e.g., (5) or (16)]. These integrals, in turn, can be evaluated in an elementary manner by using the integral representation

$$Q_l(y)=\frac{1}{2}\int_{-1}^{+1} dx\, P_l(x)(y-x)^{-1}. \tag{26}$$

Thus, with an s-state potential of the type (21) (with $l=0$) operative between 2 nucleons, the deuteron binding energy α^2/M and the s-phase shift δ_0 are given by

$$(\alpha/4\pi^2\lambda_0)=\ln(2\alpha+\beta_0/\beta_0), \tag{27}$$

and

$$k\cot\delta_0=\left(1-\frac{4\pi^2\lambda_0}{k}\tan^{-1}\frac{2k}{\beta_0}\right)\left(\frac{2\pi^2\lambda_0}{k^2}\ln\frac{\beta_0{}^2+4k^2}{\beta_0{}^2}\right)^{-1}, \tag{28}$$

corresponding to the triplet effective-range parameters

$$0.8\approx(a\alpha)^{-1}=2x^{-2}[x-\ln(1+x)], \tag{29}$$

and

$$0.2\approx\tfrac{1}{2}r_0\alpha=\ln(1+x)-\tfrac{1}{3}x, \tag{30}$$

where $x=2\alpha/\beta_0$. These relations are satisfied with $\beta_0\approx5\alpha$ corresponding to a range of 0.9×10^{-13} cm. While this value is somewhat smaller than the meson Compton wavelength, viz. $\mu^{-1}\approx1.4\times10^{-13}$ cm, it is reasonable enough to warrant more detailed calculations with such potential shapes, and such calculations are in progress.

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Subtractions in Dispersion Relations*

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The following theorem is proved: If an analytic function $f(z)$ has singularities only on the real axis and is bounded in magnitude at infinity by a finite but arbitrary power of z, then $f(z)$ has essentially the same limits everywhere at infinity. This theorem enables one to express the contribution from the infinite circle of the Cauchy contour integral in terms of the boundary values of $f(z)$ at infinity along only one of the cuts extending to infinity. The exact dispersion relation is thus determined. As examples, we derive the forward and double pion-nucleon dispersion relations, assuming that the total cross section approaches a finite limit at infinite energy. We see how the subtractions are determined completely by the theorem.

I. INTRODUCTION

In order to determine the number of subtractions in dispersion relations, we usually introduce subtractions until the dispersion integrals appear to be convergent on the basis of conjectured asymptotic behaviors of integrands along the cuts. It could, however, be that the contribution from the infinite circle of the Cauchy contour integral we started with does not yet vanish, which implies that the subtracted dispersion relation has to be supplemented by some finite terms. It could also be that the last subtraction was unnecessary since the dispersion relation prior to the last subtraction was already finite because of the cancellation of divergences among dispersion integrals (in the case when there are two cuts extending to infinity).

Obviously the best way to find out the exact dispersion relation is to deal directly with the integral over the infinite circle in the original Cauchy integral. The question then arises how we know the behavior of the function at arbitrary infinite points in the complex plane. This is exactly why we wish to prove the theorem (stated in Sec. II), which says that the behavior at infinity is essentially the same everywhere in the complex plane even when the branch cuts extend to infinity, as long as we can expect dispersion relations at all.

In Sec. II we state the theorem and the simplest form of the dispersion relation when the function approaches finite limits along one of the cuts extending to infinity. We present the proof in Sec. III. In Sec. IV are given supplementary remarks on the theorem, applying to special cases when there is crossing symmetry and when only one cut extends to infinity. We remark also how to use the theorem to get dispersion relations in the case of asymptotic behaviors other than simple finite limits.

As examples of application of the theorem, we derive forward (Sec. V) and double (Sec. VI) pion-nucleon dispersion relations, assuming asymptotic behavior of scattering amplitudes which is consistent with the finite total cross section at infinite energy. The theorems due to Pomeranchuk[1] and Amati, Fierz, and Glaser[2] follow as immediate consequences of the present theorem. The double dispersion relation is essentially the same as, but

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[1] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 725 (1958) [Soviet Phys. J.E.T.P. **34(7)**, 499 (1958)].
[2] D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters **4**, 89 (1960).

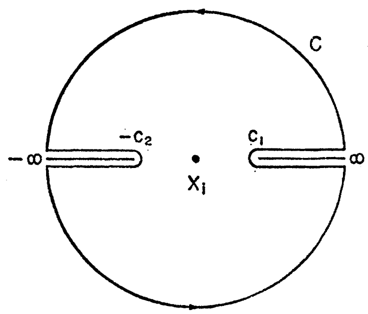

FIG. 1. Singularities of $f(z)$ in the z plane are shown. The two lines are branch cuts and the dot is a pole. The contour line C is the one to which Cauchy's theorem (4) is applied.

definitely simpler than, the one conjectured by Mandelstam.[3]

In the Appendix, we present the most technical part of our proof of the theorem. This part is, however, essential to our proof.

II. STATEMENT OF THE THEOREM

Let $f(z)$ be analytic everywhere in the complex z plane except for two cuts and poles on the real axis as shown in Fig. 1. We assume that the divergence of $f(z)$ at $|z|=\infty$ is not stronger than a large but finite power of $|z|$.[4]

If $f(z)$ has finite limits $f(\infty\pm i\epsilon)$ as $z\to\infty\pm i\epsilon$ along the c_1 cut (ϵ being a positive infinitesimal number), then the limits of $f(z)$ when z approaches infinity in any other direction are

$$\begin{aligned}\lim_{|z|\to\infty} f(z)&=f(\infty+i\epsilon),\ \text{in the upper half-plane,}\\ &=f(\infty-i\epsilon),\ \text{in the lower half-plane,}\end{aligned}\tag{1}$$

provided that $f(z)$ approaches definite (not necessarily finite) limits at $-\infty$ along the c_2 cut. The dispersion relation for $f(z)$ becomes

$$f(z)=\sum_i\frac{R_i}{z-x_i}+\frac{1}{\pi}\left(\int_{c_1}^{\infty}+\int_{-\infty}^{-c_2}\right)\frac{\Delta f(x)dx}{x-z}+\bar{f}(\infty),\tag{2}$$

where

$$\begin{aligned}\Delta f(x)&=(1/2i)[f(x+i\epsilon)-f(x-i\epsilon)],\\ \bar{f}(x)&=(1/2)[f(x+i\epsilon)+f(x-i\epsilon)],\end{aligned}\tag{3}$$

are respectively, the absorptive and dispersive parts of $f(z)$ when z approaches real x in the upper half plane and R_i is the residue at the pole at x_i. It is stressed that the behavior at the end of either cut is sufficient to yield (1) and (2). We do not have to know the limits of the $f(z)$ along both cuts simultaneously.

The proof is given in Sec. III and in the Appendix. Our proof is complete in so far as $f(z)$ satisfies the conditions implied by (9), (40), and (42) below. It is possible that the theorem is correct without these conditions. These conditions are already sufficiently weak to accommodate virtually all the cases of actual interest in physics.

III. PROOF OF THE THEOREM

The Cauchy integral theorem applied to the contour C of Fig. 1 is

$$f(z)=\sum_i\frac{R_i}{z-x_i}+\frac{1}{\pi}\int_{\text{cuts}}\frac{\Delta f(x)dx}{x-z}+\frac{1}{2\pi i}\int_{\infty}\frac{f(z')dz'}{z'-z},\tag{4}$$

where the second term is the same as the second term of (2) and the last term is the integral over the infinite circle.

Since (4) is correct for any z as long as z is inside C and, therefore, $|z'|>|z|$ in the last term of (4), we can expand $1/(z'-z)$ in a power series of z/z'. Applying the boundedness condition at $|z|=\infty$ to the integrands of the resulting series, we see that this series becomes a finite polynomial in z:

$$\frac{1}{2\pi i}\int_{\infty}\frac{f(z')}{z'-z}dz'=\sum_{n=0}^{N}\frac{z^n}{2\pi}\int_0^{2\pi}\frac{f(z')}{z'^n}d\theta=\sum_{n=0}^{N}a_nz^n,\tag{5}$$

where $z'=|z'|\exp(i\theta)$ and N is some positive integer for which $f(z)/z^N$ becomes at most finite at $|z|=\infty$.

We now see that the contribution from the infinite circle can always be eliminated by introducing $N+1$ subtractions, whereas we would need an infinite number of subtractions if the series in (5) did not terminate. Therefore, our boundedness condition is no more than is necessary in any case.[4]

We can rewrite (4) in a divergenceless form by introducing regulations,

$$\frac{1}{x-z}=\frac{z}{x(x-z)}+\frac{1}{x}=\frac{z^2}{x^2(x-z)}+\frac{z}{x^2}+\frac{1}{x}=\cdots.\tag{6}$$

If we introduce one regulation in the c_1 integral since $\Delta f(\infty)$ is finite and $N+1$ regulations in the c_2 integral and change the sign of x in all the c_2 integrals, we get

$$\begin{aligned}f(z)={}&\sum_i\frac{R_i}{z-x_i}+\frac{z}{\pi}\int_{c_1}^{\infty}\frac{\Delta f(x)dx}{x(x-z)}\\ &+(-1)^N\frac{z^{N+1}}{\pi}\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{x^{N+1}(x+z)}\\ &+\left(a_0+\int_{c_1}^{\infty}\frac{\Delta f(x)dx}{\pi x}-\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x}\right)\\ &+\left(a_1+\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x^2}\right)z+\cdots\\ &+\left(a_N+(-1)^{N+1}\int_{c_2}^{\infty}\frac{\Delta f(-x)dx}{\pi x^{N+1}}\right)z^N.\end{aligned}\tag{7}$$

[3] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[4] We do not state that $f(z)$ has no essential singularity at infinity, since the infinite point is not isolated in this case and the term, essential singularity, does not apply to such a point. The boundedness condition assumed here is equivalent to requiring that only a finite number of subtractions is necessary to obtain the exact dispersion relation, as is explained in the third paragraph of Sec. III.

Since the first two integrals are now convergent for finite z, the polynomial in (7) is also finite for finite z. Therefore, the coefficients of various powers of z are finite.

Evidently (7) contains many terms which diverge severely as $z \to \infty \pm i\epsilon$. In order that (7) as a whole approach finite limits as $z \to \infty \pm i\epsilon$, exact cancellation must occur among the various divergences. To analyze this, we first rewrite the first integral as follows:

$$\frac{z}{\pi}\int_{c1}^{\infty}\frac{\Delta f(x)dx}{x(x-z)}=\frac{\Delta f(\infty)}{\pi}\int_{c1}^{\infty}\left(\frac{1}{x-z}-\frac{1}{x+z}\right)dx$$
$$+\frac{1}{\pi}\int_{c1}^{\infty}\left(\frac{1}{x-z}-\frac{1}{x+z}\right)g(x)dx-\frac{z}{\pi}\int_{c1}^{\infty}\frac{\Delta f(x)dx,}{x(x+z)} \qquad (8)$$

where

$$g(x)=\Delta f(x)-\Delta f(\infty). \qquad (9)$$

Direct evaluation shows that the first term of (8) approaches the finite limits $\pm i\Delta f(\infty)$ as $|z| \to \infty$ in the upper and lower half planes, respectively. We show in the Appendix that the second term tends to zero as $|z| \to \infty$. The third term can be shown to diverge at most logarithmically and have in fact a logarithmic divergence as $z \to \infty$ if $\Delta f(\infty)$ is not zero. The argument is essentially the same as that which follows Eq. (39) in the Appendix.

The behavior of the second integral of (7) is now evident since it has the same structure as the third term of (8); it could diverge as strongly as $z^N \ln z$ for $z \to \infty \pm i\epsilon$ which would in fact be the case if $\Delta f(-x)/x^N$ had a nonzero limit as $x \to \infty$.

We thus have seen that the highest divergence in (7) as $z \to \infty \pm i\epsilon$ comes from the second integral. In order to reproduce the assumed behavior of $f(z)$ as $z \to \infty \pm i\epsilon$ however, the second integral can diverge only as strongly as z^N and moreover this highest divergence must exactly cancel the same divergence of the last term of (7). Therefore

$$\lim_{x\to\infty}\frac{\Delta f(-x)}{x^N}=0,$$
$$\lim_{z\to\infty} z\int_{c2}^{\infty}\frac{\Delta f(-x)dx}{x^{N+1}(x+z)}=\text{a finite number} \qquad (10)$$
$$=\int_{c2}^{\infty}\frac{\Delta f(-x)dx}{x^{N+1}}.$$

The last step of the second equation is guaranteed by a theorem quoted and explained by Amati, Fierz, and Glaser.[2] It follows from (10) that $a_N=0$ since a z^N divergence would otherwise remain in (7).

Since (10) also implies that the last regulation introduced in the c_2 integral is unnecessary, we go back to the previous stage where the whole expression for $f(z)$ becomes the same as (7), except that N is replaced by $N-1$. We can then repeat the same argument, provided $\Delta f(-x)$ has a definite limit (not necessarily finite) as $x \to \infty$, until we come to the expression (7) for $f(z)$ with $N=0$. In order to apply the same argument once more, we rewrite this last expression for $f(z)$ as

$$f(z)=\sum_i\frac{R_i}{z-x_i}+\frac{1}{\pi}\int_c^{\infty}\left(\frac{1}{x-z}-\frac{1}{x+z}\right)\Delta f(x)dx$$
$$-\frac{z}{\pi}\int_c^{\infty}\frac{\Delta f(x)-\Delta f(-x)}{x(x+z)}dx$$
$$+\left(a_0+\int_c^{\infty}\frac{\Delta f(x)-\Delta f(-x)}{\pi x}dx\right)+\cdots, \qquad (11)$$

where we have used (8) and c is arbitrary as long as $c>c_1$ and $c>c_2$ and the dots indicate integrals from c_1 and c_2 to c, which are not only finite but approach zero as $|z| \to \infty$, as can be shown easily.

We now apply the same argument to (11): The third term, the only one which could possibly diverge in (11), must have a finite limit as $z \to \infty \pm i\epsilon$. By the above quoted theorem, this limit is minus the second part of the fourth term of (11).

We now see that even the first regulations introduced in the c_1 and c_2 integrals are unnecessary. We also see that (11) implies

$$\lim_{z\to\infty\pm i\epsilon} f(z)=a_0\pm i\Delta f(\infty), \qquad (12)$$

which identifies a_0 as $\bar{f}(\infty)$ according to (3). We thus have shown that (4) or (7) reduces eventually to (2). It is remarked that the two integrals in (2) must together be finite, but do not have to be so separately.

In order finally to infer (1), we first remark that

$$\lim_{z\to\infty}\int_c^{\infty}\frac{\Delta f(x)-\Delta f(-x)}{x+z}dx=0, \qquad (13)$$

because this integral is the sum of the third term and the second part of the fourth term of (11). We now have only to prove that the integral in (13) approaches zero as $|z| \to \infty$, since all the rest of (11) becomes exactly $f(\infty \pm i\epsilon)$ as $|z| \to \infty$. (13) implies that the same integral with z replaced by $|z|$ goes to zero as $|z| \to \infty$. Therefore, we may as well prove that

$$\lim_{|z|\to\infty}\int_c^{\infty}\left(\frac{1}{x+z}-\frac{1}{x+|z|}\right)[\Delta f(x)-\Delta f(-x)]dx=0. \qquad (14)$$

We can prove (14) in almost the same way as we prove (33) in the Appendix, because both integrals have nearly the same structure. We know by now that $\Delta f(\infty)=\Delta f(-\infty)$. In fact, (14) becomes the same as (33) when $z \to -|z|$, where the proof of (14) becomes most difficult.

We add finally that $a_0=\bar{f}(\infty)$ is consistent with the definition of a_0 in (5) and the boundary conditions (1), as it should be.

IV. SUPPLEMENTARY REMARKS ON THE THEOREM

Remark 1. The theorem is correct independently of the number of poles and cuts, including cuts which do not extend to infinity. In the case when no cuts extend to infinity, the theorem becomes trivial since the infinite point is then isolated. In the case when only one cut (say, the c_1 cut) extends to infinity, (1) implies that $\Delta f(\infty)=0$, because $f(z)$ would not otherwise be continuous at $z=-\infty$. Such an example will be mentioned in Sec. VI.

Remark 2. In the case when the discontinuities across the cuts are pure imaginary, $\Delta f(x)=\mathrm{Im}f(x)$ and $\bar{f}(x)=\mathrm{Re}f(x)$, where $f(x)$ is understood to mean $f(x+i\epsilon)$.

Remark 3. In the case when we have crossing symmetry, which says essentially (see the examples in Secs. V and VI) $f(z)=\pm f(-z)$ as $z\rightarrow\infty\pm i\epsilon$, the theorem implies that either $\Delta f(\infty)$ or $\bar{f}(\infty)$ has to be zero, depending upon whether the symmetry is even or odd.

Remark 4. We assume in Secs. II and III that both limits $f(\infty\pm i\epsilon)$ are finite. This has to be always the case when the discontinuities are pure imaginary. Even in the case when, say, $f(\infty+i\epsilon)$ is finite and $f(\infty-i\epsilon)$ is infinite, the upper half of our statement (1) is correct: If $f(\infty+i\epsilon)$ is finite, then $f(z)$ has to tend to $f(\infty+i\epsilon)$ when z goes to infinity in any direction in the upper half-plane. The proof for this can be worked out in virtually the same way as is done in Sec. III; we have only to choose a contour line in Fig. 1 which consists of an infinite semicircle in the upper half plane and a path along the entire real axis.

Remark 5. If $f(z)$ is known to diverge as z goes to either or both of the limits $\infty\pm i\epsilon$, we introduce a known function $F(z)$ which diverges at least as strongly and apply our theorem to the new function $f(z)/F(z)$. Of course, $F(z)$ has to be such that $f(z)/F(z)$ still satisfies the boundedness condition in Sec. II and cannot have branch cuts or zero points elsewhere than on the real axis. If such an $F(z)$ exists, we can apply our theorem to $f(z)/F(z)$ and conclude that $f(z)$ behaves when $|z|\rightarrow\infty$ as $F(z)$ times constants which are the limits of $f(z)/F(z)$ at $z=\infty\pm i\epsilon$.

The extra factor $F(z)$ in the denominator introduces, in general, new poles and new branch cuts on the real axis. However, the residues and the discontinuities of $f(z)/F(z)$ at these new singular points can be given explicitly in terms of $f(z)$ on the real axis since $F(z)$ is known. We, therefore, can obtain the dispersion relation for $f(z)$ by first writing down the dispersion relation for $f(z)/F(z)$ according to our instruction (2) and then multiplying it by $F(z)$.

It is expected that this procedure will work in virtually all cases of interest in physics. An example in which $f(z)$ diverges linearly with respect z as $z\rightarrow\infty\pm i\epsilon$ is treated in detail in the next section.

Remark 6. If $f(z)$ is known to approach zero as $z\rightarrow\infty\pm i\epsilon$, the theorem says that $f(z)$ approaches zero everywhere at $|z|=\infty$ and the no-subtraction dispersion relation [(2) without the last term] is justified.

Remark 7. The theorem is proved only when all cuts are on the real axis. This, however, does not appear to restrict the applicability of the theorem to the cases of actual interest, including the case of double dispersion relations, as is shown in Sec. VI.

V. FORWARD PION-NUCLEON DISPERSION RELATIONS

Let $f_\pm(\omega)$ be the forward pion-nucleon scattering amplitudes as functions of pion energy ω in the laboratory system ($\pm$ referring to components symmetric or anti-symmetric with respect to pion isotopic spin). We normalize them so that the optical theorem reads

$$\mathrm{Im}f_\pm(\omega)=-(q/2)[\sigma_{p\pi^-}(\omega)\pm\sigma_{p\pi^+}(\omega)],\qquad(15)$$

where q is the pion laboratory momentum and the σ's are total cross sections in the charge channels indicated by subscripts.

Assuming that $\sigma(\infty)$ is finite, we begin with the boundary conditions that $f_\pm(\omega)/\omega$ have finite limits $f_\pm$ as $\omega\rightarrow\infty$.

We recall that there is crossing symmetry, $f_\pm(\omega)=\pm f_\pm{}^*(-\omega)$, and the discontinuities are pure imaginary in this case. We define, according to the instruction in Remark 5 of Sec. IV, new functions $f_\pm(z)/(z-a)$ which have finite limits $f_\pm$ when $z\rightarrow\infty\pm i\epsilon$, are bounded in the same sense as $f_\pm(z)$, and have the same analyticity properties as $f_\pm(z)$, except for new poles at $z=a$ with residues $f_\pm(a)$. These residues become real if a falls on the portion of the real axis between the two cuts. We see also that $f_\pm(z)/(z-a)$ still have crossing symmetry in the sense of Remark 3 of Sec. IV, with even-odd properties inverted from those of $f_\pm(\omega)$. It then follows from Remarks 2 and 3 that f_+ has to be pure imaginary, while f_- has to be pure real. In particular, the latter implies from (15) that $\sigma_{p\pi^-}(\infty)=\sigma_{p\pi^+}(\infty)$, which is nothing but the Pomeranchuk theorem.[1,2]

To obtain the dispersion relations, we apply (2) to $f_\pm(z)/(z-a)$, which yields

$$\frac{f_\pm(z)}{z-a}=\sum_i\left(\frac{R_i}{\omega_i-a}\right)\frac{1}{z-\omega_i}+\frac{f_\pm(a)}{z-a}+\frac{1}{\pi}\int_{\mathrm{cuts}}\frac{\mathrm{Im}f_\pm(\omega)d\omega}{(\omega-a)(\omega-z)}+\mathrm{Re}f_\pm,\qquad(16)$$

where $f_\pm(z)$ have poles at ω_i with residues R_i. Multiplying (16) by $(z-a)$, we obtain

$$f_{\pm}(z)=f_{\pm}(a)+\sum_i\left(\frac{R_i}{z-\omega_i}-\frac{R_i}{a-\omega_i}\right)$$
$$+\frac{1}{\pi}\int_{\text{cuts}}\left(\frac{1}{\omega-z}-\frac{1}{\omega-a}\right)\text{Im}f_{\pm}(\omega)d\omega+(z-a)\,\text{Re}f_{\pm}. \tag{17}$$

If our knowledge about ω_i, R_i, the cuts, and crossing symmetry is exploited, Eq. (17) become

$$f_+(z)=f_+(a)+\frac{2g^2\omega_0}{z^2-\omega_0^2}-\frac{2g^2\omega_0}{a^2-\omega_0^2}$$
$$+\frac{1}{\pi}\int_\mu^\infty\left[\frac{2\omega}{\omega^2-z^2}-\frac{2\omega}{\omega^2-a^2}\right]\text{Im}f_+(\omega)d\omega, \tag{18}$$

$$f_-(z)=f_-(a)-\frac{2g^2z}{z^2-\omega_0^2}+\frac{2g^2a}{a^2-\omega_0^2}$$
$$+\frac{1}{\pi}\int_\mu^\infty\left[\frac{2z}{\omega^2-z^2}-\frac{2a}{\omega^2-a^2}\right]\text{Im}f_-(\omega)d\omega+(z-a)f_-, \tag{19}$$

where $\omega_0=\mu^2/2M$, μ and M being the pion and nucleon masses and g is the renormalized pion-nucleon coupling constant. Equation (18) is nothing but the conventional once-subtracted dispersion relation,[5] which is known to be consistent with the present data. (19) is also in a subtracted form but not of the conventional type. The terms in the integral in (19) are not in a combination that enhances convergence of the integral. We can therefore conclude that the individual terms of the integral in (19) are already finite. This was first remarked by Amati, Fierz, and Glaser.[2,6]

We can now rewrite (19) in an unsubtracted form:

$$f_-(z)=-\frac{2g^2z}{z^2-\omega_0^2}+\frac{1}{\pi}\int_\mu^\infty\frac{2z\,\text{Im}f_-(\omega)}{\omega^2-z^2}d\omega+zf_-, \tag{20}$$

where the (real) constant which could in general be added to (20) has been dropped because of the odd symmetry of $f_-(z)$. We claim that (18) and (20) are the simplest forms of the dispersion relations for $f_{\pm}(z)$ when $f_{\pm}(\omega)/\omega$ approach finite limits as $\omega\to\infty$. It is added that the conventional form[5] of $f_-(z)$ follows from (20) if one (conventional) subtraction is applied to (20). We add also that (20) is an example in which the convergence of the dispersion integral does not justify the absence of additional terms in the dispersion relation.

We found however that the presently available data are all consistent with (20) without the last term. Incidentally what Goldberger, Miyazawa, and Oehme have observed in the note added in proof of their paper[5] amounts to the same claim. We then observe that (20) with $f_-=0$ follows immediately from the boundary condition with $f_-(\omega)$ have a finite limit as $\omega\to\infty$, since $f_-(\infty)$ will then have to be imaginary (Remarks 2 and 3) and (2) reduces to (20) with $f_-=0$. We now conclude that the boundary conditions

$$f_+(\omega)/\omega,\quad f_-(\omega)\underset{\omega\to\infty}{\longrightarrow}\text{finite limits} \tag{21}$$

are consistent with dispersion relations and the assumption that $\sigma(\infty)$ is finite.

VI. DOUBLE PION-NUCLEON DISPERSION RELATIONS

Let $A_{\pm}$ and $B_{\pm}$ be the invariant pion-nucleon scattering amplitudes which are related to $f_{\pm}(\omega)$ of the previous section by

$$f_{\pm}(\omega)=A_{\pm}(s,t=0)-\omega B_{\pm}(s,t=0),\quad s=M^2+\mu^2+2M\omega, \tag{22}$$

where the invariant variables are defined by

$$s=-(q+p)^2,\quad t=-(q'-q)^2,\quad u=-(p-q')^2,\quad s+t+u=2(M^2+\mu^2), \tag{23}$$

p and p' being initial and final momenta of the nucleon, and q and q' being those of the pion. According to Mandelstam,[3] the singularities are three cuts given by

$$+\infty\geqslant s\geqslant(M+\mu)^2,\quad +\infty\geqslant u\geqslant(M+\mu)^2,\quad +\infty\geqslant t\geqslant 4\mu^2, \tag{24}$$

and two poles of $B_{\pm}$ located at $s=M^2$ and $u=M^2$. Crossing symmetry is expressed by

$$A_{\pm}(s,t,u)=\pm A_{\pm}(u,t,s),\quad B_{\pm}(s,t,u)=\mp B_{\pm}(u,t,s). \tag{25}$$

We add that Eq. (25) agrees with the statement we made in Remark 3 of Sec. IV.

We argue below that plausible boundary conditions are that $A_{\pm}$ and $B_{\pm}$ all stay, at most, finite when any or all of the variables go to infinity. We start with (22) and the differential elastic cross section expressed in

[5] M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. **99**, 986 (1955).

[6] We should point out that their proof (reference 2) is imperfect in the following two respects: First, the argument leading to their Eq. (10) is misleading, since the trivial example $\sigma^+(E')=\sigma^+(\infty)+\text{const}/E'+\cdots$ gives a term $(\ln E)/E$ in addition to those of their Eq. (10); this term is, in fact, greater than any in their Eq. (10). However, their claim based on their Eq. (10) follows from our Eq. (32) which is proved in the Appendix. Secondly, their proof is based upon the conventionally subtracted dispersion relation; the validity of such a form is not guaranteed and, in fact, (20) below constitutes a counter-example.

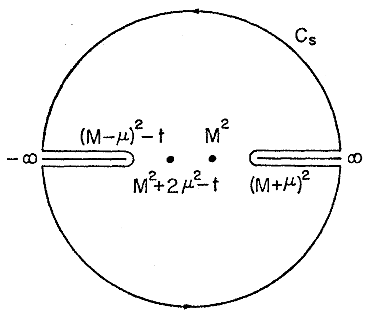

FIG. 2. Singularities of $B(s,t)$ in the s plane when t is real and greater than $-2M\mu+\mu^2$.

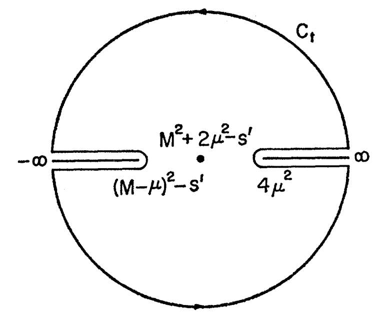

FIG. 3. Singularities of $B(s,t)$ in the t plane when s is real and greater than $(M+\mu)^2$.

terms of $A_\pm$ and $B_\pm$, as a function of the c.m. scattering angle θ, in the limit of large c.m. particle momentum q:

$$\frac{d\sigma}{d\Omega}\simeq\frac{1}{(8\pi)^2}\{\tfrac{1}{2}(1-\cos\theta)|A|^2+[2q^2(1+\cos\theta)+\tfrac{1}{2}M^2(1-\cos\theta)]\times|B|^2-\tfrac{1}{2}M(3+\cos\theta)(A^*B+B^*A)\},\quad(26)$$

which holds for $\theta=0$. A and B are linear combinations of $A_\pm$ and $B_\pm$ depending upon specific charge assignments and

$$s\simeq 4q^2,\quad t\simeq -2q^2(1-\cos\theta),\quad u\simeq -2q^2(1+\cos\theta).\quad(27)$$

The boundary condition (21) and the requirement that $d\sigma/d\Omega$ stay at most finite as $q\to\infty$ for $\theta\neq 0$ [we assume that $\sigma(\infty)$ is finite] imply that all amplitudes stay at most finite when any or all of the variables go to infinity, except that $A_+(s, t=0)$ is allowed to diverge linearly as $s\to\infty$. Those boundary conditions which do not follow from (22) and (26) can be inferred from the symmetry (25). Since we know at present no compelling reasons for allowing such a divergence in A_+, we assume in the present paper that all amplitudes stay at most finite at infinity. Incidentally, this is essentially what Mandelstam[3] derived from perturbation calculations. We shall not further elaborate the boundary condition, since the primary purpose of the present paper is to obtain the dispersion relation from the assumed boundary condition.

Consider $B(s,t)$ (we shall drop the indices $\pm$ and pole-terms in all the equations below to simplify notation) with t real and greater than $-2M\mu+\mu^2$. All the singularities of $B(s,t)$ in the s plane then appear on the real axis, as shown in Fig. 2, none overlapping any other. The dispersion relation is therefore the same as (2):

$$B(s,t)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\Delta_{s'}B(s',t)ds'}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\Delta_{u'}B(u',t)du'}{u'-u}+\bar{B}(s=\infty,t),\quad(28)$$

where the subscripts on Δ are those variables with respect to which the differences are to be taken and we have changed s' to u' in the second integral.

Next we consider $B(s',t)$, $B(u',t)$, and $B(s'=\infty,\ t)$ in (28) separately. Since the singularities of $B(s',t)$ in the t plane with s' real and greater than $(M+\mu)^2$ all appear on the real axis as shown in Fig. 3, none overlapping another, $B(s',t)$ satisfies (2). The difference with respect to s' is then given by

$$\Delta_{s'}B(s',t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Delta_{s't'}B(s',t')dt'}{t'-t}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\Delta_{s'u'}B(s',u')du'}{(s'+u')-(s+u)}+\Delta_{s'}\bar{B}(s',t=\infty),\quad(29)$$

where the pole term in $B(s',t)$ cancels out exactly in the difference. Since (29) is correct for any t inside the contour C_t of Fig. 3, which includes $4\mu^2>t>-2M\mu^2+\mu^2$, we can substitute (29) into the first integral of (28). Into the second term of (28) we can substitute the analogous expression for $\Delta_{u'}B(u',t)$. It is now simple algebra to show that these two integrals of (28) give the conventional three double integrals and the two single integrals with respect to s and u, respectively, the latter having the same structure as the t integral of (30) below.

$B(s=\infty,\ t)$ of the last term of (28) has in the t plane a single cut from $4\mu^2$ to ∞, all the rest of the singularities having moved to infinity along the negative real axis, as is seen from Fig. 3. This is an example where there is a single cut. The dispersion relation is still of the form of (2):

$$\bar{B}(s=\infty,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Delta_{t'}\bar{B}(s=\infty,t')}{t'-t}dt'+\bar{B}_{\text{av}}(s=\infty,t=\infty),\quad(30)$$

where we have taken the implied average with respect to s at infinity and $\bar{B}_{\text{av}}$ means the averages with respect to both s and t at infinity. Since (30) is also correct for any real t such that $4\mu^2>t>-2M\mu+\mu^2$, we now get the complete double dispersion relation: $B(s,t)=$three double integrals

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Delta_{t'}\bar{B}(s=\infty,t')}{t'-t}dt'$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\Delta_{s'}\bar{B}(s',t=\infty)}{s'-s}ds'$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\Delta_{u'}\bar{B}(u',t=\infty)}{u'-u}du'$$
$$+\bar{B}_{\text{av}}(s=\infty,t=\infty). \quad (31)$$

This is the final form since (31) can be continued analytically in the t plane.

We list below a few of the characteristics of the dispersion relation (31). Firstly, the three double integrals together, the three single integrals individually, and the additional constant in (31) are finite. This can be seen by observing that the single t integral and the additional constant of (30) are separately finite and then remarking that our arguments in this section are completely symmetrical with respect to the interchange of the three variables. This is one of the most noticeable differences between (31) and the representation due to Mandelstam,[3] where regulations are introduced into the integrals involved.

Secondly, we have identified all the integrands and the additional constant in (31) as the arguments given indicate explicitly; the integrands in the double integrals are obviously the double differences across the cuts concerned. Therefore, if we know that some amplitudes vanish when some of the variables go to infinity (either plus or minus infinity, but not both), we can tell which terms of (31) are missing. For example, the symmetry conditions (25) imply that the single t integral is missing for A_- and B_+ since $\bar{A}_-(s=\infty,t)$ and $\bar{B}_+(s=\infty,t)$ have to vanish (Remark 3 of Section IV). We can tell also that B_+ cannot have the additional constant in (31), since $d\sigma/d\Omega$ (26) would otherwise diverge when $q\rightarrow\infty$ for $\pi>\theta>0$, as can be seen from (27). We are unable, however, to argue that A_+ cannot have the additional constant in (31) either, which is another marked difference between our (31) and that of Mandelstam.[3] Our dispersion relation (31) contains two parameters, the residue of the pole-term [which is not given explicitly in (31)] and the additional (real) constant for A_+. It is expected that this constant is closely related to the pion-pion scattering term constant of present pion field theory.

As usual, we now can argue in (31) that all integrands and the additional constant are real, even though individual $\bar{B}$'s and ΔB's are not necessarily so.

So far we have developed our arguments on the assumption that the total cross section approaches a finite limit at infinite energy. To give support to this, we have examined the case that $f_\pm(\omega)$ both have finite limits. In this case, $f_+(\infty)$ is real, and thus (15) implies that $\sigma(\omega)$ approaches zero faster than $1/\omega$. On the other hand, the $B_\pm$ would decrease only as $1/s$ for $s\rightarrow\infty$, because they have poles in s. Thus $d\sigma/d\Omega$ for $\pi>\theta>0$ would approach zero only as $1/s$ or $1/\omega$ according to (26) and (27), which is not consistent with the original assumption.

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APPENDIX

We here prove that

$$\lim_{|z|\rightarrow\infty}\int_c^{\infty}\left(\frac{1}{x-z}-\frac{1}{x+z}\right)g(x)dx=0, \quad (32)$$

where c is an arbitrary finite number and $g(x)$ approaches zero as $x\rightarrow\infty$.

We split (32) into five parts, putting $|z|=r$ and introducing a finite positive number a smaller than 1, as

$$\int_c^{\infty}=\int_c^{\sqrt{r}}+\int_{\sqrt{r}}^{r(1-a)}+\int_{r(1+a)}^{\infty}+\int_{r(1-a)}^{r(1+a)}\frac{g(x)}{x-z}dx$$
$$-\int_{r(1-a)}^{r(1+a)}\frac{g(x)}{x+z}dx, \quad (33)$$

where the integrands are the same, when not given, as that of (32). We prove in the following that these five terms approach zero individually as $r\rightarrow\infty$. The first term behaves as follows:

$$\left|\int_c^{\sqrt{r}}\right|\leq|g(x')|\int_c^{\sqrt{r}}\frac{2rdx}{r^2-x^2}$$
$$=|g(x')|\ \ln\frac{(r+\sqrt{r})(r-c)}{(r-\sqrt{r})(r+c)}, \quad (34)$$

where x' is the point between c and $\sqrt{r}$ at which $g(x)$ assumes the maximum magnitude. Since (34) is now seen to go to zero as $r\rightarrow\infty$, the first term of (33) behaves likewise. The second and the third terms can be treated quite the same way: We can show that the magnitudes of these terms cannot exceed $|g(x)'|$ times integrals of the type of (34). The integrals now stay finite as $r\rightarrow\infty$ but $|g(x')|$ approaches zero since the lower limits of the integrals go to ∞.

The fourth term can be written as

$$\int_{1-a}^{1+a}\frac{g(rx)}{x-e^{i\theta}}dx, \quad (35)$$

where we have put $z=re^{i\theta}$ and changed the integration variable. We see that the integrand in (35) tends to zero as $r \to \infty$ everywhere in the (finite) integration region as long as $\theta \neq 0$. Therefore the fourth term approaches zero as $r \to \infty$ as long as $\theta \neq 0$. Since the fifth term has the same structure as the fourth, we see also that the fifth term goes to zero as $r \to \infty$ as long as $\theta \neq \pi$.

We now examine the fourth term when $\theta = 0$ and prove that it goes to zero as $r \to \infty$. This obviously completes our proof. In the limit of $\theta = 0$, (35) becomes

$$P\int_{1-a}^{1+a} \frac{g(rx)dx}{x-1} + i\pi g(r), \tag{36}$$

where P stands for the principal value of the integral and the second term of (36) goes to zero as $r \to \infty$. If we split the principal value integral into two parts, $1-a \to 1-\delta$ and $1+\delta \to 1+a$, and change the integration variables so that the two parts can be combined into a single integral, the first term of (36) can be written as

$$\int_0^1 \frac{g(r+arx)-g(r-arx)}{x} dx. \tag{37}$$

We have set δ equal to zero in (37); at $x=0$ the integrand becomes

$$2ar[dg(r)/dr]. \tag{38}$$

The integral (37) approaches zero as $r \to \infty$ if (38) stays at most finite as $r \to \infty$. We can in fact show that the limit of (38), if it exists whether finite or not, has to be zero in order that $g(x)$ tend to zero as $x \to \infty$: Putting $x[dg(x)/dx] = F(x)$, we get

$$g(x) = \int^x \frac{F(y)}{y} dy = F(x) \int^x \frac{dy}{y} + \int^x \frac{F(y)-F(x)}{y} dy. \tag{39}$$

The first integral would diverge as $F(x)\ln x$ as $x \to \infty$ if $F(\infty)$ did not vanish. The second integral could diverge, but the divergence of the second integral cannot exceed that of the first integral if $F(\infty) \neq 0$. This is because the lower integration limit of the second integral can be shifted to any large finite number without introducing any divergence in the limit of $x \to \infty$. Thus we may choose a lower limit for which $F(y)-F(x)$ does not exceed in magnitude, say, half of $F(\infty)$ anywhere in the integration region. We see therefore that $g(x)$ as given by (39) would diverge at least logarithmically as $x \to \infty$ if $F(\infty)$ did not vanish. Since $g(x)$ must go to zero, $F(\infty)$ has to vanish.

In order to examine the case when the limit of (38) does not exist, we separate $g(x)$ without any loss of generality, into two factors:

$$g(x) = A(x)S(x), \tag{40}$$

where $A(x)$ approaches zero as $x \to \infty$ in such a way that the limit of $x[dA(x)/dx]$ as $x \to \infty$ exists, and $S(x)$ is bounded by a finite constant but may oscillate all the way to infinity. (37) can then be written as the sum of three terms:

$$\int_0^1 \frac{A(r+arx)-A(r)}{x} S(r+arx)dx + \int_0^1 \frac{A(r)-A(r-arx)}{x} S(r-arx)dx + A(r)\int_0^1 \frac{S(r+arx)-S(r-arx)}{x} dx. \tag{41}$$

We can show that the first and second terms of (41) approach zero as $r \to \infty$ in exactly the same way we showed that (37) approaches zero when the limit of (38) exists.

To show that the third term of (41) vanishes as $r \to \infty$, we have only to prove that the integral involved stays at most finite as $r \to \infty$. To do this, let us assume an expansion

$$S(x) = \int_{-\infty}^{+\infty} e^{ikx} s(k) dk, \tag{42}$$

where $s(k)$ may include δ functions corresponding to sin and cos functions in $S(x)$. We can then rewrite the integral in question as

$$\int_{-\infty}^{+\infty} e^{ikr} s(k) dk \int_0^{ar} \frac{2i \sin ky}{y} dy. \tag{43}$$

The integrals (42) and (43) are different only in the last factor of (43), which is known to be $i\pi$, 0 or $-i\pi$ depending upon whether $k>0$, $k=0$, or $k<0$, respectively, in the limit of $r \to \infty$. Therefore we see no sign of divergence in (43) as long as (42) converges.

We do not know how the integral of the third term of (41) behaves if $S(x)$ does not allow an expansion of the type of (42). It is possible that this integral remains finite even when the expansion (42) fails to be valid. However, we shall not further elaborate this point since we feel that our conditions implied by (40) and (42) are weak enough to accommodate practically all cases of actual interest in physics. We also remark that the function $g(x)$ cannot be completely arbitrary because of the original assumptions about $f(z)$ itself.