Electric Dipole Approximation and the Canonical Formalism in Electrodynamics*

MELVIN SCHWARTZ†
Syracuse University, Syracuse, New York
(Received March 1, 1961)

We use the electric dipole approximation to study the problem of finding commuting solutions of coupled equations of motion. We point out that for a charged particle in an external radiation field, the solutions of the coupled equations cannot be considered independent in the sense of commuting with one another if the homogeneous solutions are assumed to have the commutation properties of uncoupled variables. We explicitly treat the case of a charged free particle and a charged harmonic oscillator in an external radiation field. We indicate that for a retarded (advanced) self-field, the free particle fits into a canonical formalism while the oscillator does not. For a stationary self-field, both the free particle and the oscillator fit into a canonical formalism. We show that the Fourier transforms of the configuration space solutions (based on $e^{i\mathbf{k}\cdot\mathbf{x}}$ and $e^{i\omega t}$) do not exist. In the latter connection, we point out that earlier treatments of the oscillator by Sokolov and Tumanov and Norton and Watson contain misleading results as a consequence of their using Fourier transforms.

I. INTRODUCTION

THIS paper will be the first in a series[1] dealing with the electric dipole approximation in electrodynamics.[2] We propose to discuss the problem of developing a consistent canonical formalism in a theory where variables associated with the electromagnetic field as well as the particle variables are treated as dynamical variables. With the understanding that in the quantum theory of the model to be discussed the particles are to be treated in first quantization while the radiation field is to be treated in second quantization, the discussion will be valid for both classical and quantum theory.[3]

In order to develop a consistent canonical formalism, we will have to deal with two related problems. First we will have to identify a complete set of dynamical variables. By completeness we mean that the dynamical variables form the minimal set of conjugate pairs required to determine the canonical equations of motion. Having identified a complete set of dynamical variables we are then faced with the next problem. Suppose the solutions of the inhomogeneous equations of motion are given as some functions of the solutions of the uncoupled equations. Now the identification of dynamical variables must be made for all times. Therefore the commutation relations[3] for the uncoupled solutions will have to be specified for all times. In particular, if one imposes conditions such that the solution for the particle variables reduces to the solution of the uncoupled equation at some initial time, the commutation rules for the uncoupled particle variables will then be fixed. However, the solution for the electromagnetic field variables will contain a mixture of uncoupled field and particle variables in such a way, that at the initial time, one will be left with a mixture of uncoupled field and uncoupled particle variables. One may then argue that the commutation rules for the uncoupled field variables should be determined by the requirement that they guarantee that the full set of dynamical variables satisfy the proper commutation rules. Suppose one requires that the free-field variables satisfy free-field commutation rules. Will it then be consistent with the canonical commutation rules for the dynamical variables if the free-field variables are assumed to satisfy free-field commutation rules at all times?[4]

Since it is the radiation field which will be treated in second quantization, one would hope that the above question may be answered affirmatively.

In this first paper of the series we should like to emphasize that the choice of position, momentum conjugate to position, electromagnetic vector potential, and its conjugate momentum as dynamical variables is not complete in the electric dipole approximation. In order to illustrate the preceding statement and elucidate the mathematical problems to be overcome in connection with the free-field commutation rules, we will solve the free-particle and harmonic-oscillator models exactly. At the end of the paper we will indicate the course of action to be followed in overcoming some of the problems raised.

To some extent, the problems to be raised have been dealt with by Kramers,[2] Sokolov and Tumanov,[5] and Norton and Watson.[6] However, Kramers is incomplete in that he does not discuss constraints nor does he (from our point of view) adequately discuss the free-field problem. Sokolov and Tumanov and Norton and Watson treat the oscillator model. However, they did

* Supported by the National Science Foundation.
† Present address: Adelphi College, Garden City, Long Island, New York.
[1] The other papers will be written in collaboration with R. Schiller.
[2] H. A. Kramers, *Proceedings of the Solvay Congress 1948* "*Collected Scientific Papers*" (North-Holland Publishing Company, Amsterdam, 1956); N. G. Van Kampen, Klg. Danske Vid. Sels. Mat. fys. Medd. **26**, 15 (1951).
[3] Hereafter it should be understood that commutation relation and Poisson bracket relation may be used interchangeably.
[4] A. S. Wightman and H. Epstein, Ann. Phys. **11**, 201 (1960) have considered a relativistic theory in which the above question is answered negatively.
[5] A. Sokolov and I. Tumanov, Soviet Phys. JETP **3**, 958 (1956–57).
[6] R. E. Norton and W. K. R. Watson, Phys. Rev. **116**, 1597 (1959).

not deal with the completeness question. Moreover, they incorrectly assume the existence of Fourier transforms and are led to misleading conclusions.

II. COMMUTATION RULES AND INDEPENDENT VARIABLES

We wish to consider a charged particle under the influence of an external potential and an external radiation field. We use the Coulomb gauge and assume the electric dipole approximation.[2,7] We start with the Lagrangian,

$$L_0=\tfrac{1}{2}m_0\mathbf{V}^2+e\mathbf{V}\cdot\mathfrak{A}-e\Phi-U(\mathbf{R})+\tfrac{1}{2}\int d^3x\{[\mathbf{E}(\mathbf{x},t)]^2+[\mathbf{H}(\mathbf{x},t)]^2\},\quad(1)$$

where m_0 is the bare (unrenormalized) mass, $\mathbf{R}(t)$ and $\mathbf{V}(t)$ are respectively, the position and velocity operators of the particle, $U(\mathbf{R})$ is a scalar potential energy, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ represent respectively the electric and magnetic field vectors at the space point $\mathbf{x}$, and $\mathbf{A}$ and Φ are respectively the vector and scalar potentials associated with the electromagnetic field. Units are chosen so that $\hbar=c=1$.

$$\mathfrak{A}=\int d^3x\,\rho(\mathbf{x})\mathbf{A}(\mathbf{x},t)\quad(2)$$

and

$$\Phi=\int d^3x\,\rho(\mathbf{x})\varphi(\mathbf{x},t),\quad(3)$$

where $\rho(\mathbf{x})$ is a form factor describing the charge distribution. $\rho(\mathbf{x})$ is normalized so that $\int d^3x\,\rho(\mathbf{x})=1$.

Now,

$$\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\varphi\quad\text{and}\quad\mathbf{H}=\nabla\times\mathbf{A},\quad(4)$$

so that the variation of L_0 will be determined by varying $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\partial\mathbf{A}/\partial t$, and φ. Assuming the usual boundary conditions at infinity, it is well known that $\int d^3x\partial\mathbf{A}/\partial t\cdot\nabla\varphi$, which arises from $\mathbf{E}^2$, is equivalent to a constraint with φ as a Lagrange multiplier. Upon assuming that $\mathbf{A}$ is independent of $\mathbf{V}$, the resulting Euler-Lagrange equations will be

$$m_0\ddot{\mathbf{R}}=-\nabla U-e\partial\mathfrak{A}/\partial t,\quad(5)$$

$$\nabla^2\varphi=-e\rho(\mathbf{x}),\quad(6)$$

and

$$\Box\mathbf{A}=-\mathrm{Tr}e\rho\mathbf{V}=-e\rho(\mathbf{x})\mathbf{V}+\nabla\dot{\varphi}(\mathbf{x}),\quad(7)$$

where $\mathrm{Tr}\mathbf{B}$ denotes transverse part of a vector field $\mathbf{B}$.

Since the term $e\Phi$ does not affect the particle equations of motion, we may without loss of generality drop it and $\int d^3x(\nabla\varphi)^2$ from L_0. The reduced Lagrangian is then

$$L=\tfrac{1}{2}m_0\mathbf{V}^2+e\mathbf{V}\cdot\mathfrak{A}-U(\mathbf{R})+\tfrac{1}{2}\int d^3x\{[\mathrm{Tr}(\mathbf{E})]^2-[\mathrm{Tr}(\mathbf{H})]^2\}-\int d^3x\,\varphi\nabla\cdot\frac{\partial\mathbf{A}}{\partial t},\quad(8)$$

with φ playing the role of a Lagrange multiplier in the last term and

$$\mathrm{Tr}\mathbf{E}=-\partial\mathbf{A}/\partial t.\quad(9)$$

The conjugate pairs of variables are

$$\begin{aligned}\mathbf{R}&\leftrightarrow\mathbf{P}=m_0\mathbf{V}+e\mathfrak{A},\\ \mathbf{A}&\leftrightarrow\mathbf{\Pi}=-\mathbf{E}=\partial\mathbf{A}/\partial t+\nabla\varphi.\end{aligned}\quad(10)$$

The Hamiltonian is

$$H=1/2m_0(\mathbf{P}-e\mathfrak{A})^2+U(\mathbf{R})+\tfrac{1}{2}\int d^3x[(\partial\mathbf{A}/\partial t)^2+(\nabla\mathbf{A})^2],\quad(11)$$

where Eqs. (9) and (10), $\mathbf{H}^2=(\nabla\times\mathbf{A})^2=-\mathbf{A}\cdot\nabla^2\mathbf{A}$, and an integration by parts have been used to write the Hamiltonian in the above form. As the fundamental commutation rules we adopt

$$[R_i(t),P_j(t)]=i\delta_{ij},$$

$$[A_i(\mathbf{x},t),\Pi_j(\mathbf{x}',t)]=i\left[\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')-\frac{1}{4\pi}\partial_i\partial_j'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right].\quad(12)$$

All other equal time commutators are assumed to be zero. The assumption that $\mathbf{A}$ is independent of $\mathbf{V}$ is now to be realized by the condition that $[\mathbf{R},\mathbf{A}]=0$. We proceed to investigate the validity of that assumption.

We determine $\mathbf{A}$ by solving Eq. (7). If, in order to be specific, we use the retarded Green's function, we get[8]

$$\mathbf{A}=\mathbf{A}_h+\mathrm{Tr}\frac{e}{4\pi}\int d^3x'\frac{\rho(\mathbf{x}')\mathbf{V}(t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|},\quad(13)$$

where $\mathbf{A}_h$ is an arbitrary solution of the homogeneous equation. The appearance of $\mathbf{V}(t-|\mathbf{x}-\mathbf{x}'|)$ in Eq. (13) raises the possibility that the assumption that $\mathbf{A}_h$ satisfies free field commutation rules is inconsistent with $[\mathbf{R},\mathbf{A}]=0$.[9] In the examples treated below, we will point out that such an inconsistency exists. A similar argument holds if one uses advanced or

[7] Strictly speaking, the electric dipole approximation can be made only after separating an external field from a proper field in the sense of Kramers.[2] Only then can one get the correct electric dipole approximation solutions of the field equations. However, if we simply accept the appearance of Eqs. (2) and (3) in L_0, with ρ independent of the particle position as characterizing the electric dipole approximation, and exercise care, no inconsistencies with the approximation will arise.

[8] Note that in the electric dipole approximation, φ is a static Coulomb potential so that $\dot{\varphi}=\mathbf{V}\cdot\nabla\varphi$.

[9] If the particles were treated in second quantization with ψ denoting the particle field, the analogous result would be $[\psi,A]\neq0$. However, one couldn't make such a statement without actually having solutions to the coupled field equations.

$\frac{1}{2}$(retarded+advanced) Green's functions to find the self-field.

The above conclusion was not discussed in the paper by Norton and Watson,[6] but its consideration is essential if one wants to make comparison with current formulations of quantum field theory.

If we continue to insist that $\mathbf{A}_h$ has the properties of the free field, then one is led to conclude that there are constraints as well as additional variables to be considered. In fact, the first half of Eq. (10) is a constraint relation while the appearance of $\mathbf{V}$ in Eq. (13) leads to an acceleration dependence in L. Kramers[2] has developed a theory which properly presents a complete set of dynamical variables, but he does not consider the constraint problem for his variables nor certain implications regarding the possibility of quantization. In the second paper of this series we will present a formulation alternative to Kramers.[1] Our formulation will be based on the Ostrogradsky method[10] and the constraint problem will be fully discussed. We will find the self-field arising from several different boundary conditions and discuss the possibilities for quantization in each case.

In order to get a feeling for the problems to be encountered, we will devote the next two sections of this paper to the free particle and the harmonic oscillator.

III. FREE PARTICLE

For the free particle $U(\mathbf{R})=0$. We retain Eqs. (5)–(7) as fundamental equations. It follows from Eq. (5) that

$$\mathbf{P}=m_0\dot{\mathbf{R}}+e\mathfrak{A}=\text{const.} \tag{14}$$

The particular solution of Eq. (7) will depend on the boundary conditions chosen. We will consider the cases of retarded and stationary [$\frac{1}{2}$(advanced+retarded)] boundary conditions. The solution for advanced boundary conditions is immediately obtainable from the retarded case by appropriate substitution of the advanced time.

(i) Retarded Case

The solution of Eq. (7) is given by Eq. (13). If we substitute Eq. (13) into Eq. (14) we get

$$m_0\dot{\mathbf{R}}+\frac{e^2}{4\pi}\int d^3x\,\mathrm{Tr}\int d^3x'\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\dot{\mathbf{R}}(t-|\mathbf{x}-\mathbf{x}'|)=\mathbf{P}-e\mathfrak{A}_h. \tag{15}$$

The second term on the left in Eq. (15) blows up in the point charge limit. In order to avoid the blow up, we introduce mass renormalization in a well known fashion. Define[11]

$$\mathbf{A}_0=\frac{e}{4\pi}\,\mathrm{Tr}\int d^3x'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\dot{\mathbf{R}}(t), \tag{16}$$

and

$$\mathbf{A}_1=\mathbf{A}-\mathbf{A}_h-\mathbf{A}_0. \tag{17}$$

Replace the left-hand side of Eq. (15) by $m_0\dot{\mathbf{R}}+e\mathfrak{A}_1+e\mathfrak{A}_0$. Let

$$\delta m=\frac{e^2}{6\pi}\int d^3xd^3x'\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{18}$$

denote the electromagnetic mass, and call

$$m=m_0+\delta m \tag{19}$$

the observed mass. Since $m_0\dot{\mathbf{R}}+e\mathfrak{A}_0=m\dot{\mathbf{R}}$, Eq. (15) becomes

$$m\dot{\mathbf{R}}+e\mathfrak{A}_1=\mathbf{P}-e\mathfrak{A}_h. \tag{20}$$

Having arrived at a procedure for obtaining finite equations of motion in the point charge limit, we will now pass to that limit. By restricting ourselves to point charges, we will lose no essential generality and gain immense simplification of the calculations.

For a point charge $\mathfrak{A}_1(t)=\mathbf{A}_1(0,t)=-\frac{2}{3}(e^2/4\pi)\ddot{\mathbf{R}}(t)$. Equation (20) then reduces to

$$\ddot{\mathbf{R}}(t)-\omega_0\dot{\mathbf{R}}(t)=\frac{e\omega_0}{m}\mathbf{A}_h(0,t)-\frac{\omega_0}{m}\mathbf{P}, \tag{21}$$

where $\omega_0=6\pi m/e^2$. The characteristic function of the homogeneous part of Eq. (21) is

$$D_R(\omega)=\omega(\omega-\omega_0), \tag{22}$$

where the subscript R is used to indicate that it is associated with retarded boundary conditions. Particular solutions of Eq. (22) may be obtained by means of the Green's function,

$$G_R(t,t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega(t-t')}}{D_R(i\omega)}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega(t-t')}}{\omega(\omega+i\omega_0)}$$
$$=\begin{cases}-\dfrac{1}{\omega_0}(e^{\omega_0(t-t')}-1); & t'-t>0\\ 0; & t'-t\leqslant 0\end{cases} \tag{23}$$

$G_R(t,t')$ satisfies

$$\partial^2G_R/\partial t^2-\omega_0\partial G_R/\partial t=\delta(t-t'). \tag{24}$$

A solution of Eq. (21) which satisfies the condition that $\dot{\mathbf{R}}(-\infty)=\mathbf{P}/m$ is

$$\mathbf{R}(t)=\mathbf{R}_0+\mathbf{R}_1e^{\omega_0t}-\frac{e}{m}\int_t^{\infty}dt'(e^{\omega_0(t-t')}-1)\mathbf{A}_h(0,t')+\frac{\mathbf{P}}{m}t. \tag{25}$$

The appearance of the runaway exponential in Eq. (25) is a manifestation of the fact that $\mathbf{R}$, $\mathbf{P}_R$, $\mathbf{A}$, $\mathbf{\Pi_A}$ do not form a complete set of dynamical variables, i.e., one cannot base a Hamiltonian formalism on those variables alone. Since Hamilton's equations of motion are first

---

[10] E. T. Whittaker, *Analytical Dynamics* (Dover Publications, New York, 1944).

[11] Apart from a solution of the homogeneous wave equation, the definition of $\mathbf{A}_0$ and $\mathbf{A}_1$ follows Kramers.

order, the solutions must be determined by specifying data at a single time. However, it is clear from Eq. (25) that the requirement that $\dot{\mathbf{R}}(t)$ be finite at all times requires that $\mathbf{R}_1=0$, which in turn requires specifying $\dot{\mathbf{R}}$ at some finite time as well as at $t=-\infty$. As mentioned previously, we will overcome this difficulty in the second paper of the series by use of the Ostrogradsky technique. At the present stage we will simply require that $\dot{\mathbf{R}}(\infty)$ be finite and set $\mathbf{R}_1=0$. Then

$$\mathbf{R}(t)=\mathbf{R}_0-\frac{e}{m}\int_t^\infty dt'(e^{\omega_0(t-t')}-1)\mathbf{A}_h(0,t')+\frac{\mathbf{P}}{m}t, \quad (26)$$

$$\dot{\mathbf{R}}(t)=-\frac{e\omega_0}{m}\int_t^\infty dt'\, e^{\omega_0(t-t')}\mathbf{A}_h(0,t')+\frac{\mathbf{P}}{m}, \quad (27)$$

$$\mathbf{A}(x,t)=\mathbf{A}_h+\frac{e}{4\pi}\frac{\dot{\mathbf{R}}(t-|\mathbf{x}|)}{|\mathbf{x}|}+\frac{e}{(4\pi)^2}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left[\frac{\dot{\mathbf{R}}(t-|\mathbf{x}-\mathbf{x}'|)}{|x'|^3}-\frac{3[\dot{\mathbf{R}}(t-|\mathbf{x}-\mathbf{x}'|)\cdot\mathbf{x}']\mathbf{x}'}{|\mathbf{x}'|^5}\right]. \quad (28)$$

It is worth noting that G_R determines a particular solution the first time derivative of which remains finite for all times. There exist other particular solutions which when combined with the homogeneous solution will satisfy the condition $\dot{\mathbf{R}}(-\infty)=\mathbf{P}/m$, but cannot avoid $\dot{\mathbf{R}}(t)$ blowing up when $t=\infty$. For example, another solution is

$$\mathbf{R}(t)=\mathbf{R}_0+\mathbf{R}_1e^{\omega_0 t}+\frac{e}{m}\int_{t_0}^t dt'(e^{\omega_0(t-t')}-1)\mathbf{A}_h(0,t')-\frac{\mathbf{P}e^{\omega_0 t}}{m\omega_0}+\frac{\mathbf{P}}{m}t.$$

By setting $\mathbf{R}_1=\mathbf{P}/m\omega_0$ we obtain

$$\dot{\mathbf{R}}(t)=\frac{e}{m}\int_{t_0}^t dt'\, e^{\omega_0(t-t')}\mathbf{A}_h(0,t')+\frac{\mathbf{P}}{m},$$

which satisfies $\dot{\mathbf{R}}(t_0)=\mathbf{P}/m$, but $\dot{\mathbf{R}}(\infty)$ blows up.

We turn now to the commutators. Our particular solution involves

$$\frac{1}{2\pi}\int_{-\infty}^\infty dt'\int_{-\infty}^\infty d\omega\,\frac{e^{i\omega(t-t')}}{D_R(i\omega)}\mathbf{A}_h(0,t').$$

If one does not interchange the order of integration, then straight-forward calculation shows that this particular solution contributes only to $[\mathbf{R},\mathbf{A}]$. We may then state that except for $[R,A]\neq 0$, all the initially assumed commutation rules will be satisfied.

In calculating commutators the following pitfall must be avoided. If we make the usual plane wave expansion of $\mathbf{A}_h$ and interchange orders of integration in the above particular solution without regard for the zeros of $D_R(i\omega)$, then

$$\frac{1}{2\pi}\int_{-\infty}^\infty dt'd\omega\,\frac{e^{i\omega(t-t')}}{D_R(i\omega)}\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{(2k)^{\frac{1}{2}}}[\mathbf{a}_k e^{-ikt}+\mathbf{a}_k^\dagger e^{ikt}]=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{(2k)^{\frac{1}{2}}}\left[\frac{\mathbf{a}_k e^{-ikt}}{D_R(-i\omega)}+\frac{\mathbf{a}_k^\dagger e^{ikt}}{D_R(i\omega)}\right]. \quad (29)$$

However, the zeros of $D_R(i\omega)$ introduce a nonuniform convergence with respect to t' and make the use of $\delta(\omega+k)$ to obtain the right-hand side of Eq. (29) invalid. If one were to use the right-hand side of Eq. (29) in calculating commutators, then one would obtain misleading contributions from the particular solution.

That the particular solution should contribute nothing to the $[\mathbf{R},\mathbf{P}]$ commutators is just what we would expect from the fact that it vanishes in Eq. (27) in the limit $t\rightarrow\pm\infty$. Were we to use the right-hand side of Eq. (29), we would get a time-independent contribution to the commutators—which contradicts the result in the preceding sentence.

It is now clear that the Fourier transform of the solution does not exist. If one initially assumed the existence of Fourier transforms and transformed the differential equations into algebraic ones, one would arrive at the right-hand side of Eq. (29) as a particular solution. However, the homogeneous solution for $\mathbf{R}(t)$ obviously has no Fourier transform, while the above particular solution has none either. In treating the harmonic oscillator, Sokolov and Tumanov[5] and Norton and Watson[6] used the Fourier transform method and consequently arrive at misleading conclusions. We will return to the latter point again when we discuss the harmonic oscillator.

(ii) Stationary Case

The solution of Eq. (7) is now given by

$$\mathbf{A}=\mathbf{A}_h+\mathrm{Tr}\frac{e}{8\pi}\int d^3x'\,\rho(\mathbf{x}')\times\frac{[\mathbf{V}(t-|\mathbf{x}-\mathbf{x}'|)+\mathbf{V}(t+|\mathbf{x}-\mathbf{x}'|)]}{|\mathbf{x}-\mathbf{x}'|}. \quad (30)$$

Upon going over to the point-charge limit and using Eqs. (16)–(19), we get the equation of motion

$$\dot{\mathbf{R}}(t)=-(e/m)\mathbf{A}_h(0,t)+(1/m)\mathbf{P}, \quad (31)$$

the solution of which is

$$\mathbf{R}(t)=\mathbf{R}_0-\frac{e}{m}\int_{-\infty}^t dt'\,\mathbf{A}_h(0,t')+\frac{1}{m}\mathbf{P}t. \quad (32)$$

The results regarding commutators are the same now as in the preceding case.

IV. HARMONIC OSCILLATOR

(i) Retarded Case

We now take $U(\mathbf{R})=\frac{1}{2}K\mathbf{R}^2$. The solution of Eq. (7) is again given by Eq. (13). We substitute Eq. (13) into Eq. (5). We again adopt the defining Eqs. (16)–(19). Upon passing to the point charge limit, the resulting equation of motion is

$$\frac{d^3\mathbf{R}}{dt^3}-\frac{\alpha d^2\mathbf{R}}{dt^2}-\alpha(K/m)\mathbf{R}=(e\alpha/m)\partial\mathbf{A}_h(0,t)/\partial t, \tag{33}$$

where $\alpha=6\pi m/e^2$. The characteristic function of the homogeneous part of Eq. (33) is

$$D_R(\omega)=\omega^3-\alpha\omega^2-\alpha K/m. \tag{34}$$

A particular solution of Eq. (33) may be obtained by means of the Green's function,

$$G_R(t,t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega(t-t')}}{D_R(i\omega)}$$

$$=\begin{cases}\dfrac{-e^{\omega_0(t-t')}}{(i\omega_0+i\omega_1+\omega_2)(i\omega_0+i\omega_1-\omega_2)}; & t'-t>0\\[2ex] \dfrac{1}{2\omega_2}\left[\dfrac{e^{i\omega_2(t-t')}}{(i\omega_0+i\omega_1+\omega_2)}-\dfrac{e^{-i\omega_2(t-t')}}{(i\omega_0+i\omega_1-\omega_2)}\right]\times e^{-\omega_1(t-t')}; & t'-t<0\end{cases} \tag{35}$$

where $\omega_0>0$ and $-\omega_1\pm i\omega$ with $\omega_1>0$ are the zeros of $D_R(\omega)$. $G_R(t,t')$ satisfies

$$\frac{d^3G_R}{dt^3}-\frac{\alpha d^2G_R}{dt^2}-(\alpha K/m)G_R=\delta(t-t'). \tag{36}$$

The solution of Eq. (33) may now be written

$$\mathbf{R}(t)=\mathbf{R}_0e^{\omega_0t}+\mathbf{R}_1e^{-(\omega_1-i\omega_2)t}+\mathbf{R}_1^*e^{-(\omega_1+i\omega_2)t}+\frac{e}{m}\alpha\int_{-\infty}^{\infty}dt'\,G_R(t,t')\partial\mathbf{A}_h(0,t')/\partial t'. \tag{37}$$

It follows from Eq. (35) that once again one cannot use the plane-wave decomposition for $\mathbf{A}_h(\mathbf{x},t)$ and interchange orders of integration in the particular solution to obtain the form of Sokolov and Tumanov[5] and Norton and Watson,[6,12] namely

$$\mathbf{R}(t)=-\frac{ie\alpha}{m}\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\left(\frac{k}{2}\right)^{\frac{1}{2}}\times\left[\frac{\mathbf{a}_ke^{-ikt}}{D_R(-ik)}-\frac{\mathbf{a}_k^\dagger e^{ikt}}{D_R(ik)}\right]. \tag{38}$$

If we now follow Dirac's prescription[13] for getting rid of the real exponentials in Eq. (37), we are left with just the last term on the right hand side. But $\mathbf{P}=md\mathbf{R}(t)/dt-(m/\alpha)d^2\mathbf{R}(t)/dt^2+e\mathbf{A}_h(0,t)$. A calculation then shows that

$$[\mathbf{R}(t),\mathbf{P}(t)]=i\left[1-\frac{4(\omega_1^2+\omega_2^2)}{\omega_0^2+2(\omega_1^2+\omega_2^2)}\right]. \tag{39}$$

Although Eq. (38) is not equivalent to the particular solution in Eq. (37), one can use it to calculate $[\mathbf{R}(t),\mathbf{P}(t)]$ and obtain the same result as in Eq. (39). However, the two calculations are not equivalent. Upon using the particular solution in Eq. (37) we find that $[\mathbf{R},\dot{\mathbf{R}}]=[\mathbf{R},d^2\mathbf{R}/dt^2]=0$ so that $[\mathbf{R}(t),\mathbf{P}(t)]=e[\mathbf{R}(t),\mathbf{A}_h(0,t)]$. A direct calculation shows that the preceding statement is not true if one uses Eq. (38).

In any case Eq. (39), leads us to conclude (from the next paper of the series) that (within the framework of Ostrogradsky method[10]) the above solutions are not consistent with the canonical formalism.

(ii) Stationary Case

Using Eq. (30) and following the procedure outlined above, we arrive at the equation of motion

$$\frac{d^2\mathbf{R}}{dt^2}+\omega_0^2\mathbf{R}=-\frac{e}{m}\frac{\partial\mathbf{A}_h(0,t)}{\partial t}, \tag{40}$$

where $\omega_0^2=K/m$. The characteristic function of the homogeneous part of Eq. (38) is

$$D_s(\omega)=\omega^2+\omega_0^2, \tag{41}$$

where the subscript s is used to indicate that D_s is associated with stationary boundary conditions.

Using the Green's function

$$G_s(t,t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega(t-t')}}{D_s(i\omega)}=\begin{cases}-\dfrac{\sin\omega_0(t-t')}{2\omega_0}; & t'-t>0\\[2ex] \dfrac{\sin\omega_0(t-t')}{2\omega_0}; & t'-t<0\end{cases}, \tag{42}$$

[12] In using their Fourier transform method, Norton and Watson first treat the case of a finite size charge. In $D_R(\omega)$, there then appears an integral over **k** space. This led them to conclude that $D_R(\omega)$ had only one zero. However, they overlook the fact that in the point-charge limit, their expression reduces to a cubic in ω with three nonzero roots (one real, two complex). Moreover, if they had not taken Fourier transforms in the finite-size charge case, the above-mentioned integral would appear as a double integral over **x** space. Further analysis would then show that for sufficiently small size charge ($m_0<0$), $D_R(\omega)$ has three zeros.

[13] P. A. M. Dirac, Proc. Roy. Soc. (London) **167**, 148 (1938).

the solution of Eq. (39) may be written

$$\mathbf{R}(t)=\frac{1}{m(2\omega_0)^{\frac{1}{2}}}[\mathbf{R}_0e^{-i\omega_0 t}+\mathbf{R}_0{}^*e^{i\omega_0 t}] -\frac{e}{m}\int_{-\infty}^{\infty}dt'\,G_s(t,t')\frac{\partial\mathbf{A}_h(0,t')}{\partial t'}. \quad (43)$$

Again, the particular solution on the right-hand side of Eq. (42) will contribute only to $[\mathbf{R},\mathbf{A}]$. We are therefore led into setting

$$[\mathbf{R}_0,\mathbf{R}_0{}^*]=1. \quad (44)$$

The assumption of Eq. (43) will then guarantee that except for $[\mathbf{R},\mathbf{A}]\neq 0$, all the desired commutation rules will be satisfied. This will enable us in the next paper of the series to fit the above solution into a canonical scheme based on the Ostrogradsky method.

V. SUMMARY

We have used the electric dipole approximation to study the problem of finding commuting solutions of coupled equations of motion. We pointed out that for a charged particle in an external radiation field, the solutions of the coupled equations cannot be considered independent in the sense of commuting with one another if the homogeneous solutions are assumed to have the commutation properties of uncoupled variables. As we will show in the next paper of this series, the lack of independence is a result of constraints in the theory. In that paper, we will combine the Ostrogradsky method with the modified Poisson brackets of Dirac[14] to show how the theory fits into a canonical formalism.

We have explicitly solved the case of a charged free particle and a charged harmonic oscillator in an external radiation field. We have indicated that for a retarded (advanced) self-field, the free particle fits into a canonical formalism while the oscillator does not. For a stationary self-field, both the free particle and the oscillator fit into a canonical formalism. However, we have shown that the Fourier transforms of the configuration space solutions (based on $e^{i\mathbf{k}\cdot\mathbf{x}}$ and $e^{i\omega t}$) do not exist. In the latter connection, we pointed out that one could construct particular solutions using a plane-wave expansion for the free field, but those solutions lead to misleading contributions to the commutators. In fact if one takes the stationary solution for the self-field, the integrals appearing in the commutators are ambiguous and divergent.

In the succeeding papers we will also deal with the problem of alternative Fourier decompositions and associated boundary conditions.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor R. Arnowitt, Professor P. G. Bergmann, Professor H. Kaplan, Professor A. Komar, and Professor R. Schiller for helpful discussions. We are indebted to Professor A. A. Sokolov for calling our attention to the incorrectness of our original assertion that the right-hand side of Eq. (39) was zero.

[14] P. A. M. Dirac, Can. J. Math. 2, 129 (1950); 3, 1 (1951).