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Application of Nucleon-Nucleon Dispersion Relations to Nuclear Many-Body Problem*

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A method is given of obtaining the nucleon-nucleon scattering amplitude within nuclear matter, when the nucleon-nucleon dispersion relations are known. This is attained by establishing the dispersion relation for the scattering amplitude under the influence of the Pauli exclusion principle in intermediate states. With this modified amplitude the binding energy of the nucleus is calculated using Brueckner's method. The binding energy per nucleon turned out to be -13.2 Mev, if the contribution of the three-pion exchange potential is adjusted to give the correct nuclear density. The implication of these results is discussed.

1. INTRODUCTION

RECENTLY a new approach was found for the treatment of nucleon-nucleon scattering, namely the dispersion relations,[1] which showed some success for low-energy phenomena.[2] The question then arises: Is it possible to apply this technique to the nuclear many-body problems? In the usual quantum mechanics the many-nucleon problem can be treated in principle by solving the Schrödinger equation if the nuclear forces (including many-body forces) are known, but the exact form of the nuclear potential has not yet been obtained. In this paper we shall use a different approach and calculate the binding energy of the nuclear matter using the two-nucleon dispersion relations. In doing this the many-body effect is included in the simplest form, i.e., as the Pauli exclusion principle.

2. BRUECKNER'S METHOD

As the simplest nucleus we consider the case of nuclear matter which is an idealized nucleus of infinite size, constant density, and no Coulomb forces. We use Brueckner's method[3] to calculate the binding energy of nuclear matter. This method consists in taking into account two-body correlations seriously and replacing the potential by a two-body reaction matrix or forward scattering amplitude.

Mathematically this is explained in the following way. Let Ψ be the exact wave function of nuclear matter and Φ be the wave function of the free Fermi gas of nucleons at the same density, i.e.,

$$H\Psi=E\Psi,\quad H_0\Phi=E_0\Phi,\quad H=H_0+U,$$

where H is the total Hamiltonian and H_0 is its kinetic energy part only. E_0 represents the kinetic energy of the free Fermi gas. The total energy E is

$$E=\frac{(\Phi H\Psi)}{(\Phi\Psi)}=\frac{(\Phi(H_0+U)\Psi)}{(\Phi\Psi)}=E_0+\frac{(\Phi U\Psi)}{(\Phi\Psi)}. \tag{1}$$

If the potential U is written as a sum of two-nucleon interactions V,

$$U=\tfrac{1}{2}\sum_{ij}V_{ij},$$

then the last term of Eq. (1) is

$$\tfrac{1}{2}\sum(\Phi V_{ij}\Psi)/(\Phi\Psi).$$

Within the range of V_{12}, the two-body correlation of particles 1 and 2 is very important and if one neglects correlations with other particles, Ψ may be factorized as $\Psi=\psi_{12}\psi_{34}\cdots$. Φ is of course factorized and with suitable normalization,

$$(\Phi V_{12}\Psi)=(\phi_{12}V_{12}\psi_{12})=-f(p_1,p_2),$$

where $f(p_1,p_2)$ is the forward scattering amplitude of the colliding particles p_1 and p_2. This amplitude however is not equal to that of scattering in free space. Within nuclear matter, the nuclear levels are filled up to the Fermi momentum P_F and intermediate states below this level are forbidden by the Pauli exclusion principle. Thus, such processes as Fig. 1(a), which occur in free space, are forbidden in nuclear matter if one or both of the intermediate momenta coincides with one of the already occupied states. The f in question is the amplitude with this exclusion effect taken into account. To distinguish this from the free scattering amplitude f, we write it f_N with subscript N. The formula for the binding energy is then given by

$$E=E_0-\tfrac{1}{2}\sum_{p_i,p_j\epsilon N}f_N(p_i,p_j), \tag{2}$$

where the summation extends over all pairs in the nucleus. Our problem is to calculate the forward scattering amplitude f_N.

At this point it will be worthwhile to make a comment

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[1] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. **2**, 726 (1957).

[2] Y. Hara and H. Miyazawa, Progr. Theoret. Phys. (Kyoto) **23**, 942 (1960).

[3] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958). This contains a list of references.

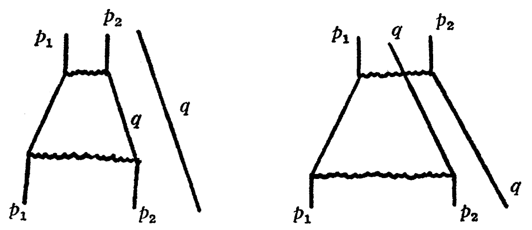

FIG. 1. Typical Feynman diagrams.

on the exclusion principle in the intermediate states. It is usually claimed that in doing a Feynman-type calculation, it is not necessary to take the exclusion principle into account for intermediate states. This is true because there are always diagrams which cancel the effect of neglecting the exclusion principle. We shall compare two standpoints: (I) to consider the exclusion principle in intermediate states; (II) to forget it. In Fig. 1, the process (*a*) is omitted from standpoint (I) while it should be included if one disregards the exclusion principle. However, process (*b*) should also be included in this case while it is actually neglected. These two terms give identical contribution but of opposite sign. The sum gives zero, so either standpoint gives the identical answer if one always sums every diagram. However, if one confines one's attention only to the two-particle correlations, the diagram Fig. 1(a) will be erroneously included in standpoint (II). In other words, the Brueckner method [standpoint (I)] automatically includes the effect of three-particle processes and, in higher order, more-particle effects while dealing with the two-particle problem. In general the effect of n-particle processes including particle exchange can be reduced to the effect of $(n-1)$-particle processes with the exclusion principle.

The amplitude f_N can be obtained as a solution of the Bethe-Goldstone equation.[4] In this paper, however, we shall obtain f_N from the dispersion relation of two-nucleon scattering.

3. DISPERSION RELATION FOR f_N

The dispersion relation for the forward scattering amplitude $f(p,p')$ in free space is written as a function of relative energy $\omega=(1/2m)(p-p')^2$. Thus

$$f(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}f(\omega')}{\omega'-\omega}d\omega'=\frac{1}{\pi}\int_{-\infty}^{0}+\frac{1}{\pi}\int_{0}^{\infty}$$
$$=-V(\omega)+\frac{1}{\pi}\int_{0}^{\infty}\frac{a(\omega')}{\omega'-\omega}d\omega'. \quad (3)$$

To save the notation, we suppressed spin and isospin indices. The imaginary part $a(\omega)$, in the physical region $\omega>0$, can be expressed by the total cross section σ,

$$a(\omega)=(v/2)\sigma(\omega),$$

where v is the relative velocity. The integral over the left-hand cut corresponds to the potential and is denoted by $-V(\omega)$. For the calculation of $a(\omega)$ in this region $\omega<0$, we consider the scattering of a particle with kinetic energy ω and momentum p_ν by a particle at rest. Here ω and p_ν are related, not by the usual relation, but

$$(\omega+m)^2=p_\nu^2-\nu.$$

If $\nu>0$, p_ν is real for all values of real ω, and the imaginary part $a(\omega,\nu)$ can be calculated from the unitary relation. Then we take analytic continuation in ν and

$$V(\omega)=\lim_{\nu\to-m^2}V(\omega,\nu)=-\lim\frac{1}{\pi}\int_{-\infty}^{0}\frac{a(\omega',\nu)}{\omega'-\omega}d\omega'. \quad (4)$$

It can be shown in the perturbation expansion that the above limit exists.

To write down the dispersion relation in nuclear matter, it is more convenient to consider the sum

$$F_N(p)=\sum_{p'\epsilon N}f_N(p,p'), \quad (5)$$

rather than $f_N(p,p')$, since what is needed in Eq. (2) is the sum and not the individual f_N's. The F_N depends only on p^2 and can be regarded as a function of the energy

$$E=p^2/2m.$$

It turns out that $F_N(E)$ has simpler analytic property than the f_N's.

We use again the technique of continuation in the mass variable ν and introduce

$$F_N(E,\nu)=\sum_{p'\epsilon N}f_N(p_\nu,p',\nu), \quad (6)$$

where $f_N(p_\nu,p',\nu)$ is the amplitude for forward scattering of a particle with the energy-momentum relation

$$(E+m)^2=p_\nu^2-\nu,$$

by the particle p'. It is shown in the Appendix that $F_N(E,\nu)$ satisfies the dispersion relation

$$F_N(E,\nu)=1/\pi\int_{-\infty}^{\infty}A_N(E',\nu)/(E'-E)dE'. \quad (7)$$

The left-hand cut can be written as a sum of individual contributions,

$$A_N(E,\nu)=\sum_{p'\epsilon N}a_N(p_\nu,p',\nu), \quad E<0. \quad (8)$$

We approximate this a_N by a, the absorptive part in free space. This means that we neglect the effect of exclusion in calculating the left-hand cut. With this

[4] H. A. Bethe, Phys. Rev. **103**, 1353 (1956); J. Goldstone, Proc. Roy. Soc. (London) **A235**, 408 (1957).

approximation, the unphysical integral in (7) is[5]

$$1/\pi\int_{-\infty}^{0} A_N(E',\nu)/(E'-E)dE' = -\sum_{p'\epsilon N} V(p,p',\nu).$$

In this form it is possible to go to the limit $\nu \rightarrow -m^2$ as in Eq. (4) and we obtain the sum of "two-body potentials" $V(\omega_{p-p'})$. The neglected terms contain the change of nuclear potential inside nuclear matter due to many-body forces. This effect will be calculated in a subsequent paper. It is expected that this neglect will not change the result drastically.

For the physical region $E>0$, in (7), it may be possible to take the limit $\nu\rightarrow -m^2$ since $A_N(E,\nu)$, $E>0$ is defined for every $\nu \geq -m^2$. Thus we have the dispersion relation

$$F_N(E) = -\sum_{p'\epsilon N} V(\omega_{p-p'}) + \frac{1}{\pi}\int_0^\infty A_N(E')/(E'-E)dE'. \quad (9)$$

$A_N(E)$, $E>0$ is related to the cross section. It is zero if E is less than the Fermi energy $E_F = P_F^2/2m$ since no real scattering occurs (except for forward scattering) due to the Pauli principle

$$A_N(E)=0, \quad 0<E<E_F.$$

If $E>E_F$, real processes occur and

$$A_N(E) = \tfrac{1}{2}\sum_{p'\epsilon N} v_{p-p'}\int P_{p,p'}d\sigma(\omega_{p-p'}), \quad E>0, \quad (10)$$

where P is the projection operator and

$P=1$ if both of the final-state nucleons are outside the Fermi surface,

$=0$ if one or both of the final-state nucleons is inside the Fermi surface. (11)

There is no difficulty, in principle, in calculating (10). However, it is simpler if a suitable average is taken and P is a function of p and p' only. P has the property that

$$P=1, \quad \text{if} \quad (p-p')^2 \gg 4P_F^2,$$

and

$$P=0, \quad \text{if} \quad (p-p')^2 \ll 4P_F^2.$$

Suppose we put

$$P=1, \quad (p-p')^2 > 4P_F^2,$$
$$P=0, \quad (p-p')^2 < 4P_F^2; \quad (12)$$

then the result is very simple and we have[6]

$$F_N(E) = \sum_{p'\epsilon N} \bar{f}_N(\omega_{p-p'}),$$
$$\bar{f}_N(\omega) = -V(\omega) + \frac{1}{\pi}\int_{\omega_F}^{\infty} \frac{\frac{1}{2}v'\sigma(\omega')}{\omega'-\omega}d\omega', \quad (13)$$
$$\omega_F = 2P_F^2/m,$$

and Eq. (2) becomes

$$E = E_0 - \tfrac{1}{2}\sum_{p,p'\epsilon N} \bar{f}_N(\omega_{p-p'}). \quad (14)$$

Two points should be discussed before concluding this section.

In (10) we put the cross section equal to that for the free case, if the final state is allowed. Actually the scattering matrix element, and therefore the cross section, is changed due to the exclusion principle. Such a change is certainly important for collisions of particles well submerged in the Fermi sea and this is the object of this paper. However, for high-energy collisions such that a real final process is possible, the exclusion effect may not be important. In the next section it is shown that errors due to this effect are in fact small.

In the actual case of nucleon-nucleon scattering, the cross section tends to a constant at high energy. In this case the dispersion integral in Eq. (13) does not converge and we have to make a subtraction,

$$\bar{f}_N(\omega) = \bar{f}_N(0) - V(\omega) + V(0) + \frac{\omega}{\pi}\int_{\omega_F}^{\infty} \frac{a(\omega')}{\omega'(\omega'-\omega)}d\omega'. \quad (15)$$

To determine the unknown constant $\bar{f}_N(0)$, we compare the above expression with the dispersion relation for free scattering,

$$f(\omega) = f(0) - V(\omega) + V(0) + \frac{\omega}{\pi}\int_0^\infty \frac{a(\omega')}{\omega'(\omega'-\omega)}d\omega' + \left(\frac{\omega C_d}{\omega_d(\omega_d-\omega)}\right), \quad (16)$$

where the last term in the parentheses is present when a bound state exists. The condition that in the high-energy limit $\bar{f}_N$ should be equal to f determines $\bar{f}_N(0)$, and

$$\bar{f}_N(\omega) = f(0) - V(\omega) + V(0) + \frac{\omega}{\pi}\int_{\omega_F}^{\infty}\frac{a(\omega')}{\omega'(\omega'-\omega)}d\omega' - \frac{1}{\pi}\int_0^{\omega_F}\frac{a(\omega')}{\omega'}d\omega' + \left(-\frac{C_d}{\omega_d}\right). \quad (17)$$

To summarize, if the scattering amplitude in free

[5] When the absorptive part A_N is a sum of Lorentz-invariant quantities a, the dispersion integral of A_N can be written as a sum of dispersion integrals of a. The formula is

$$\sum_{v\epsilon N}\int \frac{a((u-v)^2)}{(u-x)^2-(w-x)^2}du_0 = \sum_{v\epsilon N}\int\frac{a((u-x)^2)}{(u-x)^2-(w-v)^2}du_0,$$
$$\text{if } a(y^2)=0, \quad y^2 > -\nu P_F,$$

where u, v, w, and x are four-dimensional vectors, and $u^2=w^2=-\nu$, $v^2=m^2$, $x=(m,0,0,0)$. See also reference 6.

[6] To derive (13), we use the formula

$$\sum_{p'\epsilon N}\int\frac{a((p-p')^2)}{p^2-q^2}dp^2 = \sum_{p'\epsilon N}\int\frac{a(p^2)}{p^2-(q-p')^2}dp^2,$$
$$\text{if } a(p^2)=0, \quad p^2<P_F^2.$$

This is the nonrelativistic version of the formula in reference 5.

space is given by

$$f(\omega)=-V(\omega)+\frac{1}{\pi}\int_0^\infty \frac{a(\omega')}{\omega'-\omega}d\omega', \qquad (18)$$

the corresponding amplitude in nuclear matter is obtained by omitting some part of the dispersion integral if that process is excluded by the Pauli principle:

$$\bar{f}_N(\omega)=-V(\omega)+\frac{1}{\pi}\int_{\omega_F}^\infty \frac{a(\omega')}{\omega'-\omega}d\omega'. \qquad (19)$$

If one subtraction is necessary, the formula is given by (17). In either case

$$\bar{f}_N(\omega)=f(\omega)-\frac{1}{\pi}\int_0^{\omega_F} \frac{a(\omega')}{\omega'-\omega}d\omega'. \qquad (20)$$

4. NUMERICAL RESULT

The final formula for the binding energy of nuclear matter is given by (14). The binding energy per particle is given by

$$\frac{E}{A}=\frac{3P_F^2}{10m}-\frac{1}{2\pi^2}\int_0^{2P_F}[\bar{f}_{Npp}(\omega_k)+\bar{f}_{Npn}(\omega_k)]\times\left(1-\frac{3k}{4P_F}+\frac{k^3}{16P_F^3}\right)k^2dk, \qquad (21)$$

where A is the mass number, $\bar{f}_{Npp}$ and $\bar{f}_{Npn}$ are the amplitudes for unpolarized pp and pn scattering, respectively, and

$$\bar{f}_{Npp}(\omega)+\bar{f}_{Npn}(\omega)=-\frac{4\pi}{m}(\tfrac{3}{4}a_3+\tfrac{1}{4}a_1+\tfrac{1}{2}a)+\frac{2\pi}{m}\frac{f^2}{4\pi}\frac{3}{2}\left(\frac{1}{\omega+(\mu^2/2m)}-\frac{1}{(\mu^2/2m)}\right)+\frac{\omega}{\pi}\int_{-\infty}^{-2\mu^2/m}\frac{a_{pn}(\omega')+a_{pp}(\omega')}{\omega'(\omega'-\omega)}d\omega'$$
$$+\frac{\omega}{2\pi}\int_{\omega_F}^\infty\frac{v'(\sigma_{pp}+\sigma_{pn})}{\omega'(\omega'-\omega)}d\omega'-\frac{1}{2\pi}\int_0^{\omega_F}\frac{v'(\sigma_{pp}+\sigma_{pn})}{\omega'}d\omega'+\frac{2\pi}{m^2}\frac{1}{R-r_3}\frac{1}{-\omega_d},\quad \omega_d=-2|B|. \qquad (22)$$

The notation is as follows: a_3 is the triplet np scattering length; a_1, the singlet np scattering length; a, the singlet pp scattering length; f, the pion-nucleon coupling constant; μ, the pion mass; σ_{pp} and σ_{pn}, the total cross section for unpolarized pp and np scattering, respectively; R, the deuteron radius; r_3, the triplet effective range; and B, the binding energy of the deuteron.

The potential (the integral over the unphysical region) can be obtained if the scattering amplitudes f_{pp} and f_{pn} are known as functions of energy. The phase shifts for proton-proton scattering are fairly well known, but not much is known about the phase shifts for neutron-proton scattering so it is not possible to determine $V(\omega)$ in this way.

The absorptive part $a(\omega)$ in the unphysical region has been calculated for two-pion exchange in the region[2]

$$-9\mu^2/2m<\omega<-2\mu^2/m.$$

For $\omega<-9\mu^2/2m$, where three-pion exchange takes place, it is extremely difficult to calculate $a(\omega)$. We make the approximation

$$\frac{1}{\pi}\int_{-\infty}^{-9\mu^2/2m}\frac{a_{pp}(\omega')+a_{pn}(\omega')}{\omega'(\omega'-\omega)}d\omega' =\frac{1}{\pi}\int_{-\infty}^{-9\mu^2/2m}\frac{a_{pp}+a_{pn}}{\omega'^2}d\omega'\equiv -V_{3\pi}'(0), \qquad (23)$$

which is justified because $9\mu^2/2m=90$ Mev is fairly larger than the average energy $\langle\omega\rangle_{\text{av}}\cong 40$ Mev. The constant $V_{3\pi}'(0)$ is an arbitrary parameter and will be determined later. If $a_{pp}(\omega)+a_{pn}(\omega)$ are assumed constant for $\omega<-9\mu^2/2m$ and equal to $a_{pp}(-9\mu^2/2m)+a_{pn}(-9\mu^2/2m)$ (this is consistent with the value for $\omega<-2m$ where a can be related to nucleon-antinucleon cross section), we obtain

$$V_{3\pi}'(0)=2/\mu^3 \quad \text{(interpolated value)}. \qquad (24)$$

The energy per nucleon E/A thus calculated by Eq. (21) is a function of P_F or the density of nuclear matter. We look for the minimum of E/A by varying P_F to obtain the stable nuclear matter. The minimum position depends on the value of $V_{3\pi}'(0)$ and we choose it so that the minimum occurs at the observed value $P_F=2\mu$. This happens when

$$V_{3\pi}'(0)=4.7/\mu^3, \quad \text{(determined value)} \qquad (25)$$

and in this case

$$E/A=-13.2 \text{ Mev}, \quad \text{(theory)}. \qquad (26)$$

The experimental value is

$$E/A=-15 \text{ Mev}, \quad \text{(experiments)}.$$

Before concluding this section we compare $\bar{f}_N$ with f at $\omega>\omega_F$ and see if they differ very much. The results are shown in Table I.[7] We see that the amplitude, and therefore the cross section at $\omega>\omega_F$, is not very much changed by the Pauli principle. Even if a_{pp} and a_{pn} are

TABLE I. Values of $\bar{f}_N$ and f for $\omega>\omega_F$.

ω	$1.3\omega_F$	$2\omega_F$	$4\omega_F$
$\bar{f}_{Npp}$	0.75	0.85	1.0
f_{pp}	0.99	1.0	1.0

[7] From the formula (22), we have the result $\bar{f}_N(\omega_F)=\ln\infty$. This is due to the choice of a sharp step function (12). Actually P is a smooth function and such an infinity does not occur.

zero for $\omega_F<\omega<4\omega_F$, the change in E/A is about $+1.5$ Mev. The actual change will be very much smaller than 1 Mev which is negligible in our discussion. Also we can see that the result is insensitive to the shape of P which we approximated by a step function (12).

5. DISCUSSIONS

In Sec. 3 we derived a formula to obtain the scattering amplitude inside nuclear matter when the free-particle dispersion relations are known. In applying it to the calculation of nuclear binding energy, we had to introduce an undetermined parameter $V_{3\pi}'(0)$ due to our lack of knowledge about the three-pion exchange potential, but this parameter could be determined if more information on proton-neutron scattering phase shifts were obtained. If this parameter is adjusted so as to give the correct nuclear density, then the value determined by Eq. (25) is somewhat larger than the interpolated value (24). This is consistent with the fact obtained from the electromagnetic form factors of nucleons that the three-pion intermediate state will be important. Also this corresponds to the results of the current nucleon-nucleon scattering theory that a very large repulsive core is necessary at a short internucleon distance. Once this parameter was determined, the binding energy per nucleon turned out to be in reasonable agreement with the observed value. This calculated value, however, cannot be taken too seriously for several reasons.

First, the effect of renormalization of one-body propagator or the change of effective mass inside nuclear matter is not considered here. Bethe[8] estimated this effect to be about 5 Mev per nucleon for the binding energy.

Second, we have made approximations for the spectral function both on the right-hand and the left-hand cut. On the right-hand cut we have neglected the change of cross sections due to the Pauli principle. This correction is calculable in successive approximation by means of nonforward dispersion relations or partial wave amplitude dispersion relations, which we did not attempt in this paper. It is the authors' feeling that this correction is not very important.

More important is the correction on the left-hand cut or the change of nuclear potential inside of nuclear matter, which we neglected. A part of such a change was calculated by Fujita and Miyazawa,[9] who found a change of about ten percent both in the strength and the range of the potential. Drell and Walecka[10] estimated the quenching of the anomalous magnetic moment inside the nucleus and it turned out to be 7%. These indicate that a change of several percent is also expected for $V(\omega)$. Since the binding energy is a result of the cancellation between a large potential and a large kinetic energy, a change of a few percent in the potential may result in a change as much as a factor of two for the binding energy. This point will be investigated in a subsequent paper.

[8] H. A. Bethe (private communication).

[9] J. Fujita and H. Miyazawa, Progr. Theoret. Phys. (Kyoto), **17**, 360 (1957).

[10] S. D. Drell and J. D. Walecka, Phys. Rev. **120**, 1069 (1960).

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APPENDIX

Derivation of Eq. (7)

In this Appendix we use the relativistic perturbation theory of Feynman. Due to the exclusion principle in the intermediate states, the usual proof of the dispersion relation fails for f_N. For this reason it is convenient to adopt the standpoint (II) discussed in Sec. 2. Then every diagram of a scattering with exclusion principle is equivalent to a sum of diagrams of many-particle collisions without exclusion principle. Thus the $f_N(p,p')$ can be expressed as a sum of $f(p,q_1,q_2,\cdots)$ corresponding to Feynman diagrams with external momenta p, $p'(=q_1)$, q_2, $\cdots$. We introduce mass variable ν and put

$$p_0{}^2=p^2-\nu.$$

It can be shown that if $\nu>0$, $f(p,q_1,q_2,\cdots)$ as a function of p_0 (with fixed direction of p) does not have any singularities except on the real axis and a possible cut starting from $p_0=\pm i\nu^{\frac{1}{2}}$. This comes from the fact that f is a product of terms of the form

$$1/[(p-Q)^2-M^2],$$

and the denominator never vanishes except on the real axis. When we sum over all contributions to get F_N, the superficial branch points $p_0=\pm i\nu^{\frac{1}{2}}$ vanishes, since F_N is an even function of p. $F_N(E)$ has branch cuts only on the real axis. The position of the branch is determined from individual graph. The right-hand cut starts from

$$p_0=\frac{x_0(P_F{}^2+m^2)^{\frac{1}{2}}-(x_0{}^2+\nu)^{\frac{1}{2}}P_F}{m},\quad x_0=\frac{3m^2+\nu}{2m}$$

to the right; and the left-hand cut starts from

$$p_0=\frac{x_1(P_F{}^2+m^2)^{\frac{1}{2}}+(x_1{}^2+\nu)^{\frac{1}{2}}P_F}{m},\quad x_1=\frac{m^2-\nu-\mu^2}{2m},$$

to the left.