Preliminary Analysis of Photoproduction of K Mesons in the Mandelstam Representation

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(Received February 17, 1961; revised manuscript received April 24, 1961)

The preliminary analysis essential for the application of the Mandelstam representation to the photoproduction of K mesons has been carried out. The analytic properties of individual multipoles have been investigated and the positions of the singularities have been located.

I. INTRODUCTION

APPLICATION of the analytic properties of the scattering amplitude to photoproduction was initiated by Chew and Low.[1] Chew *et al.*[2] applied the fixed-momentum-transfer dispersion relation to the photoproduction of pions. They used the static approximation and exploited the (3,3) resonance. Ball[3] has extended their treatment by applying the Mandelstam representation.

In the case of photoproduction of K mesons, we have not as much information as in the photoproduction of pions. We do not know about the K–Y relative parity nor anything about the magnetic moments of hyperon. Also the phase of the pion photoproduction matrix elements is simply related to the pion scattering phase shifts by unitarity. No such simple relation exists between the photoproduction of K mesons and the K–Y scattering phase shifts as exists when there is only one channel open. Moreover, there is a large unphysical range on the physical cut, as the cut starts at $(\mu+m)^2$ and the threshold is at $(K+M)^2$. Thus even the integral on the physical cut is not simply determined by unitarity. The only simple statement of unitarity applicable for the multichannel case is that given by Feldman, Matthews, and Salam.[4]

In this paper, we have carried out the initial stages of analysis essential for the application of the Mandelstam representation to the photoproduction of K mesons. Many complicated features of the problem due to the four different masses become evident.

In Sec. II, the kinematics are discussed and invariant amplitudes are set up. In Sec. III, the Mandelstam representation is written and the residues of the poles are calculated. In Sec. IV, the multipole analysis is done and the singularities of the partial wave amplitude are determined. In the Appendix the singularities are derived.

II. KINEMATICS

Let the four-vector momenta k, q, p_1, p_2 correspond formally to the ingoing particles (Fig. 1). Define the three invariants:

$$s=(k+p_1)^2=(q+p_2)^2, \tag{2.1a}$$

$$u=(q+p_1)^2=(k+p_2)^2, \tag{2.1b}$$

$$t=(k+q)^2=(p_1+p_2)^2. \tag{2.1c}$$

Conservation of momentum gives

$$s+u+t=m^2+M^2+K^2. \tag{2.2}$$

Each of the invariants defined by Eqs. (2.1a, b, c) represents the square of the total energy in the barycentric system for the reactions:

$$\text{I.}\quad k+p_1\rightarrow -q-p_2 \quad (\gamma+N\rightarrow K+Y), \tag{2.3a}$$

$$\text{II.}\quad q+p_1\rightarrow -k-p_2 \quad (\bar{K}+N\rightarrow \gamma+Y), \tag{2.3b}$$

$$\text{III.}\quad k+q\rightarrow -p_1-p_2 \quad (\gamma+\bar{K}\rightarrow \bar{N}+Y). \tag{2.3c}$$

These three reactions must be considered together if one uses the Mandelstam representation.

We define

$$k\equiv(|\mathbf{k}|,\mathbf{k}),\quad p_1\equiv(\epsilon_1,-\mathbf{k}),$$
$$-q\equiv(\omega,\mathbf{q}),\quad -p_2\equiv(\epsilon_2,-\mathbf{q}), \tag{2.4}$$

where $\mathbf{k}$ and $\mathbf{q}$ are the initial and final three-vector momenta in the barycentric system.

For reaction I:

$$s=W^2, \tag{2.5}$$

$$u=m^2+K^2-2\epsilon_1\omega-2kq\cos\theta, \tag{2.6}$$

$$t=K^2-2k\omega+2kq\cos\theta, \tag{2.7}$$

$$k^2=\frac{(s-m^2)^2}{4s},\quad q^2=\frac{[s-(M+K)^2][s-(M-K)^2]}{4s}, \tag{2.8}$$

$$\omega=\frac{s-(M^2-K^2)}{2\sqrt{s}},\quad \epsilon_1=\frac{s+m^2}{2\sqrt{s}},\quad \epsilon_2=\frac{s+(M^2-K^2)}{2\sqrt{s}}. \tag{2.9}$$

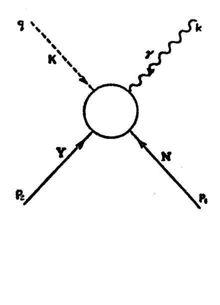

FIG. 1. Feynman diagram for photoproduction of K mesons.

[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).
[2] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957). Hereafter this will be referred to as CGLN.
[3] J. S. Ball, thesis, Lawrence Radiation Laboratory Report, UCRL—9172, 1960 (unpublished).
[4] G. Feldman, P. T. Matthews, and A. Salam, Nuovo cimento **16**, 549 (1960).

We write the T matrix as

$$T=\sum_{i=1}^{8} B_i(s,u,t)N_i, \tag{2.10}$$

$$\begin{aligned} N_1&=\gamma\cdot\epsilon, & N_5&=i\gamma\cdot\epsilon\gamma\cdot k,\\ N_2&=k\cdot\epsilon, & N_6&=ik\cdot\epsilon\gamma\cdot k,\\ N_3&=q\cdot\epsilon, & N_7&=iq\cdot\epsilon\gamma\cdot k,\\ N_4&=P\cdot\epsilon, & N_8&=iP\cdot\epsilon\gamma\cdot k. \end{aligned} \tag{2.11}$$

This is the most general T matrix allowed by Lorentz invariance. Application of gauge invariance gives only four independent functions. We select these independent functions in the same way as CGLN. Thus, for $(K$–$Y)$ even parity:

$$T=A_1M_1+A_2M_2+A_3M_3+A_4M_4, \tag{2.12}$$

$$M_1=i\gamma\cdot\epsilon\gamma\cdot k, \tag{2.13a}$$

$$M_2=2i(P\cdot\epsilon q\cdot k-P\cdot kq\cdot\epsilon), \tag{2.13b}$$

$$M_3=(\gamma\cdot\epsilon q\cdot k-\gamma\cdot kq\cdot\epsilon), \tag{2.13c}$$

$$M_4=2(\gamma\cdot\epsilon P\cdot k-\gamma\cdot kP\cdot\epsilon-im\gamma\cdot\epsilon\gamma\cdot k). \tag{2.13d}$$

For $(K$–$Y)$ odd parity:

$$T=\sum_{i=1}^{4} A_i(s,u,t)\gamma_5 M_i. \tag{2.14}$$

The A_i's are functions of s, u, and t, as well as of the isotopic spin. Denoting the isotopic index of the outgoing Σ by β, we have three isotopic spin invariants for the reaction $(\gamma+N\rightarrow K+\Sigma)$:

$$V_+{}^\beta=\tfrac{1}{2}[\tau_\beta\tau_3+\tau_3\tau_\beta]=\delta_{\beta 3}, \tag{2.15a}$$

$$V_-{}^\beta=\tfrac{1}{2}[\tau_\beta,\tau_3], \tag{2.15b}$$

$$V_0{}^\beta=\tau_\beta. \tag{2.15c}$$

Λ^0 is an isoscalar. Therefore, for the reaction

$$(\gamma+N\rightarrow K+\Lambda^0),$$

we have two isotopic spin invariants,

$$S_+=\tau_3, \tag{2.16a}$$

$$S_0=1. \tag{2.16b}$$

Thus

$$A_i=A_i{}^{V+}V_+{}^\beta+A_i{}^{V-}V_-{}^\beta+A_i{}^{V_0}V_0{}^\beta, \tag{2.17}$$

and

$$A_i=A_i{}^{S+}S_++A_i{}^{S_0}S_0, \tag{2.18}$$

for Σ and Λ^0, respectively.

The values of the V's and S's for a particular reaction are tabulated (see Table I).

III. MANDELSTAM REPRESENTATION

The Mandelstam representation for the gauge-invariant amplitudes A_i has been written by Ball[3]:

$$\begin{aligned} A_i(s,u,t)=&\frac{\Gamma_i(s)}{m^2-s}+\frac{\Gamma_i(u)}{M^2-u}\\ &+\frac{1}{\pi^2}\int_{(\mu+m)^2}^\infty ds'\int_{(\mu+M)^2}^\infty du'\frac{a_i^{12}(s',u')}{(s'-s)(u'-u)}\\ &+\frac{1}{\pi^2}\int_{(\mu+m)^2}^\infty ds'\int_{(\mu+K)^2}^\infty dt'\frac{a_i^{13}(s',t')}{(s'-s)(t'-t)}\\ &+\frac{1}{\pi^2}\int_{(\mu+M)^2}^\infty du'\int_{(\mu+K)^2}^\infty dt'\frac{a_i^{23}(u',t')}{(u'-u)(t'-t)}. \end{aligned} \tag{3.1}$$

There may be present one-dimensional integrals in the variables s and u in the spectral representation of A_i; $i=1$, 3, 4.

As shown by Mandelstam, one can easily derive one-dimensional dispersion relations with either s, u, or t held fixed. For fixed s:

$$A_i(s,u,t)=\text{poles}+\frac{1}{\pi}\int_{(\mu+M)^2}^\infty du'\frac{a_i^2(u',s)}{u'-u}+\frac{1}{\pi}\int_{(\mu+K)^2}^\infty dt'\frac{a_i^3(t',s)}{t'-t}. \tag{3.2}$$

The absorptive parts $a_i^j(x,y)$ are equal to $\mathrm{Im}A_i$ when the variables s, u, and t are in the physical region for the reaction "j" defined by Eq. (2.3). Equation (3.1) shows that

$$a_i^2(u',s)=\frac{1}{\pi}\int_{(\mu+m)^2}^\infty ds'\frac{a_i^{12}(u',s')}{s'-s}+\frac{1}{\pi}\int_{(\mu+K)^2}^\infty dt'\frac{a_i^{23}(u',t')}{t'+s+u'-\Sigma}, \tag{3.3a}$$

$$a_i^3(t',s)=\frac{1}{\pi}\int_{(\mu+m)^2}^\infty ds'\frac{a_i^{13}(t',s')}{s'-s}+\frac{1}{\pi}\int_{(\mu+M)^2}^\infty du'\frac{a_i^{23}(u',t')}{u'+s+t'-\Sigma}, \tag{3.3b}$$

TABLE I. Matrix elements of $V_{\pm,0}$ and $S_{+,0}$ for the possible charge configurations.

	$\gamma+p\rightarrow K^++\Sigma^0$	$\gamma+p\rightarrow K^0+\Sigma^+$	$\gamma+p\rightarrow K^++\Lambda^0$	$\gamma+n\rightarrow K^++\Sigma^0$	$\gamma+n\rightarrow K^++\Sigma^-$	$\gamma+n\rightarrow K^0+\Lambda^0$
V_+	1	0	···	1	0	···
V_-	0	$\sqrt{2}$	···	0	$-\sqrt{2}$	···
V_0	1	$\sqrt{2}$	···	-1	$\sqrt{2}$	···
S_+	···	···	1	···	···	-1
S_0	···	···	1	···	···	1

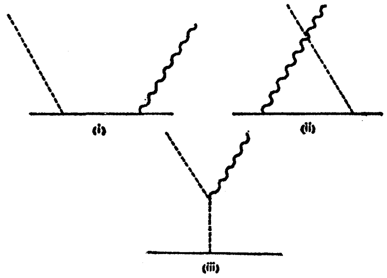

FIG. 2. The diagrams which give rise to poles.

where

$$\Sigma = M^2 + m^2 + K^2. \tag{3.4}$$

The spectral functions a_i^{ik} actually vanish over parts of regions of integration in Eqs. (3.1) and (3.3).

The poles may arise due to diagrams shown in Fig. 2. For a particular reaction, either diagrams (i), and (ii), or (i) and (iii), or (ii) and (iii) contribute. Ordinary perturbation theory calculations show that any two of the diagrams combine to give a gauge-invariant combination. The diagram (iii) contributes only to B_3. We get the same result if we consider that there are poles due only to diagrams (i) and (ii), whose residues are as given below.

Case I. (K–Y) Relative Parity Odd

$\Gamma_i(s)$ is given by

$$\Gamma_1{}^{V\pm,0} = g_\Sigma e/2,$$

$$\Gamma_2{}^{V\pm,0} = g_\Sigma e/(t-K^2),$$

$$\Gamma_3{}^{V\pm} = g_\Sigma(\mu_p - \mu_n)/2, \quad \Gamma_3{}^{V_0} = g_\Sigma(\mu_p + \mu_n)/2,$$

$$\Gamma_4{}^{V\pm} = -g_\Sigma(\mu_p - \mu_n)/2, \quad \Gamma_4{}^{V_0} = -g_\Sigma(\mu_p + \mu_n)/2,$$

$$\Gamma_1{}^{S+,0} = g_\Lambda e/2,$$

$$\Gamma_2{}^{S+,0} = g_\Lambda e/(t-K^2),$$

$$\Gamma_3{}^{S+} = g_\Lambda(\mu_p - \mu_n)/2, \quad \Gamma_3{}^{S_0} = g_\Lambda(\mu_p + \mu_n)/2,$$

$$\Gamma_4{}^{S+} = -g_\Lambda(\mu_p - \mu_n)/2, \quad \Gamma_4{}^{S_0} = -g_\Lambda(\mu_p + \mu_n)/2.$$

In order to calculate $\Gamma_i(u)$ we have to consider the electromagnetic vertex of the hyperon.

$$\langle -p_2 | j_\mu{}^\dagger | n \rangle \sim i[G_1\gamma_\mu + G_2\sigma_{\mu\nu}(-p_1 - q - p_2)_\nu].$$

Then

$$\Gamma_1 = G_1 g_Y + 2(M-m)G_2 g_Y,$$

$$\Gamma_2 = G_1 g_Y / q\cdot k,$$

$$\Gamma_3 = -G_2 g_Y,$$

$$\Gamma_4 = -G_2 g_Y.$$

The electromagnetic current due to the hyperon is

$$j_\mu \sim \bar{\Sigma}\gamma_\mu T_3 \Sigma + \bar{\Lambda}^0\gamma_\mu \Sigma_3 + \bar{\Lambda}^0 \gamma_\mu \Lambda^0.$$

Since Σ is an isovector, it only contributes to the isovector part. The first and second terms contribute to the Σ part and second and third to the Λ^0 part. If the (Σ,Λ) parity is odd, the second term, which gives rise to the transition moment, becomes $\bar{\Lambda}^0\gamma_\mu\gamma_5\Sigma_3$. It is only here that the relative (Σ,Λ) parity enters. Thus

$$G_1{}^{V\pm} = e\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad G_1{}^{V_0} = 0,$$

$$G_2{}^{V\pm} = \begin{bmatrix} \mu_+ \\ (g_\Lambda/g_\Sigma)\mu_T \\ \mu_- \end{bmatrix}, \quad G_2{}^{V_0} = 0,$$

$$G_1{}^{S+,0} = 0,$$

$$G_2{}^{S+} = (g_\Sigma/g_\Lambda)\mu_T, \quad G_2{}^{S_0} = \mu_{\Lambda^0},$$

where μ_+, μ_-, and μ_{Λ^0} are the magnetic moments of Σ^+, Σ^-, and Λ^0 and μ_T is the transition magnetic moment. Thus

$$\Gamma_1{}^{V\pm} = g_\Sigma e\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2(M-m)g_\Sigma \begin{bmatrix} \mu_+ \\ (g_\Lambda/g_\Sigma)\mu_T \\ \mu_- \end{bmatrix}, \quad \Gamma_1{}^{V_0} = 0,$$

$$\Gamma_2{}^{V\pm} = 2g_\Sigma e/t - K^2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \Gamma_2{}^{V_0} = 0,$$

$$\Gamma_3{}^{V\pm} = \Gamma_4{}^{V\pm} = -g_\Sigma \begin{bmatrix} \mu_+ \\ (g_\Lambda/g_\Sigma)\mu_T \\ \mu_- \end{bmatrix},$$

$$\Gamma_3{}^{V_0} = \Gamma_4{}^{V_0} = 0,$$

$$\Gamma_1{}^{S+} = 2(M-m)\mu_T g_\Sigma, \quad \Gamma_1{}^{S_0} = 2(M-m)\mu_{\Lambda^0} g_\Lambda,$$

$$\Gamma_2{}^{S+,0} = 0,$$

$$\Gamma_3{}^{S+} = \Gamma_4{}^{S+} = -g_\Sigma \mu_T,$$

$$\Gamma_3{}^{S_0} = \Gamma_4{}^{S_0} = -g_\Lambda \mu_{\Lambda^0}.$$

Case II. (K–Y) Parity Even

$$\Gamma_i(s) \to -i\Gamma_i(s) \text{ (odd)},$$

$$\Gamma_1(u) = -i[G_1 + 2(M+m)G_2]G_Y,$$

$$\Gamma_2(u) \to -i\Gamma_2(u) \text{ (odd)},$$

$$\Gamma_{3,4}(u) \to i\Gamma_{3,4}(u) \text{ (odd)}.$$

IV. ANGULAR MOMENTUM ANALYSIS AND LOCATION OF SINGULARITIES

For (K–Y) parity even, we write the cross section in the barycentric system as

$$\frac{d\sigma}{d\Omega} = \frac{q}{k}|\chi_Y{}^\dagger G \chi_N|^2, \tag{4.1}$$

$$G = i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\mathbf{k}}{k}G_1 + i\frac{\mathbf{q}\cdot\boldsymbol{\varepsilon}}{q}G_2 + i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\mathbf{q}}{q}G_3 + i\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{q^2 k}G_4, \tag{4.2}$$

$$G_1 = \frac{1}{4\pi}\frac{W-m}{2W}[(\epsilon_1+m)(\epsilon_2+M)]^{\frac{1}{2}} \times\left[A_1 - (W+m)A_4 - \frac{t-K^2}{2(W+m)}(A_3+A_4)\right], \tag{4.3}$$

$$G_2=\frac{1}{4\pi}\frac{W-m}{2W}[(\epsilon_1+m)(\epsilon_2+M)]^{\frac{1}{2}}q\times[A_3+A_4-(W+m)A_2], \quad (4.4)$$

$$G_3=-\frac{1}{4\pi}\frac{W-m}{2W}\left(\frac{\epsilon_1+m}{\epsilon_2+M}\right)^{\frac{1}{2}}q\times\left[A_1+(W-m)A_4+\frac{t-K^2}{2(W-m)}(A_3+A_4)\right], \quad (4.5)$$

$$G_4=\frac{1}{4\pi}\frac{W-m}{2W}\left(\frac{\epsilon_1+m}{\epsilon_2+M}\right)^{\frac{1}{2}}q^2\times[A_3+A_4+(W-m)A_2]. \quad (4.6)$$

If $(K-Y)$ parity is odd, then as in CGLN, we have

$$\frac{d\sigma}{d\Omega}=\frac{q}{k}|\chi_Y{}^\dagger F\chi_N|^2, \quad (4.7)$$

$$F=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}F_1+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})}{qk}F_2+i\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{qk}F_3+i\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{q^2}F_4. \quad (4.8)$$

The F_i's are related to the A_i's as in CGLN. In the rest of analysis we shall assume $(K-Y)$ parity odd.

The multipole analysis has been done by CGLN. Inverting their equations, we have

$$M_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx\left[F_1P_l(x)-F_2P_{l+1}(x)-F_3\frac{P_{l-1}(x)-P_{l+1}(x)}{2l+1}\right] \quad (4.9)$$

or

$$M_{l+}=\frac{1}{2(l+1)}\Big\{\alpha[A_l{}^1+(W-m)A_l{}^4]-\beta[A_{l+1}{}^1-(W+m)A_{l+1}{}^4]-\frac{\gamma}{2l+1}[(W-m)(A_{l-1}{}^2-A_{l+1}{}^2)+(A_{l-1}{}^3-A_{l+1}{}^3)+(A_{l-1}{}^4-A_{l+1}{}^4)]+\alpha'(A_l{}^3+A_l{}^4)+\alpha''(xA_l{}^3+xA_l{}^4)+\beta'(A_{l+1}{}^3+A_{l+1}{}^4)+\beta''(xA_{l+1}{}^3+xA_{l+1}{}^4)\Big\}, \quad (4.10)$$

$$A_l{}^i=\int_{-1}^{1}dx\,P_l(x)A_i, \quad (4.11)$$

$$xA_l{}^i=\frac{l+1}{2l+1}A_{l+1}{}^i+\frac{l}{2l+1}A_{l-1}{}^i. \quad (4.12)$$

α, β, etc., are kinematical factors. There are similar expressions for M_{l-}, E_{l+}, and E_{l-}.

M_{l+} contains singularities due to the kinematical factors α, β, etc., in addition to the singularities due to the $A_l{}^i$'s. The $A_l{}^i$'s have kinematical as well as dynamical singularities. The kinematical singularities arise due to the fact that for small qk, $A_l{}^i$ goes like $(qk)^l$, as can be seen from Eq. (4.14). Therefore $A_l{}^i$ for odd l has a branch cut in the W plane, which is not related to the singularities due to the vanishing of denominators in Eq. (4.14) but is due to the relation between t or u and $\cos\theta$. Hence if we consider the characteristic amplitude

$$\frac{W^4M_{l+}}{[(W+M)^2-K^2]^{\frac{1}{2}}(qk)^l},$$

it will be free of all the kinematical singularities in the W plane. This amplitude also has the correct threshold behavior. α, β, etc., now become

$$\alpha\to\frac{W^2}{16\pi}\frac{W^2-m^2}{(qk)^l},$$

$$\beta\to\frac{W^4}{4\pi}\frac{1}{[(W+M)^2-K^2](qk)^{l-1}},$$

$$\gamma\to-\frac{W^3}{8\pi}\frac{1}{(qk)^{l-1}},$$

$$\delta\to-\frac{W}{32\pi}\frac{(W^2-m^2)[(W-M)^2-K^2]}{(qk)^l},$$

$$\alpha'\to-\frac{1}{64\pi}\frac{(W+m)(W^2-m^2)[W^2-(M^2-K^2)]}{(qk)^l},$$

$$\alpha''\to\frac{W^2}{16\pi}\frac{W+m}{(qk)^{l-1}},$$

$$\beta'\to-\frac{W^3}{8\pi}\frac{(W-m)[W^2-(M^2-K^2)]}{[(W+M)^2-K^2](qk)^{l-1}},$$

$$\beta''\to\frac{W}{32\pi}\frac{(W-m)(W^2-m^2)[(W-M)^2-K^2]}{(qk)^l}.$$

The dynamical singularities in $A_l{}^i$ arise due to the vanishing of the denominators in Eqs. (3.3a), (3.3b), and (3.2). The first term in Eqs. (3.3a) and (3.3b), gives rise to physical cuts in the regions $W\geq m+\mu$ and $W\leq -(m+\mu)$. The meaning of the latter can be understood by the symmetry relations[3]:

$$M_{l+}(-W)=\frac{1}{l+1}[(l+2)M_{(l+1)-}(W)+E_{(l+1)-}(W)], \quad (4.13a)$$

$$E_{l+}(-W)=\frac{1}{l+1}[M_{(l+1)-}-lE_{(l+1)-}(W)]. \quad (4.13b)$$

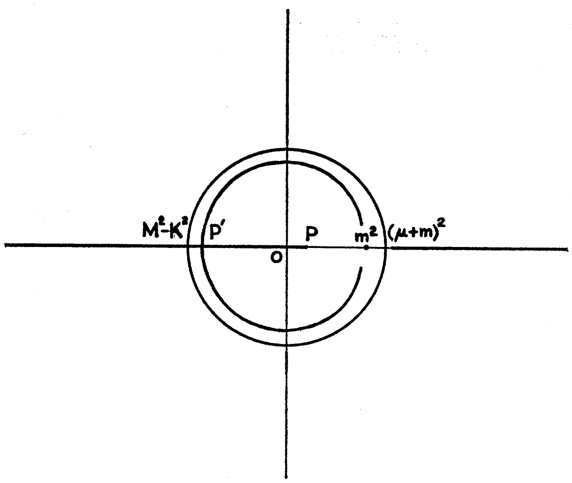

FIG. 3. The position of the dynamical singularities in the s plane of the partial-wave amplitude. The branch cuts are indicated by heavy lines.

The second term in Eqs. (3.3a) and (3.3b) does not give rise to any singularity.

Let us now continue the enumeration of these singularities in the $s=W^2$ plane. These singularities come from the vanishing of the denominators in Eq. (3.2). Let us first consider the denominator containing the variable u. Then

$$A_l{}^i=\int_{-1}^{1}\frac{\Gamma_i(u)P_l(x)dx}{M^2-(m^2+K^2-2\epsilon_1\omega-2kqx)}+\int_{(\mu+M)^2}^{\infty}du'\int_{-1}^{1}\frac{a_i{}^2(u',s)P_l(x)dx}{u'-(m^2+K^2-2\epsilon_1\omega-2kqx)}. \quad (4.14)$$

The first term gives the following branch cuts:

$$-(M^2-K^2)\leq s\leq 0,$$
$$-\infty\leq s\leq -(M^2-K^2).$$

The second term gives the branch cuts

$$0\leq s\leq\frac{2M^2m^2+(M^2-m^2-K^2)2mK-4m^2K^2}{2(2mK+M^2+K^2)}\ (P),$$
$$-\infty\leq s\leq P.$$

In addition s also becomes complex in a small region which is very close to the real axis and can be approximated as being coincident with the real axis. Finally there are singularities due to the vanishing of $t'-t$. This gives two cuts:

$$(-P')-\frac{m[(M+m)M-K^2]}{M+m}\leq s\leq 0,\quad -\infty\leq s\leq -P',$$

and a curve

$$(M^2-m^2-K^2)[x^2+y^2-m^2(M^2-K^2)]\times[(x-m^2)^2+y^2]+K^2(x^2+y^2-m^4)^2=0,$$

which reduces to a circle of radius m^2 centered around the origin, if we put $M^2\approx m^2+K^2$.

In A_2, we get a term of the form

$$\frac{g_Ye}{K^2-t}\left(\frac{1}{m^2-s}+\frac{1}{M^2-u}\right),$$

which can be put in the form

$$\frac{g_Ye}{(m^2-s)(K^2-t)}\quad\text{or}\quad\frac{g_Ye}{(m^2-s)(M^2-u)},$$

or their linear combination. The cuts due to the vanishing of M^2-u have already been discussed. The vanishing of K^2-t gives rise to the following singularities:

$$-(M^2-K^2)\leq s\leq 0,$$
$$-\infty\leq s\leq -(M^2-K^2),$$

and a circle of radius (M^2-K^2) centered around the origin. These dynamical singularities are sketched in Fig. 3.

The use of these results are under investigation and will be the subject matter of a second paper.

ACKNOWLEDGMENTS

This problem was suggested by Dr. P. T. Matthews to whom the author is deeply indebted for encouragement, advice, and very helpful discussions. He is also indebted to Professor A. Salam for encouragement. A Fellowship awarded by the Dr. Wali Muhammad Trust and a grant from the Pakistan Atomic Energy Commission are gratefully acknowledged.

APPENDIX

I

$$\int_{(\mu+M)^2}^{\infty}du'\int_{-1}^{1}\frac{a_i{}^2(u',s)P_l(x)dx}{u'-(m^2+K^2-2\epsilon_1\omega-2kqx)}.$$

Changing the variables to

$$z'=u'-m^2-K^2,$$
$$z=-2\epsilon_1\omega-2kqx,$$

we have

$$-\frac{1}{2kq}\int_{a'}^{\infty}dz'\int_{Q}^{Q'}\frac{a_i{}^2(z',s)P_l(-(z+2\epsilon_1\omega)/2kq)dz}{z'-z},$$

$$Q=-(2\epsilon_1\omega-2kq),$$
$$Q'=-(2\epsilon_1\omega+2kq),$$
$$a'=(\mu+M)^2-m^2-K^2.$$

Let $Q<Q'$. Then $Q<z<Q'$ and $a'<z'<\infty$. Therefore a singularity occurs, when

$$a'\leq Q',\quad Q\leq\infty\quad\text{or}\quad\infty>Q>a'.$$

Now

$$s=\frac{-[Q^2-(M^2-m^2-K^2)Q-2M^2m^2]\pm[Q-(M^2-m^2-K^2)][(Q+2mK)(Q-2mK)]^{\frac{1}{2}}}{2(Q+M^2+K^2)}.$$

In the range

$$2mK<Q<\infty,$$

s is real, and we get two cuts,

$$0\leq s\leq P, \quad \text{and} \quad -\infty\leq s\leq P.$$

In the range

$$a'<Q<2mK,$$

s is complex. But this region is small and close to the real axis and can be taken as coincident with the real axis.

II

$$\int_{(\mu+K)^2}^{\infty} dt' \int_{-1}^{1} \frac{a_i^3(t',s)P_l(x)dx}{t'-(K^2-2k\omega+2kqx)}.$$

Changing the variables, we have

$$\frac{1}{2kq}\int_{a'}^{\infty} dz' \int_{Q}^{Q'} \frac{a_i^3(z',s)P_l((z+2k\omega)/2kq)dz}{z'-z},$$

$$Q=-(2k\omega+2kq),$$

$$Q'=-(2k\omega-2kq),$$

$$a'=\mu^2+2\mu K.$$

Let $Q<Q'$. Then, as before, a singularity occurs when

$$\infty>Q>a'.$$

Now

$$s=\frac{-[Q^2-(M^2+m^2-K^2)Q-2m^2K^2]\pm Q\{[Q-(M^2+m^2-K^2+2Mm)][Q-(M^2+m^2-K^2-2Mm)]\}^{\frac{1}{2}}}{2(Q+K^2)}.$$

When

$$(M^2+m^2-K^2+2Mm)<Q<\infty,$$

s is real, and we get two cuts,

$$-P'\leq s\leq 0, \quad -\infty\leq s\leq -P'.$$

When

$$a'<Q<(M^2+m^2-K^2+2Mm),$$

s is complex, and we get a curve

$$(M^2-m^2-K^2)[x^2+y^2-m^2(M^2-K^2)]\times[(x-m^2)^2+y^2]+K^2(x^2+y^2-m^4)^2=0.$$

Similar considerations show that the vanishing of M^2-u and K^2-t gives the singularities as given in the text.