PHYSICAL REVIEW VOLUME 123, NUMBER 5 SEPTEMBER 1, 1961

Electromagnetic Sources in General Relativity Theory

THOMAS R. WAITE*
Atomics International, Canoga Park, California
(Received January 10, 1961)

The simplest, most direct method of unifying Maxwell's theory of electromagnetism and Einstein's theory of gravitation was formulated by Rainich in 1925. That theory applies only to charge-free space. However, in regions of space in which the electromagnetic field invariant corresponding to $\mathbf{E}\cdot\mathbf{B}$ vanishes, the two sets of Maxwell's equations are independent for Rainich's unified theory. The Rainich theory may be modified to allow for nonvanishing charge and current density in such regions. The electromagnetic sources and fields obey Maxwell-Lorentz theory and the electromagnetic matter-energy obeys the laws of Einstein's general relativity theory. The necessary and sufficient conditions which one must impose on the metric tensor and its derivatives in order to assure the existence of a unique antisymmetric tensor obeying the Maxwell-Lorentz laws in the presence of charges and currents have been derived.

I. INTRODUCTION

A UNIFIED classical theory of gravitation and electromagnetism is important for two reasons: (1) Quantum theories are in general formulated by quantization of an otherwise complete classical theory. (2) Although the existence of isolated purely electro-magneto-gravitational objects in the real world seems unlikely, the theoretical possibilities have not been fully investigated.

The simplest, most direct method of unifying Maxwell's theory of electromagnetism and Einstein's theory of gravitation was formulated by Rainich[1] in 1925. Rainich considered Riemannian spaces containing an antisymmetric tensor field, F_{jk}, obeying Maxwell's equations. The metric and curvature of the space were fixed by equating the Einstein tensor to the electromagnetic stress-energy-momentum tensor, $T_k{}^j$. The only matter-energy considered was that in the electromagnetic field. The field equations are:

$$F^{k\alpha}{}_{,\alpha}=0, \tag{1}$$

$$F_{jk,l}+F_{lj,k}+F_{kl,j}=0, \tag{2}$$

$$T^j{}_k=-F^{\alpha j}F_{\alpha k}+\tfrac{1}{4}(F^{\alpha\beta}F_{\alpha\beta})\delta^j{}_k, \tag{3}$$

$$R^j{}_k-\tfrac{1}{2}R\delta^j{}_k=-(8\pi G)T^j{}_k, \tag{4}$$

G is the Newtonian gravitation constant. R_{jk} is the Ricci curvature tensor and R is the scalar curvature. Rainich considered the questions:

(1) What are the necessary and sufficient conditions which one must impose on the metric tensor and its derivatives in order that an antisymmetric tensor F_{jk} satisfying Eqs. (1)–(4) exists?

(2) To what extent do the metric tensor and Eqs. (1)–(4) determine the electromagnetic field F_{jk}?

In this formalism, charges and currents can be considered only as singular points, lines, or surfaces on which the field equations fail. Recently, Misner and Wheeler[2] have attempted to introduce charges and currents by considering topological "wormholes" in a multiply connected space.

The purpose of the present work is to consider the possibility of introducing charges and currents simply by dropping the first of Maxwell's equations [Eq. (1)] and defining:

$$J^k=F^{\alpha k}{}_{,\alpha}. \tag{5}$$

The four-vector current density $(-g)^{\frac{1}{2}}J^k$ may be considered in the classical sense as some primordial ooze (which is conserved by virtue of its definition) flowing with local velocity v. Alternatively, J^k may be considered simply as a differential property of the electromagnetic field defined by Eq. (5). It is in no sense a singularity of the fields or the space.

II. RAINICH'S CHARGE-FREE SPACE

Rainich[1] showed that the following are the necessary and sufficient conditions which one must impose on the metric tensor in order to insure the existence of a non-null electromagnetic field satisfying Eqs. (1) to (4).

$$R=0, \tag{6}$$

$$R^0{}_0<0, \tag{7}$$

$$R^j{}_\alpha R^\alpha{}_k-\tfrac{1}{4}(R^\alpha{}_\beta R^\beta{}_\alpha)\delta^j{}_k=0, \tag{8}$$

$$q_{j,k}-q_{k,j}=0, \tag{9}$$

$$q_j=(-g)^{\frac{1}{2}}(R^u{}_vR^v{}_u)^{-1}\epsilon_{j\alpha\beta\gamma}R^{\alpha\sigma,\beta}R_\sigma{}^\gamma. \tag{10}$$

g is the determinant of the metric tensor. $\epsilon_{j\alpha\beta\gamma}$ is the totally antisymmetric tensor density whose components are 0 or ±1. Equations (6)–(8) are algebraic equations which must be satisfied in order that F_{jk} exist and satisfy Eq. (3). Equation (9) is a fourth-order differential equation which must be satisfied in order that F_{jk} satisfy Maxwell's equations, Eqs. (1) and (2). Rainich also showed that except for a constant determined by specifying the ratio of the field invariants at one point, the metric tensor determines the electromagnetic field uniquely.

* This work was begun while the author was a National Science Foundation Postdoctoral Fellow at California Institute of Technology, Pasadena, California.

[1] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[2] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).

III. NONVANISHING ELECTROMAGNETIC SOURCES

It is well known that given Eqs. (3) and (4) with the Bianchi identity, not all 8 of Maxwell's equations are independent.[1,3,4] Equations (3) and (4) with the Bianchi identity imply

$$-T^{\alpha}{}_{k,\alpha}=F_{\alpha k}J^{\alpha}+\mathfrak{F}_{\alpha k}\mathcal{J}^{\alpha}=0, \tag{11}$$

where

$$\mathfrak{F}_{jk}=\frac{1}{2!}(-g)^{\frac{1}{2}}\epsilon_{jk\alpha\beta}F^{\alpha\beta}, \tag{12}$$

$$\mathcal{J}^{k}=\frac{1}{3!(-g)^{\frac{1}{2}}}\epsilon^{k\alpha\beta\gamma}(F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}). \tag{13}$$

Since Eq. (2) requires that $\mathcal{J}^k$ vanish, Eqs. (2), (3), and (4) imply that[5]

$$F_{\alpha k}J^{\alpha}=0. \tag{14}$$

Clearly J^k must vanish unless

$$\det|F_{jk}|=0, \tag{15}$$

which is equivalent to

$$S=\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}=0. \tag{16}$$

S corresponds to $\mathbf{E}\cdot\mathbf{B}$ in a flat space.

When S vanishes, the dependencies implied by the Bianchi identity exist entirely within the two individual sets into which Maxwell's equations are usually divided [Eqs. (1) and (2)]. These two sets are independent of one another and the charge and current densities need not vanish. Previous investigators[1–4,6] have overlooked or ignored this point (except for a few speculations regarding null electromagnetic fields).

In order to reduce Eqs. (2)–(4) to a purely geometric theory, note that any antisymmetric tensor satisfying Eq. (16) can be expressed in the form:[7]

$$F_{jk}=\xi_j\eta_k-\eta_j\xi_k. \tag{17}$$

There is no loss in generality by assuming that

$$\xi^{\alpha}\xi_{\alpha}=-1, \tag{18}$$

$$\xi^{\alpha}\eta_{\alpha}=0. \tag{19}$$

For the moment, the null case ($\eta^{\alpha}\eta_{\alpha}=0$) is excluded. From Eqs. (3) and (17), one obtains

$$T^{j}{}_{k}=[\eta^{j}\eta_{k}-\tfrac{1}{2}(\eta^{\alpha}\eta_{\alpha})\delta^{j}{}_{k}]-(\eta^{\alpha}\eta_{\alpha})\xi^{j}\xi_{k}. \tag{20}$$

Clearly, Rainich's algebraic conditions on $T^{j}{}_{k}$, Eqs. (6), (7), and (8) must still be satisfied. The fourth-order differential condition, Eq. (9) is however, replaced by a third-order differential condition.

To obtain the third-order condition note that

$$T^{\alpha}{}_{k}q_{\alpha}+\tfrac{1}{2}(\eta^{\beta}\eta_{\beta})q_{k}=-F_{\alpha k}\mathcal{J}^{k}, \tag{21}$$

which may be proven by expressing each of the tensors explicitly in terms of ξ_j and η_k. Clearly, if the second set of Maxwell's equations, Eq. (2), is satisfied then:

$$T^{\alpha}{}_{k}q_{\alpha}+\tfrac{1}{2}(\eta^{\beta}\eta_{\beta})q_{k}=0. \tag{22}$$

Conversely, if Eqs. (11), (16), and (22) are satisfied, then the second set of Maxwell's equations are satisfied identically. The vectors necessary to express F_{jk} and $\mathfrak{F}_{jk}$ in the form of Eq. (17) are all orthogonal and independent. Given Eqs. (6), (7), and (8) there always exists an F_{jk} with $S=0$ which satisfies Eqs. (3) and (4).[1,2]

The necessary and sufficient conditions which one must impose on the metric tensor and its derivatives in order that there exists an antisymmetric field tensor F_{jk} satisfying only Eqs. (2), (3), and (4) are Eqs. (6)–(8), and (22). The electromagnetic field tensor F_{jk} (or its dual) are determined uniquely by the metric tensor.

Rainich's theory may be modified to include charges and currents simply by replacing Eq. (9) by Eq. (22). The vector q_k must be an eigenvector of the stress-energy-momentum tensor $T^{j}{}_{k}$. In regions of space where the curl of this eigenvector vanishes, both sets of Maxwell's equations are obeyed. If the curl of the eigenvector q_k does not vanish, then only one set of Maxwell's equations are obeyed and the charge-current density is nonzero. Perhaps a more general class of spaces with direct physical interpretation may be obtained by requiring either Eq. (9) or Eq. (22) but not necessarily both (except on the boundaries between regions of the two types).

The electromagnetic fields satisfying the above conditions fall into three classes depending on whether η_j is a spacelike, timelike, or null vector. The null case is singular and is discussed in Sec. IV. The vector ξ_j is by definition spacelike.

If η_j is timelike, then q_k and J_k are spacelike. There exists a coordinate system appropriate to each point such that (1) the field is purely electric and (2) the charge density vanishes at that point.

If η_j is spacelike, then J_k and q_k are orthogonal and one of them is timelike (unless they are equal and null). There exists a coordinate system appropriate to each point such that at that point (1) the field is purely magnetic, (2) the charge density may or may not be

[3] A. Einstein, Sitzber. preuss. Akad. Wiss., Physik.—Math. Kl. (1919) [translation: A. Einstein *et al. The Principle of Relativity* (Dover Publications, New York, 1924)].

[4] Louis Witten, Phys. Rev. **115**, 206 (1959).

[5] The local vanishing of the Lorentz force density need not imply that the force exerted on a charged object by an external field vanishes. The field acting on the charge-current density may be (conceptually) divided into two parts: (1) the fields arising from the object's own charges and currents and (2) the external field. The integrated force due to the first may be interpreted as an inertial force or the rate of change of the object's mechanical momentum. The integrated force due to the second is the usual Lorentz force. They are equal in magnitude.

[6] Bruno Bertotti, Phys. Rev. **115**, 742 (1959).

[7] J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), p. 35.

zero, and (3) the current is zero or parallel to the magnetic field.

Four eigenvectors of $T^j{}_k$ are ξ_k, η_k, q_k, and J_k. They are independent unless q_k and J_k are equal and null.

IV. NULL ELECTROMAGNETIC FIELDS

The considerations of the previous sections break down for null electromagnetic fields because many of the significant quantities vanish identically. In this case F_{jk} is of the form given in Eq. (17) where

$$\eta^\alpha \eta_\alpha = 0, \tag{23}$$

and therefore,

$$T^j{}_k = \eta^j \eta_k. \tag{24}$$

The metric tensor and its derivatives determine η_j uniquely. There will exist a null electromagnetic field satisfying Eqs. (2), (3), and (4), if and only if there exists a vector ξ_k satisfying the first-order linear differential equations:

$$\epsilon^{k\alpha\beta\gamma}[(\xi_\alpha \eta_\beta - \eta_\alpha \xi_\beta)_{,\alpha}] = 0. \tag{25}$$

The conditions which one must impose on the field η_j in order to assure existence and/or uniqueness of solutions to Eq. (25) are not considered here. However, Eq. (25) is satisfied identically if both η_j and ξ_k are gradients of scalars.

There has been some speculation as to whether an electromagnetic field which is null in some region of space must be null everywhere because of Maxwell's equations.[6] Since Maxwell's equations are independent in such regions, it is perhaps more interesting to focus attention on the invariant S and the single set of Maxwell's equations.

V. EUCLIDEAN ELECTROMAGNETISM

If one neglects the effects of gravitation, the concepts considered in the previous sections can be presented in a more familiar physical notation. The field equations are

$$\nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \tag{26}$$

$$\nabla \cdot \mathbf{E} = \rho; \quad \nabla \times \mathbf{B} - \partial \mathbf{E}/\partial t = \mathbf{j}, \tag{27}$$

$$\mathbf{B} \times \mathbf{j} - \rho \mathbf{E} = 0. \tag{28}$$

Equation (26) corresponds to the second of Maxwell's equations, Eq. (2). The first of Maxwell's equations, Eq. (1), is discarded and the differential quantities merely define the charge current densities, Eq. (27). The vanishing of the Lorentz force density, Eq. (28), corresponds to the vanishing of the divergence of the electromagnetic stress-energy-momentum tensor. It may be considered to be the limit of Eq. (4) and the Bianchi identity as the gravitational constant goes to zero. It assures that the energy and momentum of the electromagnetic field are conserved and therefore, electromagnetic matter-energy obeys the laws of relativistic mechanics. The local vanishing of the Lorentz force by no means implies that charges will not be accelerated by external fields. It asserts only that this theory accounts for no masses other than electromagnetic energy (and gravitational energy in the more general case).

These equations may be combined into a single set of four equations for the electromagnetic potential,

$$(\phi_{j,\beta} - \phi_{\beta,j})\phi^\beta{}_{,\alpha}{}^\alpha = 0, \tag{29}$$

$$\phi^\alpha{}_{,\alpha} = 0, \tag{30}$$

which are also valid in the modified Rainich theory, if covariant derivatives are implied. The imposition of the Lorentz gauge, Eq. (30), is no restriction.

The familiar solutions of Maxwell's equations for charge-free space satisfy these field equations. They correspond to the vanishing of ρ and $\mathbf{j}$ in Eqs. (26)–(28) or to the vanishing of $\phi^j{}_{,\alpha}{}^\alpha$ in Eq. (29).

It is not difficult to find solutions for which ρ and $\mathbf{j}$ do not vanish everywhere. For example, in cylindrical coordinates, the field generated by the charge current density

$$\begin{aligned} \rho &= \rho(r), \\ j_r &= \rho(r), \\ j_z &= j_\theta = 0, \end{aligned} \tag{31}$$

is a solution. It is, of course, not physically significant and it is unstable to perturbation by external fields due to the nonlinearity of Eq. (28) or Eq. (29).

VI. CHARGED ELECTROMAGNETOGRAVITATIONAL OBJECTS

A charged electromagnetogravitational object would be described by a space of the above type in which the charge-current and energy densities were not everywhere zero and were concentrated in the neighborhood of some average world line of the object. At any time the charge, energy, linear momentum, and angular momentum of such an object would each be an integral over the region of three-dimensional space in which the corresponding densities were concentrated. These integral quantities may be interpreted as mechanical properties Q, E, $\mathbf{P}$, and $\mathbf{L}$ which are conserved by virtue of the field equations. If such an object interacts with a field generated by some other external source, only the totals for the two systems are conserved. Any attempt to describe the one object independently of the other will lead to variations in the individual mechanical properties with time due to the "interactions." If the object under consideration is a localized charge distribution such that the external field is "almost uniform" over the region of space which contains "nearly all" of the electromagnetic energy, momentum, and charge of the object during the very short time interval $\Delta\tau$, then the change in energy and momentum ΔE and ΔP during $\Delta\tau$ can be expressed as two-dimensional surface integrals over the space-like sphere containing the object at that time. These surface integrals giving the time rate of change of energy and momentum can, except for gravitational effects, be expressed in terms of the external fields and the object's total charge and average velocity. This new "Lorentz Force" acting on the total object in interaction with an external field will in general not vanish. However, it is by no means certain whether any such physically significant solutions exist. An examination of the question of existence proceeds most directly by imposing boundary conditions pertinent to "physically significant solutions."

The problem of boundary conditions for electromagnetogravitational objects is difficult, and this author does not intend to pursue it further. However, it is worth pointing out that it has not even been proven that Eqs. (26)–(28) have no solutions similar to an electron because the boundary conditions usually imposed are not consistent with the present concept of the electron. About sixty years ago, serious attempts were made to describe the electron as a purely electromagnetic object. Maxwell's equations for charge-free space were imposed everywhere except on some singular surface or region. Since the vanishing of the Lorentz Force, Eq. (28), was not required in the singular regions, these objects failed to obey the laws of relativistic mechanics. Pauli[8] in 1921 pointed out that a purely electromagnetic object must satisfy all of Eqs. (26)–(28). He then argued that no solutions exist which satisfy the boundary conditions then believed to be relevant to the electron; i.e., a stationary localized charge distribution. Clearly these boundary conditions are not consistent with the present view of the electron. The electron possesses a magnetic dipole moment and, if it is described by a charge distribution ρ, it must also have a current distribution $\mathbf{j}$. A superimposed electric monopole field and magnetic dipole field possess angular momentum as well as energy. Even worse, it is generally accepted today that there is no such thing as a localized, stationary electron. An initially stationary electron would fill all space and an initially localized electron would after a time fill all space. It is no test of a field theory to search for solutions satisfying boundary conditions appropriate to the nineteenth century concept of a particle.

[8] W. Pauli, *The Theory of Relativity* (Pergamon Press, New York, 1958).

VII. DISCUSSION AND CONCLUSION

The simplest most direct method of unifying Maxwell's theory of electromagnetism and Einstein's theory of gravitation was formulated by Rainich in 1925. He equated the stress-energy-momentum tensor of a Maxwell field to the Einstein tensor and investigated the restrictions which this imposes on the metric. This theory, however, applies only to charge-free space.

In regions of space in which the electromagnetic field invariant corresponding to $\mathbf{E}\cdot\mathbf{B}$ vanishes, the two sets of Maxwell's equations are independent and one of them may be dropped from Rainich's theory. In such regions the charge and current density may not vanish. The electromagnetic sources and fields obey Maxwell-Lorentz theory and the matter-energy obeys the laws of Einstein's general relativity theory. The necessary and sufficient conditions which one must impose on the metric tensor and its derivatives in order to assure the existence of an antisymmetric tensor obeying the Maxwell-Lorentz laws in the presence of charges and currents have been derived [Eqs. (6)–(8) and (22)].

The question of the existence of physically significant solutions to the field equations presented can be attacked only after defining "physically significant" solutions and the corresponding boundary conditions. This problem has not been considered here.

ACKNOWLEDGMENTS

The author wishes to thank Professor H. P. Robertson, C. Iddings, S. Coleman, and C. Engelbrecht for helpful discussions. He would also like to express appreciation for the support of the National Science Foundation during the early stages of this work and during the period of study which preceded it.