\mathbf{k}}^*]\Psi_{\text{vac}}, \tag{6.2}$$

where $b_{\mathbf{k}}^*$ is the creation operator for the pair $\mathbf{k}, -\mathbf{k}$ with parallel (antiparallel) spins for odd (even) l configurations. To simplify the matter, we shall only consider here the case of antiparallel pairing; the same results would be found in the case of parallel pairing, only with the opposite spin combination. Expanding the operators n in Fourier series, we obtain immediately:

$$\langle n(\mathbf{x},\sigma)n(\mathbf{x}',\sigma')\rangle=\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\langle\Psi_g|\,c_{\mathbf{k}_1\sigma}^*c_{\mathbf{k}_2\sigma}c_{\mathbf{k}_3\sigma'}^*c_{\mathbf{k}_4\sigma'}\,|\Psi_g\rangle\times e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{x}}e^{i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}'}. \tag{6.3}$$

We are only interested in the space average of this correlation function, which depends only upon the relative distance $\mathbf{r}=\mathbf{x}'-\mathbf{x}$ and the spins:

$$\langle n(\sigma)n(\sigma')\rangle_{\text{av}}=\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\langle\Psi_g|\,c_{\mathbf{k}_1\sigma}^*c_{\mathbf{k}_2\sigma}c_{\mathbf{k}_3\sigma'}^*c_{\mathbf{k}_4\sigma'}\,|\Psi_g\rangle\times e^{i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{r}}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4). \tag{6.4}$$

In the case of antiparallel pairing, we obtain in a straightforward fashion:

$$\langle n(\uparrow)n(\uparrow)\rangle_{\text{av}}=n[n+P_F(r)],$$
$$\langle n(\uparrow)n(\downarrow)\rangle_{\text{av}}=n^2+\tfrac{1}{4}[\textstyle\sum_{\mathbf{k}}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}]\times[\sum_{\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{-i(\mathbf{k}'\cdot\mathbf{r}+\psi_{\mathbf{k}'})}]. \tag{6.5}$$

Here n is the total density of particles of one spin (up or down) and $P_F(r)$ is the exchange correlation or "Fermi hole." Note that the above results include the correlation effects due to the statistics of the particles and to the condensation but not the spin independent "correlation hole" due to the direct interaction. This many-body effect is, however, included in the formalism from the very beginning, since $c_{\mathbf{k}}^*$ creates a quasi-particle, that is to say, a particle dressed by its cloud of correlated particles. We have implicitly assumed that this rather short-range many-body effect, which determines the effective mass of the quasi-particles, is essentially unaffected by the transition into a condensed state. In other words, we are specifically considering the case where the condensation is a long-range phenomenon in which the close region (in position space) does not play a major role: the short-range "correlation hole" and the long-range correlation due to condensation:

$$\langle nn\rangle=\tfrac{1}{4}|\textstyle\sum_{\mathbf{k}}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}|^2, \tag{6.6}$$

are well decoupled in position space. Let us remember that this correlation (6.6) affects particles with opposite spins in the case of antiparallel pairing and particles with the same spin for parallel pairing. In any case, we see that this extra term in expression (6.5) is the contribution of the terms $\mathbf{k}_1=-\mathbf{k}_3=\mathbf{k}$ and $\mathbf{k}_2=-\mathbf{k}_4=\mathbf{k}'$ in the sum on the right-hand side of (6.4). The same terms also provide a finite contribution to the density-current density correlation $\langle n\mathbf{j}\rangle$ in the condensed state (this term would vanish otherwise). The same derivation leads to

$$\langle n\mathbf{j}\rangle=-\frac{\hbar}{8}\{[\sum_{\mathbf{k}}\mathbf{k}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}][\sum_{\mathbf{k}'}\sin\chi_{\mathbf{k}'}e^{-i(\mathbf{k}'\cdot\mathbf{r}+\psi_{\mathbf{k}'})}]+\text{c.c.}\}$$
$$=-\frac{\hbar}{8}\{\mathbf{J}I^*+\mathbf{J}^*I\}, \tag{6.7}$$

where we have defined the integrals:

$$I(\mathbf{r})=\textstyle\sum_{\mathbf{k}}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}, \tag{6.8a}$$
$$\mathbf{J}(\mathbf{r})=\textstyle\sum_{\mathbf{k}}\mathbf{k}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}. \tag{6.8b}$$

We note that the second integral is the gradient of the first multiplied by $(-i)$ so that we shall only need to compute $I(\mathbf{r})$. Let us first remark that if $I(\mathbf{r})$ is written

$$I(\mathbf{r})=|I(\mathbf{r})|e^{i\gamma(\mathbf{r})}, \tag{6.9a}$$

then:

$$\mathbf{J}(\mathbf{r})=-ie^{i\gamma(\mathbf{r})}\nabla_{\mathbf{r}}|I(\mathbf{r})|+I(\mathbf{r})\nabla_{\mathbf{r}}\gamma(\mathbf{r}). \tag{6.9b}$$

Since the first term of this expression of $\mathbf{J}(\mathbf{r})$ cancels in (6.7), we obtain thereby a general relation between the correlation functions $\langle nn\rangle$ and $\langle n\mathbf{j}\rangle$:

$$\langle n\mathbf{j}\rangle=\tfrac{1}{4}\hbar|I|^2\nabla_{\mathbf{r}}\gamma(\mathbf{r})=\hbar\langle nn\rangle\nabla_{\mathbf{r}}\gamma(\mathbf{r}). \tag{6.10}$$

This relation is really all what we need to know if the condensed state under consideration has rotational symmetry around its anisotropy axis 0z (case of a simple lm configuration, such as the ground state for p-type condensation). Indeed one sees directly from the expression (6.8a) by performing the rotation in momentum space which brings the $0zx$ plane onto the radius vector $\mathbf{r}$, that the phase factor of $I(\mathbf{r})$ is simply $m\varphi$ (φ is the azimuth angle of $\mathbf{r}$ around the z axis). Then:

$$\langle nj\rangle_{\varphi}=(m\hbar/r)\langle nn\rangle. \tag{6.11}$$

That is to say, that in this simple case each particle P of the condensed fluid attracts a small "hump" of correlated particles and that this hump rotates around the anisotropy axis originating at P, according to the law

of equal swept areas (velocity decreasing as $1/r$). The total angular momentum of this correlation current is therefore proportional to the total number of correlated particles n_c:

$$n_c=\int\int\int_{\mathbf{k}\mathbf{k}'\mathbf{r}} \sin\chi_{\mathbf{k}} \sin\chi_{\mathbf{k}'} e^{i(\psi_{\mathbf{k}}-\psi_{\mathbf{k}'})} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \frac{d\tau_{\mathbf{k}}}{8\pi^3}\frac{d\tau_{\mathbf{k}'}}{8\pi^3}d\tau$$

$$=\int\int_{\mathbf{k}\mathbf{k}'} \sin\chi_{\mathbf{k}} \sin\chi_{\mathbf{k}'} e^{i(\psi_{\mathbf{k}}-\psi_{\mathbf{k}'})}\delta^3(\mathbf{k}-\mathbf{k}')\frac{d\tau_{\mathbf{k}}}{8\pi^3}d\tau_{\mathbf{k}'}$$

$$=\int_{\mathbf{k}} \sin^2\chi_{\mathbf{k}'}\frac{d\tau_{\mathbf{k}}}{8\pi^3}. \tag{6.12}$$

A straightforward integration shows that n_c is of the order of $N_0\epsilon_0$, that is to say of the order of ϵ_0/ϵ_F times the total density of particles n. The total angular momentum of the correlation currents in a volume V of the condensed fluid, is in this case:

$$I=m\hbar n_c V\cong m\hbar(\epsilon_0/\epsilon_F)nV. \tag{6.13}$$

We notice, however, that this spontaneous circulation in the ground state is not a volume effect [to the contrary of what expression (6.13) seems to suggest] because the correlation current around one particle inside the fluid is exactly canceled by the currents around the neighboring particles. On the other hand, this cancellation does not obtain on the boundary of the sample and, if the same anisotropy axis is retained on the surface of the fluid (completely ordered state, presumably stable at absolute zero), these uncanceled correlated particles currents add up to form a sheet current on the surface, the total angular momentum of which is just I as given by the relation (6.13).

This discussion is not applicable to the true $l=2$ ground state (4.12), however, since the corresponding angular dependent energy gap function $C(\theta,\varphi)$ has no rotational symmetry. We need therefore to compute $I(\mathbf{r})$ explicitly. The interesting region in position space is the long-range region, where $k_F r$ is much larger than 1; this suggests computing $I(\mathbf{r})$ by the so-called stationary phase technique (see Appendix C). We have found:

$$I(\mathbf{r})=\frac{2N_0\cos(k_F r)}{ik_F r}K_0\left(\frac{r|C(\hat{r})|}{\hbar v_F}\right)C(\hat{r}), \tag{6.14}$$

where K_0 is the modified Bessel function of the second kind and order zero. This remarkable result shows that the long-range correlation $\langle nn\rangle$ in a given direction depends only upon the value of the gap function $C(\theta,\varphi)$ in that direction and, of course, upon the distance r. We note particularly that the correlation vanishes in the directions of the nodes of $|C(\hat{r})|$. The radial dependence is dominated, in the long-range limit, by the exponential K_0 function, the extension of which is of the order of $\hbar v_F/|C|$. The range of the correlation function $\langle nn\rangle$ is therefore anisotropic and extends to infinity in the direction of the nodes of $|C(\hat{r})|$. We may, however, define an average coherence length r_c:

$$r_c\simeq\frac{\hbar v_F}{\Delta}\simeq\frac{\epsilon_F}{\epsilon_0}k_F^{-1}. \tag{6.15}$$

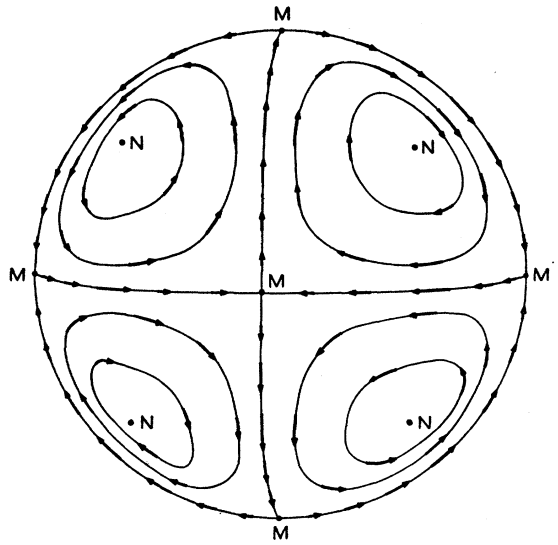

FIG. 4. Correlation current pattern at a fixed distance from a particle for the d-type ground-state configuration (4.12). The points N are the nodes of the angle-dependent gap function $|C(r)|$; M are its maxima.

Going back to the general expression (6.10) of the correlated particles' current density, we note that the only complex factor in the expression (6.14) of $I(r)$ is $C(\hat{r})$; consequently, the correlation current has no radial component. It is straightforward to compute the tangential components from (6.14) for any gap function $C(\theta,\varphi)$. For the ground-state configuration (4.12), we find:

$$\langle nj\rangle_\varphi=\hbar\langle nn\rangle\frac{5\sqrt{3}}{16\pi|f|^2}\sin^2\theta(3\cos^2\theta-1)\cos2\varphi,$$
$$\langle nj\rangle_\theta=\hbar\langle nn\rangle\frac{5\sqrt{3}}{16\pi|f|^2}\sin2\theta\sin2\varphi. \tag{6.16}$$

We have sketched the correlation current pattern on a sphere for this configuration (Fig. 4). Note that the current flows around the nodes of the gap function $C(\hat{r})$ and vanishes at both the nodes and the maxima of $C(\hat{r})$. These currents add up in such a way that the whole distribution has no net angular momentum about any axis. Under these circumstances, there will be considerable, if not complete, cancellation of the correlated particle currents at the surface unless the surface is strongly curved. In conjunction with the results of the next section, it appears that the anisotropy of the d-type ground-state configuration (4.12) is of such a

subtle character that it may be rather difficult to detect it in physical measurements.

VII. FLOW PROPERTIES, SUPERFLUIDITY

As we mentioned in the Introduction, the experimental investigation of liquid helium-3 properties has not yet disclosed any departure from a normal Fermi liquid behavior; particularly, it has been found that the flow properties of helium-3 are those of a normal fluid, that is to say that helium-3 does not participate in the superfluid flow of helium-4 in mixtures of both isotopes.[23] If, however, a condensation of the type we have been discussing above does indeed occur at very low temperature, one expects that liquid helium-3 is not a normal fluid near the absolute zero. The analogy with the Meissner effect of superconducting electrons condensed into the $l=0$ configuration indicates that condensed helium-3 would be superfluid. We shall show by a straightforward generalization of the BCS derivation of the Meissner effect that the alteration of the excitation energy spectrum brought about by the condensation entails superfluidity, in accordance with the suggestion of Brueckner *et al.*[3] The same result has been found by Glassgold and Sessler by a somewhat different method.[8]

The most convenient theoretical approach to superfluidity is to consider the flow of the fluid in a rotating container. Since we do not know how to treat the interaction between the fluid and the rotating wall of the container in the laboratory coordinate system, we shall transform our problem into the rotating coordinate system in which the container is at rest; as Blatt *et al.*[24] pointed out we have to perform this canonical transformation to rotating coordinates in order to do statistical mechanics at all. This, of course, is identical to the "cranking model" procedure proposed by Inglis[25] for nuclei. The new (transformed) Hamiltonian is

$$H'=H-\boldsymbol{\omega}\cdot\mathbf{L}, \tag{7.1}$$

where $\mathbf{L}$ is the total angular momentum of all particles. Note that H' is not numerically equal to the Hamiltonian H of the system at rest; the extra term is a perturbation of the first order in the rotation speed ω, corresponding to the Coriolis forces. Using the relation:

$$\boldsymbol{\omega}\cdot(\mathbf{r}_i\times\boldsymbol{\nabla}_i)=(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\boldsymbol{\nabla}_i, \tag{7.2}$$

the perturbation can be written

$$-\boldsymbol{\omega}\cdot\mathbf{L}=i\hbar\sum_{\text{all particles}}(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\boldsymbol{\nabla}_i. \tag{7.3}$$

Let us introduce the vector field $\mathbf{A}(\mathbf{r})$, with $\mathbf{A}$ as the speed of a point attached to the rotating coordinate system with respect to the fixed coordinate system:

$$\mathbf{A}(\mathbf{r})=\boldsymbol{\omega}\times\mathbf{r}. \tag{7.4}$$

In second quantization, the perturbation to the Hamiltonian is then

$$H_R=\int\Psi^*[i\hbar\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\nabla}]\Psi d\tau$$
$$=-(2\pi)^{\frac{3}{2}}\sum_{\mathbf{k},\mathbf{q},\sigma}\hbar\mathbf{k}\cdot\mathbf{a}(\mathbf{q})c_{\mathbf{k}+\mathbf{q},\sigma}{}^*c_{\mathbf{k}\sigma}, \tag{7.5}$$

where we have used the plane wave expansions of the operators Ψ and Ψ^*, and where $\mathbf{a}(\mathbf{q})$ is the (formal) Fourier transform of $\mathbf{A}(\mathbf{r})$. This procedure is not strictly correct since the plane waves fail to satisfy the boundary conditions of the system (cylindrical container) and a Fourier-Bessel expansion of the operators Ψ and Ψ^* should be used instead. We expect, however, that the conclusions we shall draw from the comparison of the (formal) Fourier transform of the current operator $\mathbf{j}(\mathbf{r})$ with the (formal) Fourier transform of the speed field $\mathbf{A}(\mathbf{r})$ will be strongly indicative of the exact behavior of the model. Let

$$\mathbf{j}(\mathbf{r})=\frac{\hbar}{2m}\sum_{\mathbf{k},\mathbf{q},\sigma}(2\mathbf{k}+\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{r}}c_{\mathbf{k}+\mathbf{q},\sigma}{}^*c_{\mathbf{k}\sigma} \tag{7.6}$$

be the expansion of the current density operator (6.2). Since the expectation value of the current vanishes in the system at rest, this expectation value in the rotating system is, in the first order in ω,

$$\mathbf{j}(\mathbf{r})=\sum_i\left[\frac{\langle\Psi_0|H_R|\Psi_i\rangle\langle\Psi_i|j|\Psi_0\rangle}{E_0-E_i}+\text{c.c.}\right], \tag{7.7}$$

where Ψ_0 is the ground-state wave function at the temperature T (energy E_0):

$$\Psi_0=\prod_{\text{ground pairs}}\left(\cos\frac{\chi_{\mathbf{k}}}{2}e^{i\psi_{\mathbf{k}}/2}+\sin\frac{\chi_{\mathbf{k}}}{2}e^{-i\psi_{\mathbf{k}}/2}b_{\mathbf{k}}{}^*\right)$$
$$\times\prod_{\text{excited pairs}}\left(\cos\frac{\chi_{\mathbf{k}'}}{2}e^{-i\psi_{\mathbf{k}'}/2}b_{\mathbf{k}'}{}^*-\sin\frac{\chi_{\mathbf{k}'}}{2}e^{i\psi_{\mathbf{k}'}/2}\right)$$
$$\times\prod_{\text{single particles}}(c_{\mathbf{k}''}{}^*)|\Psi_{\text{vac}}\rangle. \tag{7.8}$$

The summation in (7.7) extends over all excited states Ψ_i (energy E_i). Fortunately, states corresponding to different anisotropy axes are orthogonal,[26] so that the angular degeneracy can be disregarded. We remark furthermore that the perturbation H_R due to the rotation of the coordinate system is formally identical to the magnetic perturbation considered by BCS. We shall

[23] J. G. Daunt, R. E. Probst, H. L. Johnson, and L. T. Aldrich, Phys. Rev. **72**, 502 (1947), and J. Chem. Phys. **15**, 759 (1947).
[24] J. M. Blatt, S. T. Butler, and M. R. Schafroth, Phys. Rev. **100**, 481 (1955).
[25] D. R. Inglis, Phys. Rev. **96**, 1059 (1954).
[26] The scalar product of two states Ψ_i and Ψ_i', corresponding to different z axes, is an infinite product of factors including at least one zero (for the individual particle state $\mathbf{k}$ such that $\chi_{\mathbf{k}}=\chi_{\mathbf{k}}'=\pi/2$, $\psi_{\mathbf{k}}=\psi_{\mathbf{k}}'\pm\pi$).

therefore obtain a formally identical result:

$$\mathbf{j}'(\mathbf{r})=\frac{\hbar^2}{2m}(2\pi)^{\frac{3}{2}}\sum_{\mathbf{k},\mathbf{q}}(2\mathbf{k}+\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{r}}[\mathbf{a}(-\mathbf{q})\cdot\mathbf{k}]\times L(\mathbf{k},\mathbf{k}+\mathbf{q}), \quad (7.9)$$

where

$$L(\mathbf{k},\mathbf{k}')=2\sum_i\frac{|\langle\Psi_0|c_{\mathbf{k}'}{}^*c_{-\mathbf{k}}{}^*c_{-\mathbf{k}'}|\Psi_i\rangle|^2}{E_0-E_i}. \quad (7.10)$$

Even for nonspherically symmetric distributions, $L(\mathbf{k},\mathbf{k}')$ has a simple expression in the limit $\mathbf{k}'\to\mathbf{k}$:

$$\lim_{\mathbf{k}'\to\mathbf{k}}L(\mathbf{k},\mathbf{k}')=-\frac{df_{\mathbf{k}}}{dE_{\mathbf{k}}}=\frac{e^{\beta E_{\mathbf{k}}}}{(1+e^{\beta E_{\mathbf{k}}})^2}. \quad (7.11)$$

The (formal) Fourier transform of the current is then in the long-wavelength limit:

$$\lim_{\mathbf{q}\to0}\mathbf{j}'(\mathbf{q})=\frac{2\hbar^2\beta}{m}\sum_{\mathbf{k}}\mathbf{k}[\mathbf{a}(\mathbf{q})\cdot\mathbf{k}]\frac{e^{\beta E_{\mathbf{k}}}}{(1+e^{\beta E_{\mathbf{k}}})^2}. \quad (7.12)$$

Since the statistical factor is strongly peaked at $E=0$, the integration here reduces practically to an integration in a shell near the Fermi surface. For a normal Fermi fluid, we can replace $\mathbf{k}[\mathbf{a}(\mathbf{q})\cdot\mathbf{k}]$ by the angular average $\frac{1}{3}k_F{}^2\mathbf{a}(\mathbf{q})$, and we find the expected result:

$$\lim_{\mathbf{q}\to0}\mathbf{j}_n'(\mathbf{q})=2\beta\frac{k_F{}^3}{3\pi^2}\mathbf{a}(\mathbf{q})\int_0^\infty\frac{e^{\beta\epsilon}d\epsilon}{(1+e^{\beta\epsilon})^2}=n\mathbf{a}(\mathbf{q}). \quad (7.13)$$

This result (7.13) means, of course, that the thermodynamically stable state of a fluid in a rotating container is such that the fluid is at rest with respect to the container, that is to say, rotates at the same speed as the container:

$$\mathbf{j}_n(\mathbf{r})=n\mathbf{A}(\mathbf{r}). \quad (7.13a)$$

We shall show that it is not so for a condensed fluid and more precisely that the statistical average $\mathbf{j}'$ of the current density vanishes in the low-temperature limit for finite rotation speed of the container; in other words, the fluid at large (i.e., as far as large-scale motions, corresponding to the long-wavelength Fourier components, are concerned) does not participate in the motion of the container at 0°K. In accordance with the usual meaning of superfluidity in the context of oscillating-disk[27] experiments, for example, we can characterize this behavior as "superfluid behavior."

For the anisotropic condensed state (4.12), we cannot use an angular average of the term $\mathbf{k}(\mathbf{a}\cdot\mathbf{k})$ as above and we must instead compute separately the three components of $\mathbf{j}'$. Since $E_{\mathbf{k}}$ depends only upon $\sin^2 2\varphi$, all but the following terms cancel upon integration over φ:

$$\lim\begin{pmatrix}j_x'\\ j_y'\\ j_z'\end{pmatrix}=\frac{\beta k_F}{4\pi^3}\int d\Omega\int_0^\infty d\epsilon\begin{pmatrix}a_xk^2\sin^2\theta\\ a_yk^2\sin^2\theta\\ 2a_zk^2\cos^2\theta\end{pmatrix}\frac{e^{\beta E_{\mathbf{k}}}}{(1+e^{\beta E_{\mathbf{k}}})^2}. \quad (7.14)$$

Because the statistical factor is strongly peaked at $E_{\mathbf{k}}=0$ in the low-temperature limit, the domain of integration practically reduces to the eight subdomains Σ_1 near the nodes of $|C(\theta,\varphi)|$ (see the last paragraph of Sec. IV). Replacing $k^2\sin^2\theta$ and $k^2\cos^2\theta$ by their average values at the node, respectively $\frac{2}{3}k_F{}^2$ and $\frac{1}{3}k_F{}^2$, we can therefore replace (7.14) by

$$\lim\mathbf{j}'(\mathbf{q})=4\beta\frac{k_F{}^3}{3\pi^3}\mathbf{a}(\mathbf{q})\int d\Omega\int_0^\infty d\epsilon\frac{e^{\beta E_{\mathbf{k}}}}{(1+e^{\beta E_{\mathbf{k}}})^2}=4J_2n\mathbf{a}(\mathbf{q})(kT/\epsilon_0)^2 \quad (7.15)$$

or, more conveniently:

$$\lim_{q\to0,\,T\to0}\left(\frac{\mathbf{j}'(\mathbf{q})}{\mathbf{j}_n'(\mathbf{q})}\right)=1.08(T/T_c)^2. \quad (7.15a)$$

Note that the variation of the long-wavelength Fourier components of the current density versus temperature is approximately the same as the variation of the paramagnetic susceptibility (see Fig. 3).

The reason why (7.15) is isotropic is, of course, that the ground-state (4.12) has cubic symmetry for $|f(\theta,\varphi)|^2$ and thus for E_k. It appears that this is a special circumstance of the $l=2$ case; but we do expect that qualitatively the stage will attempt to be as isotropic as is consistent with l.

VIII. CONDENSED STATE OF A NONIDEAL FERMION GAS

We shall now specialize to the case of liquid helium-3 and indicate the properties of the condensed phase in the low-temperature limit. Previous analysis by Brueckner *et al.*[3] and Emery and Sessler[4] have indicated that the most favorable condensed state configuration is d-type ($l=2$). The result of Brueckner *et al.* is given in terms of the K-matrix formalism of Brueckner and Gammel[28] which takes into account the usual many-body effects; it is found that the $l=2$ coefficient of the expansion of the K matrix in terms of spherical harmonics is positive and the largest[29] and therefore the largest condensation energy is obtained for a p-type configuration. On the other hand, Emery and Sessler have solved Eq. (5.5) in the limit $T\to T_c$ (where this equation becomes linear) and have found that the critical temperature for the transition into an $l=2$ con-

[27] E. L. Andronikashvili, J. Phys. Moskow **10**, 201 (1946); see also K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959), for further references.

[28] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[29] The magnitude of the $l=2$ coefficient is approximately equal to the magnitude of the (repulsive) $l=0$ coefficient; this gives us a very useful indication of the relative magnitude of the coefficients U_0 and U_2.

densed state is the highest; these authors have estimated the critical temperature T_c on the basis of the best phenomenological potential known and of the quasi-particle excitation spectrum computed by Brueckner and Gammel[28]; they have found a probable value of 0.07°K. It must be emphasized, however, that this result is quite sensitive to even small discrepancies between the phenomenological potential (adjusted to fit low-pressure compressibility data) and the true interaction potential in a dense phase. Particularly, any screening would have the effect of reducing the long-range attractive part of the potential and therefore decreasing the transition temperature. We shall nevertheless use this value 0.07°K for T_c, as an indication of the strength of the effective interaction U_2 (possibly too large by a significant amount).

Thermal Properties in the Low-Temperature Limit

Knowing the ground-state configuration (4.12) for d-type condensation, we can correlate the thermal properties of the fluid near 0°K, with the interaction parameter U_2 or equivalently, the critical temperature T_c. We have found that the individual particle excitation spectrum is strongly modified in a shell of thickness ϵ_0 near the Fermi surface:

$$\epsilon_0 = 2.46kT_c = 0.17°\text{K}. \tag{8.1}$$

Since ϵ_0 is not truly negligible with respect to the normal fluid Fermi energy $\epsilon_F = 2.5°\text{K}$,[30] the weak-coupling assumption on which this treatment is based does not hold well. One expects, however, that the rather large coupling does not affect significantly the results derived in the low-temperature limit, since only excitation modes very close to the Fermi level are important in this limit. One can, for example, estimate accurately the amount of condensation energy per particle at 0°K with the help of (4.13):

$$\frac{W_0}{n} = -\frac{3}{32\pi}\frac{\Delta^2}{\epsilon_F} \simeq 2\times 10^{-3}\ °\text{K}. \tag{8.2}$$

On the other hand, higher energy individual particle excitations $b_{\mathbf{k}}^*$ or $c_{\mathbf{k}}^*$ (excitation energy of the order of ϵ_0) are no longer well-behaved quasi-particles: they rapidly lose their individuality and disintegrate into many-particle excitation modes by the process of inelastic scattering. The relaxation time τ for this process can be estimated from the spin diffusion data reported by Wheatley and his co-workers[5]: one finds that τ is still as small as 6×10^{-11} second at 0.03°K (corresponding to an energy broadening $\hbar/\tau = 0.12°\text{K}$, of the order of magnitude of ϵ_0). Consequently, our treatment breaks down at temperatures of the order of T_c where excitations with an energy of the order of ϵ_0 become important; we shall see particularly that the scattering has the effect of destroying the pairs almost as soon as they are formed and, therefore, prevents their accumulation and the transition of the system into a condensed state. One expects, then, that the actual transition temperature will be significantly below the critical temperature T_c for an ideal gas.

[30] We use here the value: $n = 1.64\times 10^{22}$ particles/cm³ derived from the molecular volume data obtained by L. Goldstein [Phys. Rev. **117**, 375 (1960)] and the effective mass $m^* = 2m$ computed by Emery and Sessler (see reference 4).

Effect of Scattering on the Transition Temperature

Following an argument first proposed by Suhl,[6] we shall represent the scattering by adding an imaginary part $i(h/\tau)$ to the individual particle excitation energy $\epsilon_{\mathbf{k}}$ and we shall investigate the condition for stability of the Fermi state with respect to the formation of pairs. Let then

$$[H, b_{\mathbf{k}}^*] = \omega b_{\mathbf{k}}^* = 2\left(\epsilon_{\mathbf{k}} + i\frac{h}{\tau}\right) b_{\mathbf{k}}^* - 2(1-2f_{\mathbf{k}}) \sum_{\mathbf{k}'}^{\text{in}} U_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'}^* \tag{8.4}$$

be the equation of motion of the operator $b_{\mathbf{k}}^*$ creating an excited pair, in the presence of the Fermi sea; $f_{\mathbf{k}}$ is the Fermi-Dirac distribution function. In this scheme, the appearance of an imaginary eigenvalue ω characterizes the onset of instability of the Fermi state with respect to the formation of pairs. Defining

$$\Omega = \omega - 2i(h/\tau), \tag{8.5}$$

we derive from (8.4), with the help of the expansion (3.9):

$$b_{\mathbf{k}}^* = -\frac{2\tanh(\beta\epsilon_{\mathbf{k}}/2)}{\Omega - 2\epsilon_{\mathbf{k}}} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=+l} 2\pi U_l Y_{lm}(\theta,\varphi) \times \sum_{\mathbf{k}'}^{\text{in}} Y_{lm}^*(\theta',\varphi') b_{\mathbf{k}'}^*. \tag{8.6}$$

Multiplying both sides of this equation by $Y_{lm}^*(\theta,\varphi)$ and summing over all $\mathbf{k}$, we obtain therefore the dispersion relations for Ω:

$$1 = 4\pi U_l \sum_{\mathbf{k}}^{\text{in}} \frac{|Y_{lm}(\theta,\varphi)|^2 \tanh(\beta\epsilon_{\mathbf{k}}/2)}{\Omega - 2\epsilon_{\mathbf{k}}}. \tag{8.7}$$

Taking advantage of the symmetry with respect to the Fermi level ($\epsilon_{\mathbf{k}} \rightarrow -\epsilon_{\mathbf{k}}$), we can replace the sum on the right-hand side of (8.7) by a sum over the states above the Fermi level only:

$$1 = 4\pi U_l \sum_{|\mathbf{k}|>k_F}^{\text{in}} |Y_{lm}(\theta,\varphi)|^2 \frac{4\epsilon_{\mathbf{k}} \tanh(\beta\epsilon_{\mathbf{k}}/2)}{4\epsilon_{\mathbf{k}}^2 - \Omega^2}. \tag{8.8}$$

Let us first consider the case of negligible scattering where Ω reduces to ω. We have plotted the variation of the right-hand side of (8.8) versus ω^2 in this case (Fig. 5). The discussion of Eq. (8.8) follows immediately:

If $U_l<0$ (repulsive), the equation has only real solutions and consequently, the Fermi state is always stable with respect to the formation of l-type pairs.

If $U_l>0$ (attractive), (8.8) has only real solutions at high temperature but has two conjugate imaginary solutions ($\omega^2<0$) below the critical temperature T_c defined by the relation:

$$1=4\pi U_l \sum_{|\mathbf{k}|>k_F}^{\text{in}} |Y_{lm}(\theta,\varphi)|^2 \frac{\tanh(\beta_c\epsilon_k/2)}{\epsilon_k},$$

identical to (5.6).

These imaginary solutions $+i\omega_2$ and $-i\omega_2$ correspond to a damped perturbation and a diverging perturbation, respectively; in other words, the instability of the Fermi state is characterized by the appearance of an eigenvalue of Eq. (8.8) in the lower half of the complex plane. Let us now go back to the actual case where the scattering is important and the imaginary part $i(h/\tau)$ of the excitation energy is not negligible. Above the critical temperature T_c, the eigenvalues of (8.8) are complex: $\pm\omega_1+2i(h/\tau)$ and describe a damped oscillation (with the relaxation time $\tau/2$). Below T_c, on the other hand, (8.8) has two pure imaginary solutions:

$$\omega=\pm i\omega_2+2i(h/\tau), \tag{8.9}$$

which are *both* in the upper half of the complex plane if the energy broadening h/τ is larger than $\omega_2/2$. We have plotted the variation of both quantities versus temperature (Fig. 6); ω_2 increases rapidly near T_c and then

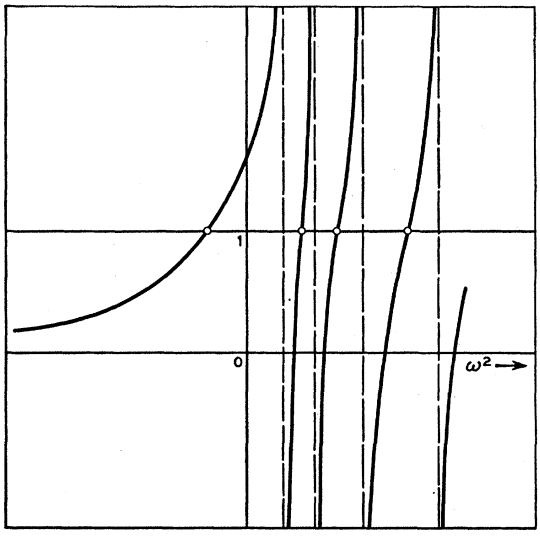

FIG. 5. Plot of the function on the right-hand side of (8.8) versus ω^2. The intersections with the straight line $y=1$ determine the solutions of this equation.

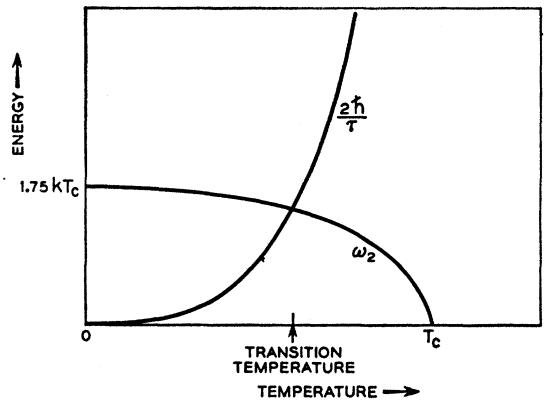

FIG. 6. Plot of the inverse scattering relaxation time and of the imaginary solution of Eq. (8.8) versus temperature. The Fermi state becomes unstable below the crossing point of these two curves (arrow).

levels off and approaches the value $1.75kT_c$ near 0°K; h/τ, on the other hand, follows a $T^{\frac{3}{2}}$ law down to 0.03°K. At this temperature, the lowest attained experimentally, $2h/\tau$ is about twice as large as the limiting value of ω_2 and consequently the Fermi state is still stable. If we can make the reasonable assumption that the $T^{\frac{3}{2}}$ law holds in the 0.01°–0.03°K range, we find that the two curves cross at 0.02°K approximately. One expects therefore, on the basis of this argument, that the actual transition temperature for helium-3 is close to 0.02°K.

Note that for the typical (low temperature) superconductors, the electron-electron scattering is quite small in the range of temperature where the transition occurs, so that the condensation does take place substantially at the critical temperature T_c computed in the weak-coupling limit.

Flow Properties

Since the density-current density correlation does not vanish in the ground state, there are spontaneous macroscopic currents on the boundary of the fluid at rest at 0°K; however, these currents are small (even in one single ordered domain) since they involve only a fraction n_c/n of the particle, that is to say, about 1%. One may argue furthermore that the amount of energy necessary to cause a very gradual rotation of the anisotropy axis is quite small indeed (see the paragraph on Specific Heat in Sec. V); one expects therefore that there will be no long-range order with respect to the orientation of the anisotropy axis, even at extremely low temperature. Since the currents in randomly oriented domains cancel each other, one does not expect that such a spontaneous circulation could be observed. All this is valid for a pure m state; for the cubically symmetric state (4.12) it is true *a fortiori*.

On the other hand, the appearance of a condensed state would be striking if the fluid is in a moving coordinate system (rotating container) since the con-

densed phase is superfluid, according to (7.15). One has found that the condensed fluid current vanishes like T^2 for a given rotation speed in the low-temperature limit. The ground state for d-type condensation is such that this temperature dependence is isotropic (independent of the relative disposition of the anisotropy axis Oz and the rotation axis).

Magnetic Properties

It is interesting to note that the paramagnetic susceptibility χ of the condensed fluid also vanishes like T^2 in the low-temperature limit, following the same variation law as the current in a rotating container:

$$\chi/\chi_n = j/j_n. \tag{8.10}$$

This suggests a two-fluid model of the condensed system. The condensed state is then described as a mixture of a "normal" phase (the relative amount of which is χ/χ_n and vanishes at 0°K) and a superfluid phase which does not participate in the flow of the fluid and has zero paramagnetic susceptibility.

IX. CONCLUSION

Our subject may be separated into two divisions: first, the general question of Fermi gas with reasonably weak interactions which are attractive in pair states $l \neq 0$, and second, the specific properties of liquid helium-3.

As far as the first division is concerned, we have concluded, by means of a generalization of the BCS theory, that there is an anisotropic condensed state of the gas with various characteristic superfluid properties, that is lower in energy than the normal Fermi gas state. This state may be best described as a condensation of Cooper pairs with zero linear momentum, but with an orbital angular momentum depending on the interaction potential. For angular momentum $l=1$, the gas as a whole has a net angular momentum, and may be described as a kind of "orbital ferromagnet," but for $l=2$ the net angular momentum is zero; the ground state of the pairs is most probably such that the physical properties will have cubic symmetry. It can be conjectured that the same techniques will lead to similar states for $l>2$ also, although we have not computed such states because they probably have no physical realization. (Note that this cubically symmetric state is quite different from what we believed to be the ground state in earlier publications.)

There still remains the question, even in a relatively weakly interacting Fermi gas, as to whether the methods we have used are correct or complete. The next most urgent problem is to study the collective oscillation spectrum around the ground state which we have found. In particular, by this means (or by the equivalent technique of Thouless[2]) we could determine whether our state is stable as compared to similar configurations of finite rather than zero linear momentum. Unfortunately, the study of collective oscillations leads into complicated mathematics which we have not yet worked out.

The second division of our paper concerned itself primarily with liquid helium-3. The forces in He^3 are rather strong. The most serious question is whether we can rely on a "Fermi liquid" type of picture at sufficiently low temperatures, having quasi-particle rather than pure single-particle excitations with a spectrum similar to that of an ideal Fermi gas. Our best reason for relying on this picture is that it does qualitatively explain the present measurements of various properties of helium-3 rather well; and also, that it can be demonstrated that the forces in the metallic Fermi gas are not weak either—perhaps not as strong by a factor of 2 as in helium—but nonetheless in many cases the Fermi liquid picture appears to be valid.

If so, we have demonstrated reasonably well that some condensed state will probably form. Whether the state is a BCS-like one or something more complicated depends on the validity of our approximations; at what temperature, on the specific assumptions we and others have made about forces and effective masses, but most specifically on the amount of scattering. It is necessary in order for the condensed pairs to stay together that $h/\tau \sim \epsilon_0 \sim kT_c$, and this can only be satisfied below 0.02°K. This has never been conclusively shown to be more than a necessary condition on the transition temperature, so we must consider it an upper limit.

If, in fact, the transition is $l=2$ and is to the cubic state we suggest, it is interesting to speculate on how the specific nature of the ground state might be observed. Tensor properties such as moment of inertia, susceptibility, etc., will be isotropic; only a nontensorial property such as surface tension or surface current will show the effect. Since even in the pure m state the surface current was found to be nearly unmeasurably small, it appears that the nature of the ground state will be difficult to elucidate. Perhaps the clearest result will be the density of states near zero energy, which will appear in low-temperature susceptibility and specific heat.

Nonetheless, since this anisotropy is the newest and most fundamental property of the ground state, it is to be hoped that it can be directly observed. One might add that the appearance of anisotropy in a liquid is not physically a surprising result. The majority of phase transitions have the same feature, that the symmetry below the transition temperature is not as high as that above. That symmetry changes have not in the past been associated with superfluidity appears to be of no particular significance.

We have investigated this new type of superfluidity both as an interesting purely mathematical possibility and because it may apply to helium-3. It is interesting to speculate whether any further examples may arise. In nuclear matter, experimentally shell effects appear to obscure the question of what might occur in a

sufficiently extended volume; preliminary calculations showed[3] that actually the latter might well prefer a d state for like pairs, a triplet s state for $T=1$ unlike pairs. The complications of the situation in nuclear matter—long-range Coulomb forces, boundary effects, isotopic spin effects, etc.—do not seem to lend themselves well to our formalism in any case.

In metals it is not entirely impossible that our anisotropic BCS states could occur. A necessary generalization for metals is to realize that the Fermi surface and the forces have only the invariance of the crystal point group, not of the full rotation group. The forces must now be classified according to representations of this point group, and the type of state which appears depends on which type of representation has the most attractive potential. There are three situations which might arise:

(1) The natural generalization of BCS is the case in which the most attractive potential belongs to the identity representation. The energy gap parameter may be somewhat anisotropic in this case but will be everywhere real and, in general, everywhere finite (at least near the Fermi surface). The behavior is normal BCS with anisotropic gap.

(2) A case which does not occur in the spherical gas is a one-dimensional representation of odd parity. For instance, in an axial crystal the solution like $l=1, m=0$ might be lower either than $l=0$ or $l=1, m=\pm 1$. Here the gap will be real but will in general have zeros and we can expect triplet pairing rather than singlet, so that the Knight shift would not change much. (Zeros may not occur if k values on the appropriate reflection plane are not represented on the Fermi surface. Note that in this case zeros are not limited to null points, so that the density of states near zero energy may be high.)

(3) Finally, we come to the case of multidimensional representations. Here, in general, the gap function is complex and has point zeros most usually. Representations may be odd (triplet) or even (singlet).

There are, of course, a number of experimental facts which one would like to attribute to the presence of cases 2 and 3. The existence of zeros in the energy gap function is a natural explanation of the severe curvatures in the specific heat characteristic observed in a number of cases—Pb, Hg, and most recently, Mo-Re alloys.[31] The Knight shift characteristic of a triplet state is also a tempting explanation of some data. The difficulty, however, is that superconductivity has been found to be so extremely insensitive to impurity and boundary scattering, which destroy the intrinsic symmetry of the crystal and should spoil rather easily any coherence which changes in sign or phase from one portion of the Fermi surface to the next—i.e., all but case (1). Thus we must reserve judgment on whether cases (2) and (3) have been observed, while realizing that in all likelihood there is no real reason why the forces in some metal or group of metals cannot be favorable to their occurrence. The differences from the simple case (1) are rather subtle and difficult to determine experimentally.

[31] F. J. Morin (private communication).

ACKNOWLEDGMENTS

We should like to acknowledge gratefully the help of D. J. Thouless and E. Jakeman, particularly in indicating the need for a new $l=2$ ground state, and of H. Suhl in explaining to us his ideas on scattering before publication. We also acknowledge discussions and preprints from W. M. Fairbank, K. A. Brueckner, T. Soda, A. Glassgold, V. J. Emery, A. M. Sessler, J. C. Fisher, B. T. Matthias, and V. M. Galitskii.

APPENDIX A

We intend to evaluate here the error for the ground-state configuration $C(\theta,\varphi)$, and for the ground-state energy E_g, involved in using the simplified Eq. (4.4) instead of (4.2). Let then $C(\theta,\varphi)$ and E_g be the quantities computed in Sec. IV with the help of the approximate Eq. (4.4) and let $C(\theta,\varphi)+\delta C(\theta,\varphi)$ and $E_g+\delta E$ be the true values. From the expression (3.7) of the ground-state energy, we derive at once the expression for δE,

$$\delta E=-\sum_{\mathbf{k}}^{\text{in}}\epsilon_{\mathbf{k}}\delta(\cos\chi_{\mathbf{k}}+\tfrac{1}{2}\sin\chi_{\mathbf{k}}\tan\chi_{\mathbf{k}})$$
$$=-\sum_{\mathbf{k}}^{\text{in}}\frac{|C(\theta,\varphi)|^2\delta|C(\theta,\varphi)|^2}{4[\epsilon_{\mathbf{k}}^2+|C(\theta,\varphi)|^2]^{\frac{3}{2}}}. \tag{A.1}$$

After integrating over the energy, we find then the rather convenient expression:

$$\delta E=-\frac{N_0}{8\pi}\int\delta C(\theta,\varphi)C^*(\theta,\varphi)d\Omega+\text{c.c.} \tag{A.2}$$

On the other hand, we can derive from Eq. (4.2) the expression for the error $\delta C(\theta,\varphi)$ resulting from the term $U_l P_l(\hat{k}\cdot\hat{k}')$ of the expansion (3.9):

$$\delta C(\theta,\varphi)=\sum_{\mathbf{k}'}^{\text{in}}\frac{2l'+1}{2}U_{l'}P_{l'}(\hat{k}\cdot\hat{k}')\frac{C(\theta',\varphi')}{E_{\mathbf{k}'}}$$
$$+\sum_{\mathbf{k}'}^{\text{in}}\left[\frac{2l+1}{2}U_lP_l(\hat{k}\cdot\hat{k}')\right.$$
$$\left.+\frac{2l'+1}{2}U_{l'}P_{l'}(\hat{k}\cdot\hat{k}')\right]\frac{\delta C(\theta',\varphi')}{E_{\mathbf{k}'}}. \tag{A.3}$$

(We have neglected terms of the order of Δ^2/ξ^2, negligible in the weak coupling limit, as well as higher than first-order terms in δC.) Now, one can expect that the

sum K,

$$K=2\pi N_0^{-1}\sum_{\mathbf{k}}^{\text{in}}\frac{Y_{l'm}^*Y_{lm}}{E_{\mathbf{k}}}, \tag{A.4}$$

is rather small on account of the orthogonality of spherical harmonics (it would vanish if $E_{\mathbf{k}}$ had no angular dependence). We conclude that: (1) the leading term of $\delta C(\theta,\varphi)$ is the first term on the right-hand side of (A.3):

$$\delta C^{(1)}=N_0U_{l'}\sum_m Y_{l'm}(\theta,\varphi)\times\int\int\int\frac{Y_{l'm}^*(\theta',\varphi')C(\theta',\varphi')}{E_{\mathbf{k}'}}d\epsilon'd\Omega'. \tag{A.5}$$

This term vanishes if l and l' have different parity (no coupling between the even and odd parts of the potential) and is small otherwise (of the order of K). (2) $\delta C(\theta,\varphi)$ has a predominantly l'-type angular dependence: $\delta C^{(1)}$ is a combination of $Y_{l'm}$ and therefore does not contribute to δE. (3) There is a small l-type contribution (of the order of $|\delta C^{(1)}|^2/|C|$) coming from the second term on the right-hand side of (A.3). This second-order perturbation alone contributes to δE. We shall therefore use the following procedure for computing the contribution to the ground-state energy of the l' term of the expansion of $U_{\mathbf{k}\mathbf{k}'}$: First, we evaluate the leading part $\delta C^{(1)}$ of the perturbation $\delta C(\theta,\varphi)$ with the help of (A.5), then we carry this expression into (A.3) to compute the second-order perturbation $\delta C^{(2)}$:

$$\delta C^{(2)}=N_0U_l\sum_m Y_{lm}(\theta,\varphi)\times\int\int\int\frac{Y_{lm}^*(\theta',\varphi')\delta C^{(1)}(\theta',\varphi')}{E_{\mathbf{k}'}}d\epsilon'd\Omega, \tag{A.6}$$

and finally, we carry this last expression into (A.2). We shall consider three cases (corresponding to the situation found for the $l=0$, 1, and 2 condensations).

$C(\theta,\varphi)=$ constant

This occurs for the s-type condensed state studied by BCS. In this case, Eq. (4.2) is linear with respect to angular dependence and there is no coupling with higher order terms of the expansion of $U_{\mathbf{k}\mathbf{k}'}$.

$C(\theta,\varphi)=\Delta Y_{lm}(\theta,\varphi)$

This occurs for p-type condensation (ground state: $C=\Delta Y_{11}$). In this case, Eq. (4.5) reduces to

$$\delta C^{(1)}=N_0U_{l'}\Delta Y_{l'm}(\theta,\varphi)\times\int\int\int\frac{Y_{l'm}^*Y_{lm}}{E_{\mathbf{k}'}}d\epsilon'd\Omega'=N_0U_{l'}\Delta KY_{lm}(\theta,\varphi)$$

and therefore

$$\delta C^{(2)}=N_0^2U_lU_{l'}\Delta K^2Y_{lm}(\theta,\varphi).$$

The energy perturbation is then

$$\delta E=-(N_0\Delta^2/8\pi)(N_0U_l)(N_0U_{l'})2K^2.$$

In the $l=1$ case, coupling occurs only with the $l=3$, 5, 7 . . ., etc., terms of (3.9); the numerical factor $2K^2$ is then

$$\begin{aligned}&0.07 \quad \text{for} \quad l'=3,\\ &0.0056 \quad \text{for} \quad l'=5,\\ &0.001 \quad \text{for} \quad l'=7.\end{aligned}$$

Note that the factor $N_0U_{l'}$ can be of the same order of magnitude as or smaller than N_0U_l (the ground state would be a l'-state otherwise). On the other hand, N_0U_l may be significantly smaller than one if ξ can be chosen much larger than kT_c.

$$C(\theta,\varphi)=\Delta[aY_{20}+b(Y_{22}-Y_{2,-2})]$$

This configuration, which is the d-type ground state configuration, is particularly interesting because it allows coupling with the first term U_0 of the expansion of $U_{\mathbf{k}\mathbf{k}'}$. Because this first term is large (in fact, of the same order of magnitude as U_2) and repulsive,[32] we must make sure that the (positive) energy perturbation resulting from this $l=2$, and $l'=0$ coupling does not offset the small energy difference between this configuration (4.11) and the simple configurations ΔY_{21} or ΔY_{22}. Now, the Y_{20} part of $C(\theta,\varphi)$ alone is coupled with Y_{00}, for $E_{\mathbf{k}}$ depends only upon $\sin^2 2\varphi$; therefore:

$$\begin{aligned}\delta C^{(1)}&=\frac{a}{4\pi}N_0U_0\Delta\int\int\int\frac{Y_{20}}{E_{\mathbf{k}}}d\epsilon d\Omega\\ &=-\frac{a}{4\pi}\left(\frac{5}{16\pi}\right)^{\frac{1}{2}}N_0U_0\Delta\int\int(3\cos^2\theta-1)\ln C(\theta,\varphi)d\Omega\\ &\simeq 0.007(N_0U_0)\Delta.\end{aligned}$$

We are thus satisfied that $\delta C^{(1)}$ is considerably smaller than $|C|$ and therefore that $\delta C^{(2)}$ is truly negligible with respect to $\delta C^{(1)}$. We can then proceed:

$$\begin{aligned}\delta C^{(2)}&=N_0U_2\delta C^{(1)}Y_{20}(\theta,\varphi)\int\int\int\frac{Y_{20}}{E_{\mathbf{k}}}d\epsilon d\Omega\\ &\simeq 0.00084(N_0U_0)(N_0U_2)\Delta Y_{20}(\theta,\varphi),\end{aligned}$$

and finally:

$$\delta E=0.0013(N_0\Delta^2/8\pi)(-N_0U_0)(N_0U_2). \tag{A.7}$$

[32] See reference 3, Fig. 1. The K matrix used in this reference is approximately equivalent to our effective potential U, in the limit of large "cutoff" parameter ξ.

This coupling produces indeed a positive perturbation of the ground state energy of the order of 1/1000th of the condensation energy and therefore much smaller than the energy difference between the simple Y_{22} configuration and the true ground state (4.6%). Let us finally remark that expression (A.7) is truly independent of the cutoff parameter ξ, although it involves the ξ-dependent U_0 and U_2; in fact, we have seen in Sec. III that

$$1/|N_0U_0| = -1/N_0U_0 = -\ln\xi + \text{constant}. \quad \text{(A.8)}$$

This relation, together with (3.13), indicates that the product (N_0U_0) (N_0U_2) is independent of ξ.

APPENDIX B

Our purpose here is to find all independent solutions of Eq. (4.1), which we shall write for studying conveniently its invariance properties:

$$C(\hat{k}) = \frac{2l+1}{2} U_l \sum_{\mathbf{k}'}^{\text{in}} P_l(\hat{k}\cdot\hat{k}') \frac{C(\hat{k}')}{[\epsilon_{k'}{}^2 + |C(\hat{k}')|^2]^{\frac{1}{2}}}. \quad \text{(B.1)}$$

$C(\hat{k})$ is in general a complex function of the angular coordinates θ and φ of the unit vector $\hat{k}$; let $C_r(\hat{k})$ and $C_i(\hat{k})$ be its real and complex parts, respectively. The Eq. (B.1) is invariant under all transformations T in momentum space, which conserve the modulus k and the scalar product $\hat{k}\cdot\hat{k}'$: that is to say, rotations and reflections. We have indeed:

$$C(T\hat{k}) = \frac{2l+1}{2} U_l \sum_{\mathbf{k}'}^{\text{in}} P_l(T\hat{k}\cdot\hat{k}') \frac{C(\hat{k}')}{[\epsilon_{k'}{}^2 + |C(\hat{k}')|^2]^{\frac{1}{2}}}$$
$$= \frac{2l+1}{2} U_l \sum_{T\mathbf{k}'}^{\text{in}} P_l(\hat{k}\cdot\hat{k}') \frac{C(T\hat{k}')}{[\epsilon_{k'}{}^2 + |C(T\hat{k}')|^2]^{\frac{1}{2}}}. \quad \text{(B.2)}$$

We conclude that if $C(\hat{k})$ is a solution, so are the functions $C(T\hat{k})$ derived from $C(\hat{k})$ by rotations or reflections. Moreover, the real and imaginary parts are coupled only through $|C|^2$ in (B.2); therefore, if the square of the real part $C_r{}^2(\hat{k})$ is invariant under the transformation T:

$$C_r(T\hat{k}) = \pm C_r(\hat{k}), \quad \text{(B.3a)}$$

then the square of the imaginary part $C_i{}^2(\hat{k})$ must also be invariant:

$$C_i(T\hat{k}) = \pm C_i(\hat{k}). \quad \text{(B.3b)}$$

The relations (B.3) can only be fulfilled simultaneously, one being the consequence of the other. Finally, we see that (B.1) is also invariant under gauge transformations [multiplying $C(\hat{k})$ by a constant phase factor] since the nonlinear factor on the right-hand side of this equation involves only $|C(\hat{k})|^2$.

p-Type Solutions ($l=1$)

The most general combination of p-type spherical harmonics is:

$$C(\hat{k}) = Ax + By + Cz + i(A'x + B'y + C'z), \quad \text{(B.4)}$$

where x, y, and z are the three Cartesian coordinates of the unit vector $\hat{k}$. The real part represents a plane which can always be brought by a suitable rotation onto the Oxy plane, for example; in this case, $C_r{}^2$ is invariant by reflection in the planes Oxy, Oyz, and Ozx. The rotation must therefore bring the plane represented by C_i onto one of the three planes Oxy, Oyz, or Ozx (unless C_i is identically zero). On account of rotational equivalence, we find that the independent solutions are

$$Az \quad \text{or} \quad \Delta_0 Y_{10},$$
$$Ax + iBy \quad \text{or} \quad \Delta_1 Y_{11} + \Delta_{-1} Y_{1,-1}.$$

A rather simple computation shows that the mixed solution ($\Delta_1 = \Delta_{-1}$) or the simple solution $\Delta_0 Y_{10}$ are not as favorable as $\Delta_1 Y_{11}$ (see Table I).

d-Type Solutions ($l=2$)

The most general combination of d-type spherical harmonics is

$$C(\hat{k}) = A(2z^2 - x^2 - y^2) + B(x^2 - y^2) + Cxy + Dyz + Ezx + i[A'(2z^2 - x^2 - y^2) + B'(x^2 - y^2) + C'xy + D'yz + E'zx]. \quad \text{(B.5)}$$

Both the real and imaginary parts are homogeneous, traceless quadratic forms representing two quadrics; the real part, for example, is diagonalized by the rotation in momentum space which brings the coordinate axes onto the principal axes of the corresponding quadrics. That is to say, that the real part can always be transformed into the even form:

$$C_r(\hat{k}) = A(2z^2 - x^2 - y^2) + B(x^2 - y^2). \quad \text{(B.6a)}$$

C_r can also be transformed into an odd expression:

$$C_r(\hat{k}) = Cxy \quad \text{or} \quad Dyz \quad \text{or} \quad Ezx, \quad \text{(B.6b)}$$

in the special case where the corresponding quadric is degenerate and reduces to two perpendicular planes. This is rotationally equivalent to $A=0$ but is convenient for computational purposes. As we have seen above, the rotation which transforms C_r into one of the expressions (B.6) must simultaneously transform C_i into one of the expressions:

$$C_i(\hat{k}) = A'(2z^2 - x^2 - y^2) + B'(x^2 - y^2),$$
$$\text{or} \quad C'xy,$$
$$\text{or} \quad D'yz,$$
$$\text{or} \quad E'zx. \quad \text{(B.7)}$$

We conclude that the d-type solutions of (B.1) are the 16 combinations of (B.6) and (B.7); not all these com-

binations are independent solutions, fortunately (on account of rotational and gauge invariance). We shall now study all independent combinations individually.

(1) $C(\hat{k})=(A+iA')(2z^2-x^2-y^2)+(B+iB')(x^2-y^2)$

On account of gauge invariance, one can multiply $C(\hat{k})$ by a suitable phase factor to make the coefficient of the first term real ($A'=0$). This solution is therefore:

$$C(\hat{k})=\Delta_0 Y_{20}+\Delta_2(Y_{22}+Y_{2,-2}), \tag{B.8}$$

where Δ_0 is real and Δ_2 may *a priori* be complex. Carrying this expression into Eq. (4.4), we find that Δ_0 and Δ_2 must satisfy the following set of nonlinear equations:

$$\begin{aligned}\Delta_0&=a_{00}\Delta_0+2a_{02}\Delta_2,\\ \Delta_2&=a_{02}\Delta_0+(a_{22}+b_{22})\Delta_2,\end{aligned} \tag{B.9}$$

where we have defined:

$$\begin{aligned}a_{\mu\mu'}&=N_0U_l\int_0^{\xi}d\epsilon\int d\Omega\frac{Y_{2\mu}{}^*Y_{2\mu'}}{E_k}\\ b_{\mu\mu}&=N_0U_l\int_0^{\xi}d\epsilon\int d\Omega\frac{Y_{2\mu}{}^2}{E_k}.\end{aligned} \tag{B.10}$$

The imaginary part cancels since $|C(\hat{k})|$ is invariant under the transformation $\varphi\rightarrow-\varphi$, and therefore, Δ_2 is real. This is true unless $1-a_{00}=a_{02}=1-a_{22}-b_{22}=0$ which happens to be satisfied when Δ_2 is pure imaginary. This solution will be discussed under (2). Using the notation (4.10) and (4.11), we can write (B.10) in a form convenient for numerical computations:

$$\begin{aligned}a_{\mu\mu'}-\delta_{\mu\mu'}&=-N_0U_l\Big[\delta_{\mu\mu'}\ln\Gamma\\ &\quad+\int\int \mathrm{Re}(Y_{2\mu}{}^*Y_{2\mu'})\ln|f|d\Omega\Big],\\ b_{\mu\mu}&=-N_0U_l\int\int \mathrm{Re}(Y_{2\mu}{}^2)\ln|f|d\Omega.\end{aligned} \tag{B.11}$$

The numerical tabulation of these coefficients versus Δ_0/Δ_2 indicates that (B.9) has four solutions:

$\Delta_0=\Delta$,	$\Delta_2=0$,	$\ln\Gamma=1.02$;
$\Delta_0=\sqrt{3}/2\Delta$,	$\Delta_2=1/\sqrt{8}\Delta$,	$\ln\Gamma=0.98$;
$\Delta_0=1/2\Delta$,	$\Delta_2=\sqrt{3}/\sqrt{8}\Delta$,	$\ln\Gamma=1.02$;
$\Delta_0=0$,	$\Delta_2=\Delta$,	$\ln\Gamma=0.98$.

A 90° rotation brings the first into the third, the second to the fourth, so these are only two independent solutions. These solutions are therefore not favorable (see Table I) and this could be related to the fact that the zeros of the "gap" $|C(\hat{k})|$ form continuous lines (instead of being discrete nodes as for the most favorable solutions).

(2) $C(\hat{k})=A(2z^2-x^2-y^2)+B(x^2-y^2)+iCxy$

This can be written in the equivalent form:

$$C(\theta,\varphi)=\Delta_0 Y_{20}+\Delta_2 Y_{22}+\Delta_{-2}Y_{2,-2}, \tag{B.12}$$

where the three coefficients Δ_0, Δ_2, and Δ_{-2} are real and given by the set of nonlinear equations:

$$\begin{aligned}\Delta_0&=a_{00}\Delta_0+a_{02}(\Delta_2+\Delta_{-2}),\\ \Delta_2&=a_{02}\Delta_0+a_{22}\Delta_2+b_{22}\Delta_{-2},\\ \Delta_{-2}&=a_{02}\Delta_0+b_{22}\Delta_2+a_{22}\Delta_{-2}.\end{aligned} \tag{B.13}$$

Discounting the solution (B.8) studied above, this system is equivalent to

$$\begin{aligned}(a_{00}-1)\Delta_0+a_{02}(\Delta_2+\Delta_{-2})&=0,\\ a_{02}\Delta_0+(a_{22}-1)(\Delta_2+\Delta_{-2})&=0,\\ a_{22}-b_{22}-1&=0.\end{aligned} \tag{B.14}$$

(B.14) has the following solutions[33]:

$\Delta_0=0$,	$\Delta_2=\Delta$, $\Delta_{-2}=0$,	$\ln\Gamma=1.131$;
$\Delta_0=0$,	$\Delta_2=-\Delta_{-2}=\Delta/\sqrt{2}$,	$\ln\Gamma=0.98$;
$\Delta_0=\Delta/\sqrt{2}$,	$\Delta_2=-\Delta_{-2}=\Delta/2$,	$\ln\Gamma=1.154$;

and possibly also solutions of the general type (B.12) with $-\Delta_2\neq\Delta_{-2}$.

Note added in proof. V. Emery has indeed brought to our attention the possibility that solutions displaying a low degree of symmetry may exist. This author analyzes Eq. (5.5) in the limit of a vanishing gap, corresponding to the situation near the critical temperature. He finds that the first order expansion of (5.5) in powers of $C(\hat{k})$ has, in addition to the solutions C_1 to C_5 below, the two following solutions:

$$C(\hat{k})=\Delta[1/\sqrt{2}Y_{20}+1/2(Y_{21}-Y_{2,-1})] \quad \text{(a)}$$

$$C(\hat{k})=\Delta[1/2(Y_{21}-Y_{2,-1})+1/2(Y_{22}+Y_{2,-2})], \quad \text{(b)}$$

with 6 and 4 nodes, respectively. Both configurations belong to group (2); (a) and (b) can indeed be brought to the form (B.12) with $\Delta_2\neq\Delta_{-2}$ by a 90° rotation.

[33] It is interesting to note that the condition $a_{00}-1=a_{22}-b_{22}-1=0$, for the existence of a solution of the type $C(\theta,\varphi)=\Delta_0Y_{20}+\Delta_2(Y_{22}-Y_{2,-2})$, is exactly fulfilled when $|C(\theta,\varphi)|$ has cubic symmetry. Indeed, the above condition can be written:

$$\int(Y_{20})^2\ln|f|d\Omega=\int 2\sin^2 2\varphi\,|Y_{22}|^2\ln|f|d\Omega$$

with

$$|f(\theta,\varphi)|^2=\tfrac{1}{2}Y_{20}{}^2+\sin^2 2\varphi\,|Y_{22}|^2.$$

After performing a $\pi/4$ rotation around the z axis ($\varphi\rightarrow\varphi+\frac{1}{4}\pi$), we have:

$$|f|^2=\frac{5}{8\pi}[x^4+y^4+z^4-z^2y^2-y^2z^2-z^2x^2].$$

Using the cubic symmetry of $|f|$, we find indeed that:

$$\begin{aligned}&\int 3(x^2-y^2)^2\ln|f|d\Omega\\ &\quad=\int[2(x^2-y^2)^2+2(y^2-z^2)^2-(x^2-y^2)^2]\ln|f|d\Omega\\ &\quad=\int(2z^2-x^2-y^2)^2\ln|f|d\Omega.\end{aligned}$$

It has not been demonstrated yet whether these solutions satisfy Eq. (5.5) to all order of its expansion, or not (private communication).

(3) $C(k)=A(2z^2-z^2-y^2)+B(x^2-y^2)+iCzy$

This is equivalent to set (2) under a 90° rotation $x\rightarrow z$.

(4) $C(k)=Azx+iBzy$

This is equivalent to

$$C(\theta,\varphi)=\Delta_1 Y_{21}+\Delta_{-1}Y_{2,-1}, \tag{B.15}$$

where the two coefficients Δ_1 and Δ_{-1} are real and solutions of

$$\Delta_1=a_{11}\Delta_1+b_{11}\Delta_{-1},$$
$$\Delta_{-1}=b_{11}\Delta_1+a_{11}\Delta_{-1},$$

or rather

$$\begin{aligned}\Delta_1+\Delta_{-1}&=(a_{11}+b_{11})(\Delta_1+\Delta_{-1}),\\ \Delta_1-\Delta_{-1}&=(a_{11}-b_{11})(\Delta_1-\Delta_{-1}).\end{aligned} \tag{B.16}$$

One sees immediately that (B.18) has only four possible solutions:

$$\begin{aligned}&\Delta_1=\Delta,\ \Delta_{-1}=0, && \ln\Gamma=1.131;\\ &\Delta_1=0,\ \Delta_{-1}=\Delta, && \ln\Gamma=1.131;\\ &\Delta_1=\Delta_{-1}, && \ln\Gamma=0.98;\\ &\Delta_1=-\Delta_{-1}, && \ln\Gamma=0.98.\end{aligned} \tag{B.17}$$

[The last two are the same (x^2-y^2) solution we found under (1).]

The final result then is that we have found five inequivalent solutions, all highly symmetrical but not particularly simple. In terms of direction cosines, these are:

$$\begin{aligned}&C_1=xy \text{ or } x^2-y^2=Y_{21}\pm Y_{2,-1}, && \ln\Gamma=0.98;\\ &C_2=2z^2-x^2-y^2=Y_{2,0}, && \ln\Gamma=1.02;\\ &C_3=(x+iy)z=Y_{2,1}, && \ln\Gamma=1.131;\\ &C_4=(x+iy)^2=Y_{2,2}, && \ln\Gamma=1.131;\\ &C_5=z^2+\epsilon x^2+\epsilon^2y^2(\epsilon=\sqrt[3]{1}), && \ln\Gamma=1.154.\end{aligned} \tag{B.18}$$

The most favorable solution we have found thus far [that is to say, excluding possibly a more complicated solution of the general type (B.12)] is the configuration C_5. It appears that the value of $\ln\Gamma$ and therefore the condensation energy is linked to the multiplicity and distribution of the zeros of $|C(\hat{k})|$. On this account, C_5 is better than C_2, for example, since the former has only eight discrete nodes, whereas the latter has two lines of nodes.

It is likely that the special configuration C_5 is the ground-state configuration on account of its high degree of symmetry. On the other hand, we have not quite excluded the three-parameter solutions (B.12) which we have not studied completely, and they may be slightly more favorable than (4.12); these solutions also have discrete nodes and therefore we do not expect that they could improve much upon the configuration (4.12). Furthermore, the low-temperature properties derived for such configurations would be essentially the same as those we have found for the configuration (4.12). Thus, in any case the configuration (4.12) yields accurate results about the properties of the ground state for a d-type condensed system.

APPENDIX C

The purpose of this appendix is to compute the integral:

$$I(\mathbf{r})=\sum_{\mathbf{k}}\sin\chi_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}+\psi_{\mathbf{k}})}=\sum_{\mathbf{k}}\frac{C(\hat{k})}{[\epsilon_{\mathbf{k}}^2+|C(\hat{k})|^2]^{\frac{1}{2}}}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{C.1}$$

in the long-range limit ($k_F r\gg 1$), by the stationary phase method. Here $C(\hat{k})$ is the angle-dependent gap function appropriate to any solution we may choose to consider; we think particularly of the $l=2$ ground state (4.12). To do this, let us express the integrand in terms of new angular coordinates: the cosine u of the angle between $\mathbf{k}$ and $\mathbf{r}$, and the azimuth angle φ of $\mathbf{k}$ about $\mathbf{r}$:

$$I(\mathbf{r})=\frac{N_0}{4\pi}\int_{-\xi}^{+\xi}d\epsilon\int_0^{2\pi}d\varphi\int_{-1}^{+1}du\frac{C(u,\varphi)}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}e^{ikru}. \tag{C.2}$$

Let us consider first the integration with respect to the variable u. Now, if $\epsilon\neq 0$, the integrand is an analytic function of u and we can therefore replace the integration from -1 to $+1$ on the real axis by integration on the contour Γ in the complex plane (see Fig. 7); we have then:

$$\begin{aligned}\int_{-1}^{+1}\frac{C(u,\varphi)e^{ikru}}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}du&=\int_{-1}^{-1+i\infty}\frac{C(u,\varphi)e^{ikru}}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}du\\ &\quad-\int_{1}^{1+i\infty}\frac{C(u,\varphi)e^{ikru}}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}du.\end{aligned} \tag{C.3}$$

Since $kr\gg 1$, the exponential factor is vanishingly small everywhere but near the real axis; neglecting then the

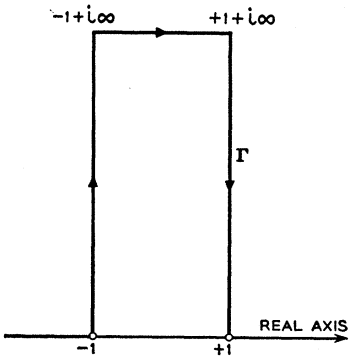

FIG. 7. Contour for integrating the function (C.3) by the stationary phase method.

variation of the factor $C(u,\varphi)/(\epsilon^2+|C|^2)^{\frac{1}{2}}$ in the small interval where the integrand is sizeable, we obtain:

$$\int_{-1}^{+1}\frac{C(u,\varphi)e^{ikru}}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}du=\frac{1}{ikr}\frac{C(+1)e^{ikr}}{[\epsilon^2+|C(+1)|^2]^{\frac{1}{2}}}-\frac{C(-1)e^{-ikr}}{[\epsilon^2+|C(-1)|^2]^{\frac{1}{2}}}. \quad (C.4)$$

Note that this result is independent of the variable φ and involves only the value of the gap function $C(\hat{k})$ in the direction of the radius vector $\mathbf{r}$, that is to say, $C(\hat{r})$. Since this function has the same parity as l, the expression (C.4) can be written:

$$\int_{-1}^{+1}\frac{C(u,\varphi)e^{ikru}}{[\epsilon^2+|C(u,\varphi)|^2]^{\frac{1}{2}}}du=\frac{C(\hat{r})}{ikr}\times\frac{e^{ikr}\pm e^{-ikr}}{[\epsilon^2+|C(\hat{r})|^2]^{\frac{1}{2}}}, \quad (C.5)$$

where the + sign corresponds to odd configurations and the − sign to even configurations.

The deformation of the contour of integration over u upon which (C.5) is based is not allowable if ϵ is near zero and $C(u,\varphi)$ goes through a zero in the range $-1<u<1$. In that case there is a branch point near the real axis beyond which we may not deform the u contour. It can be verified easily that the contribution from integrating around this branch point is negligible, as follows: carrying out first the ϵ integration by the method used later in getting (C.10), we obtain a function like $C(u,\varphi)K_0(|C|r/\hbar v_F)$. This behaves like $C\ln|C|$ at $|C|=0$. It is then easy to verify that the contribution from the branch point behaves like

$$\int du\int d\varphi C(u,\varphi)\ln|C|e^{ikFru}$$

which is of order $(1/k_Fr)^2$ smaller than (C.10) in the interesting case in which the zeros of $|C|$ are isolated points on the sphere.

We may write then:

$$I(\mathbf{r})=\frac{N_0C(\hat{r})}{2ir}\int_{-\xi}^{+\xi}\frac{e^{ikr}\pm e^{-ikr}}{[\epsilon^2+|C(r)|^2]^{\frac{1}{2}}}\frac{d\epsilon}{k}. \quad (C.6)$$

Observing now that k varies only slightly in the interval of integration (near the Fermi surface), we shall use its expansion in powers of ϵ:

$$k=k_F+\epsilon/\hbar v_F+\cdots. \quad (C.7)$$

Hence:

$$e^{ikr}=e^{ikFr}[\cos(r\epsilon/\hbar v_F)+i\sin(r\epsilon/\hbar v_F)]. \quad (C.8)$$

The second term on the right-hand side is odd and therefore cancels in the integration. Neglecting second-order imaginary terms, we obtain in the same fashion as BCS (see Appendix D of reference 1),

$$I(\mathbf{r})=\frac{N_0C(\hat{r})}{k_Fr}\int_{-\xi}^{+\xi}\cos\left(\frac{r\epsilon}{\hbar v_F}\right)\frac{d\epsilon}{[\epsilon^2+|C(\hat{r})|^2]^{\frac{1}{2}}}\times\begin{Bmatrix}\sin k_Fr, & l\text{ even}\\ -i\cos k_Fr, & l\text{ odd}\end{Bmatrix}. \quad (C.9)$$

The integrand is the product of a monotonically decreasing function of ϵ, by a very rapidly oscillating factor (in the long-range limit); we can then extend the interval of integration to $(-\infty,+\infty)$ without introducing a significant error [the error involved in this approximation is of the order of the area of the first arch of the interval $(+\xi,+\infty)$, i.e., $\epsilon_F/\xi k_Fr$]. We obtain finally:

$$I(\mathbf{r})=\frac{2N_0C(\hat{r})}{k_Fr}K_0\left(\frac{r|C(\hat{r})|}{\hbar v_F}\right)\begin{Bmatrix}\sin k_Fr, & l\text{ even}\\ -i\cos k_Fr, & l\text{ odd}\end{Bmatrix}, \quad (C.10)$$

where K_0 is the modified Bessel function of the second kind and order zero.[34] $K_0(z)$ diverges like $\ln z$ when z approaches zero so that $I(\mathbf{r})$ vanishes like $|C(\hat{r})|\ln|C(\hat{r})|$ near the nodes of $|C(\hat{r})|$. On the other hand, the leading term of the asymptotic expansion of K_0 is

$$K_0\simeq(\pi/2z)^{\frac{1}{2}}e^{-z},\quad z\to\infty.$$

This exponential factor, $\exp[-r|C(\hat{r})|/\hbar v_F]$, dominates the dependence of $I(\mathbf{r})$ upon the distance r in the long-range limit.

[34] E. T. Whittaker and G. N. Watson, *Course of Modern Analysis* (Cambridge University Press, New York, 1946), Chap. XVII, Example 40.