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# Evolution of a Quasi-Stationary State*

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To elucidate the time development of quasi-stationary states, a simple barrier penetration problem has been studied. Both approximate expressions and numerical results for some parameters were obtained for the decay rate. First, irregular oscillations occur for a short time. Second, the exponential region follows. Third, further oscillations occur during which the decay rate dips to negative values, so that the probability of finding the undecayed system increases briefly at several times. Fourth and finally, the decay rate decreases like an inverse power of the time.

THERE have appeared lately several studies[1–5] of nonexponential evolution of quasi-stationary states. Khalfin[1] has proved that all states that have a lowest energy in their spectrum eventually must decay more slowly than exponentially. Here, we examine a simple model to illustrate the various stages that occur in the time development of such systems.

Consider a one-dimensional, nonrelativistic problem, with a potential given by

$$V(x)=\infty, \qquad x<-a,$$
$$=U\delta(x), \quad x\geq -a.$$

We choose as initial wave function

$$\Psi(x,0)=(2/a)^{\frac{1}{2}}\sin(n\pi x/a), \quad -a\leq x\leq 0,$$
$$=0,\ x<-a \text{ or } 0<x,$$

and study the leakage through the barrier at the origin. Such a delta-function barrier has, for particles of mass $m$ and energy $E$, a transmission coefficient $(1+mU^2/2E\hbar^2)^{-1}$. Its use, rather than use of a barrier of finite thickness, simplifies the calculations. It also removes the need to decide whether to treat the density within the barrier as part of the undecayed system.[6] One can think of this problem as a simple picture of alpha decay; it is similar to the model used by Petzold.[3]

To obtain $\Psi(x,t)$, we first find the energy eigenfunctions $\phi_E(x)$, which are orthonormal in the sense that $(\phi_{E'},\phi_E)=\delta(E'-E)$. Then we expand $\Psi(x,0)$ in terms of the $\phi_E$, determine the expansion coefficients $C(E)$, and obtain the wave function

$$\Psi(x,t)=\int_0^\infty dE\, C(E)\phi_E \exp(-iEt/\hbar).$$

With the notation

$$q\equiv[a(2mE)^{\frac{1}{2}}]/\hbar,$$
$$T\equiv\hbar t/2ma^2,$$
$$l\equiv x/a,$$
$$G\equiv 2maU/\hbar^2,$$

the result is

$$\Psi(x,t)=2n(2/a)^{\frac{1}{2}}$$
$$\times\int_0^\infty \frac{dq\,\exp(-iTq^2)q\,\sin q[q\sin(l+1)q+f]}{(q^2-n^2\pi^2)(q^2+Gq\sin 2q+G^2\sin^2 q)}, \quad (1)$$
$$f\equiv 0,\ -1\leq l\leq 0 \text{ (inside the well)},$$
$$\equiv G\sin q\sin lq,\ 0<l \text{ (outside the well)}.$$

## ANALYTIC APPROXIMATIONS

If $G\gg 1$, the barrier has low transmittance, and the state is quasi-stationary. The wave function for the times of exponential decay can then be found from the contour shown in Fig. 1. The pole $P$ that is paramount in the determination of the wave function lies, to terms in $1/G^2$, at

$$n\pi[1-1/(G+1)-in\pi/G^2+\cdots].$$

The line $N$ of the contour introduces negligible error because the exponential is small on it except at extremely early times. The result for the inside $(-1\leq l\leq 0)$ function is

$$\Psi_{ei}=(2/a)^{\frac{1}{2}}e^{-i\epsilon T-T/2\tau}\{[1-n^2\pi^2(l+1)^2/G^2]\sin n\pi l$$
$$-[1+(in\pi-1)/G][n\pi(l+1)/G]\cos n\pi l\}. \quad (2a)$$

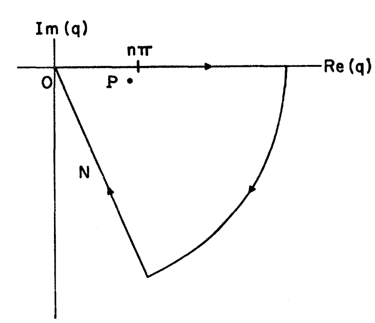


FIG. 1. Momentum plane contour for the determination of the exponential region wave function.

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[1] L. A. Khalfin, Soviet Phys.—JETP **6**, 1053 (1958); Soviet Phys. Doklady **5**, 515 (1960).
[2] G. T. Trammell, Oak Ridge National Laboratory Report ORNL 2204, 1956 (unpublished).
[3] J. Petzold, Z. Physik **155**, 422; **157**, 122 (1959).
[4] F. J. Lynch, R. E. Holland, and M. Hamermesh, Phys. Rev. **120**, 513 (1960).
[5] G. Höhler, Z. Physik **152**, 546 (1958); M. Levy, Nuovo cimento **13**, 115 (1959); P. T. Mathews and A. Salam, Phys. Rev. **115**, 1079 (1959); E. Kazes, Nuovo cimento **15**, 537 (1960); C. S. Wu, Y. K. Lee, N. Benczer-Koller, and P. Simms, Phys. Rev. Letters **5**, 432 (1960).
[6] R. G. Winter, Nature **180**, 1471 (1957).

For the outside ($l>0$) function it is

$$\Psi_{eo}=(2/a)^{\frac{1}{2}}e^{-i\epsilon T-T/2\tau}(\cos n\pi l+i\sin n\pi l)(n\pi/G). \quad (2b)$$

The energy $\epsilon$ in units of $\hbar^2/2ma^2$ and the mean life $\tau$ in units of $2ma^2/\hbar$ are given by

$$\epsilon=n^2\pi^2(1-2/G), \quad (3)$$

$$1/\tau=(4n^3\pi^3/G^2)(1-4/G). \quad (4)$$

Higher powers of $1/G$ and $l/G$ have been neglected. The approximations used here are, therefore, not valid at large distances from the barrier.

These expressions are wrong not only at very small $T$, but also at very large $T$. Everywhere, except in the neighborhood of $q=0$, the oscillations of $\exp(-iTq^2)$ in (1) eventually become so rapid that the contribution to the integral there becomes negligible. The behavior of the integrand near $q=0$ then controls the result. We expand the integrand in powers of $q$, and use the Riemann-Lebesgue lemma[7] in the form

$$\int_0^\infty dq\exp(-iTq^2)(b_0+b_2q^2+b_4q^4+\cdots)$$
$$=(1-i)(\pi/8T)^{\frac{1}{2}}(b_0-ib_2/2T+3b_4/4T^2+\cdots).$$

The result for the large-time inside ($-1\leq l\leq 0$) function is

$$\Psi_{Li}=\frac{(1+i)(l+1)}{2n\pi G^2(\pi a)^{\frac{1}{2}}T^{\frac{3}{2}}}\left\{1-\frac{3i}{2T}\left[\frac{1}{n^2\pi^2}-\frac{l^2}{6}-\frac{l}{3}\right]+\cdots\right\}. \quad (5a)$$

For the outside ($l>0$) function it is

$$\Psi_{Lo}=\frac{(1+i)(Gl+1)}{2n\pi G^2(\pi a)^{\frac{1}{2}}T^{\frac{3}{2}}}\times\left\{1-\frac{3i}{2T}\left[\frac{1}{n^2\pi^2}-\frac{l^2}{6}-\frac{l^2}{3}(Gl+1)\right]+\cdots\right\}. \quad (5b)$$

Again, only the leading terms in $1/G$ have been kept.

We now seek an expression that gives, at least qualitatively, the behavior of the wave function at all except very early times. For $G\gg 1$, the resonance in the integrand of (1) near $q=n\pi$ is sharp. Then the neighborhood of $q=0$ becomes important as soon as the effect of the resonance is made negligible by the rapid oscillations of $\exp(-iTq^2)$ there. The wave function for all except early times is approximated therefore inside the well by the sum of (2a) and (5a), and outside by the sum of (2b) and (5b):

$$\Psi\simeq\Psi_e+\Psi_L. \quad (6)$$

The current $j$ outside the well is given then by

$$\frac{2ma^2}{\hbar}j(l,T)=\frac{e^{-T/\tau}}{\tau}+\frac{1+3l+3l^2(4n^3\pi^3\tau)^{\frac{1}{2}}+4n^3\pi^3\tau l^3}{32n^8\pi^9\tau^2T^4}+\frac{e^{-T/2\tau}}{2n^3\pi^{7/2}\tau T^{\frac{3}{2}}}$$
$$\times[\sin(n\pi l-\epsilon T-\pi/4)+n\pi l\cos(n\pi l-\epsilon T-\pi/4)]. \quad (7)$$

The barrier parameter $G$ has been eliminated through (4). The first term is dominant in the exponential region and is the consequence of (2b) alone. The second term controls the very large time behavior; inverse power-time dependence is a characteristic of the $T\rightarrow\infty$ behavior of all states with a nonsingular energy spectrum.[1,2] This contribution comes from (5b) alone.

The third part of (7) is the cross term between (2b) and (5b), and is of importance in the transition from exponential to power-law behavior. It oscillates in time with frequency equal to the energy found in the exponential region. The oscillations can be violent enough to drive the current negative. It is easy to prove, for any $G$ and any $l$ within the region of validity of the approximations, that there will indeed be a time at which the current dips to negative values. Such negative currents are not as absurd as they might appear; $\Psi(x,0)$ contains negative as well as positive momenta everywhere. For high energy, the frequency of the oscillations can easily be too great to permit their detection; only the time average of $j(l,T)$, given by the first two terms of (7), would be observed then.

An interesting picture of the decay is obtained by examining the "mean momentum"

$$\bar{P}\equiv mj(0,T)/|\Psi(0,T)|^2. \quad (8)$$

The time average of $\bar{P}$, taken over a long time, approximates the expectation value of the momentum of the particles that emerge from the well. During the time in which exponential decay is valid,

$$\bar{P}\simeq nh/2a, \quad (9)$$

the magnitude of the momentum that would be found in the well if the barrier were impenetrable and the state stationary. At very large times, when only the second term in (7) is important,

$$\bar{P}\simeq\hbar/2aT=ma/t. \quad (10)$$

The quantity $\bar{P}$ will be examined further in the next section.

## NUMERICAL STUDIES

The approximations used above are good for large $G$, that is, for $\Delta E/E\ll 1$. For such "narrow" states, the second and third terms in (7) do not become important until very many mean lives have elapsed. If one begins with any reasonable number of decaying systems, the in-

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[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Macmillan Company, New York, 1947), American ed., p. 172.

tensity would become negligible long before the large-time deviations from exponential decay occur. It is interesting, therefore, to examine numerically and without approximations somewhat wider states. We want $G$ large enough to let us consider the state quasi-stationary, but small enough to given an exponential region of reasonable length. Calculations were made with $n=1$ for $G=6$ and $G=20$. The wave functions were obtained by evaluating the integral (1) with an IBM 650 computer through use of a five-point Gaussian integration formula.[8] The results are given in Figs. 2 through 7. Points at which calculations were made are shown; the curves connecting these points were sketched in.

For $G=6$, the full width of the energy spectrum at half-maximum, divided by the resonance energy, gives $\Delta E/E=0.13$. Figure 2 shows that, for about 10 mean lives, the current decays roughly exponentially with a mean life of $0.644\times 2ma^2/\hbar$. The exponential region is preceded and followed by oscillations that are brought out in Fig. 3, where the current times $\exp(T/0.644)$ is plotted as a function of time. For precisely exponential decay, this plot would show a horizontal line. At very early times, we see instead irregularities qualitatively like those observed by Lynch, Holland, and Hamermesh.[4] Since the wave function is zero at the barrier at $T=0$, the current must be zero initially. In the early time region, the wave function adjusts to the well and evolves into the function from which approximately exponential decay can take place. The current in this region is sensitive to details of $\Psi(x,0)$: Admixture of 1% of the $n=2$ initial function can double the current

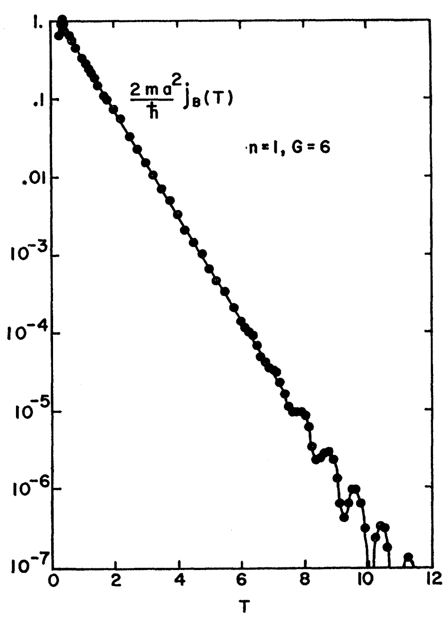


FIG. 2. The current at the barrier, $j_B$, for $G=6$.

[8] Z. Kopal, *Numerical Analysis* (Chapman and Hall, Ltd., London, 1955).

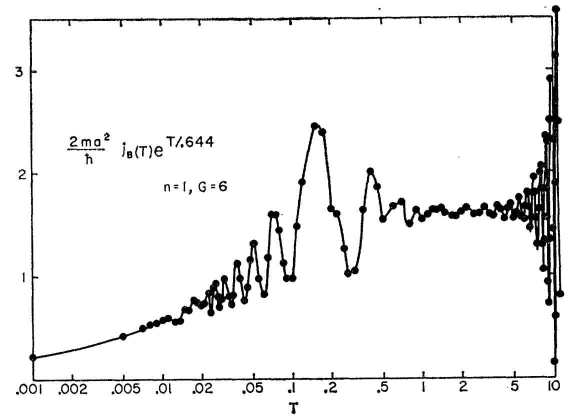


FIG. 3. The current at the barrier multiplied by $\exp(T/\tau)$ for $G=6$.

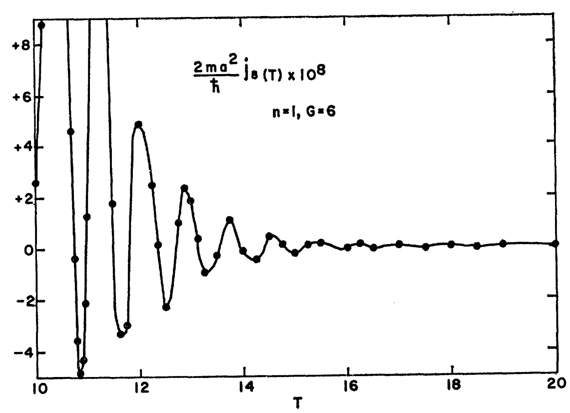


FIG. 4. The current at the barrier, for $G=6$, in the large time region. Note the negative dips.

at some instants. There follows then the approximately exponential region, which, in turn, gives way to a region of violent oscillations that are described qualitatively by the last term of (7). The remainder of the decay is displayed in Fig. 4. Between $T=10$ and 20, the negative dips in the current occur. As a check, the first dip around $T=10.85$ was studied in some detail. A numerical integration of $\Psi\Psi^*$ over the well was made at $T=10.75$ and $T=10.95$. The increase of the probability of finding the particle inside equals the time integral of the negative current, within the 2% accuracy of the calculation. These oscillations do not depend strongly on details of the initial state: With optimum choice of phase,an admixture of 45% of the $n=2$ function is necessary to drive the current positive at $T=10.85$. Finally, near $T=20$, the power-law behavior of the second term in (7) appears. The numerical results are inaccurate here because the current in the very large time region is given by the small difference of large quantities.

The mean momentum $\bar{P}$, defined in (8), is displayed in Figs. 5 and 6. Since the current rises from zero faster than the density at very small times, $\bar{P}\to\infty$ as $T\to 0$.

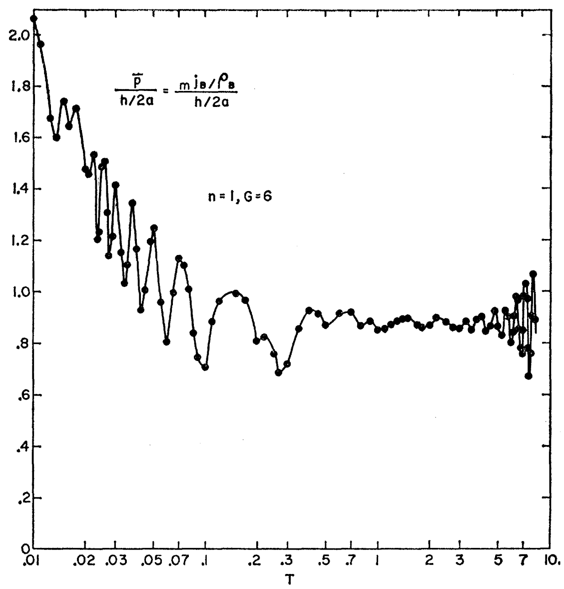


FIG. 5. The mean momentum $\bar{P}$ for $G=6$ during the early and middle times of the decay.

After the early-time oscillations, $\bar{P}$ hovers around 0.88 $h/2a$, then goes through the large time oscillations, and finally goes to zero as in (10). The oscillations of $\bar{P}$ near the middle of the exponential region can be connected with the width of the state; between $T=1$ and $T=4$, $(\bar{P}_{\max}{}^2-\bar{P}_{\min}{}^2)/(\bar{P}_{\mathrm{av}})^2\simeq 0.12$. Note that $\hbar$ times the angular frequency of the large-time oscillations, $7.6\hbar/2ma^2$, does equal the energy found in the exponential region if we take that energy to be $\bar{P}^2/2m$. An examination of the evolution of $\bar{P}$, with oscillations averaged out, suggests a description that is intuitively

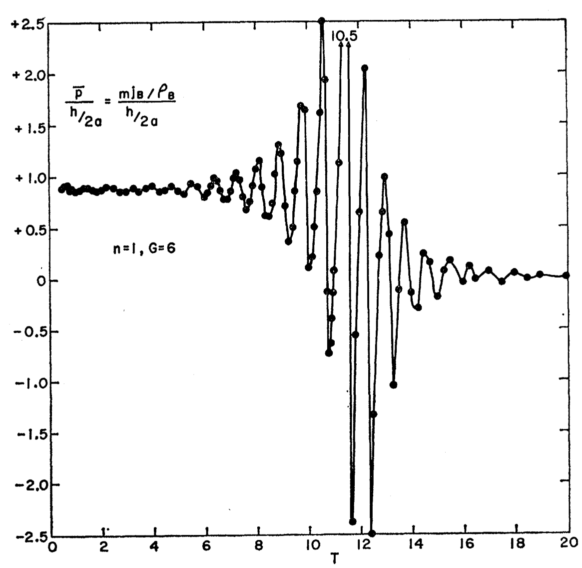


FIG. 6. The mean momentum $\bar{P}$ for $G=6$ during the middle and late times of the decay.

plausible. At very early times, the high-momentum components leave the well rapidly. Then, during the exponential region, the bulk of the components that have approximately the resonance energy determines the decay. The decay is exponential because only the size, not the shape, of the wave function changes appreciably. Finally, after the components near the resonance are depleted, the very low momentum components come out. Khalfin's theorem[1] can be viewed as follows: As the state evolves over long times, the high-energy components are depleted preferentially, and the mean energy of the emitted particles approaches the lowest energy in the spectrum. Such lowering of the mean energy will cause a steady lengthening of the mean life, which is equivalent to a slower than exponential decay.

For $G=20$, $\Delta E/E=0.018$. The decay was followed in any detail only through the beginning of the exponential region. Enough points were calculated at later times to locate roughly the end of the exponential

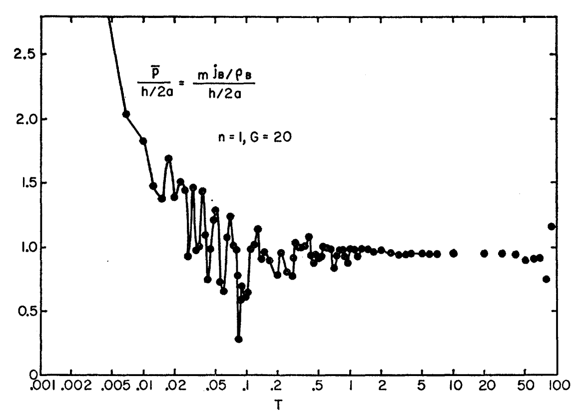


FIG. 7. The mean momentum $\bar{P}$ for $G=20$ during the early and middle times of the decay.

region. The results for $\bar{P}$ are shown in Fig. 7. The very early time region is qualitatively like that shown for $G=6$ in Fig. 5. In the exponential region, $\bar{P}$ remains close to $0.95h/2a$, and the mean life is $4.05\times 2ma^2/\hbar$. The large time deviations from exponential decay occur after about 20 mean lives.

The computer results do not, of course, agree in detail with (7) because $G=6$ and $G=20$ are too small to permit neglect of all but the lowest powers of $1/G$. There is, however, fair agreement regarding the location of the exponential, oscillatory, and power-law regions, and regarding the magnitude of the terms in (7).

## MEASURABILITY

The detailed results displayed above are, of course, valid only for our rather artificial model, but many quasi-stationary states will develop in a similar fashion. One can usually expect that there will first be a short time during which the initially specified state adjusts to the interactions that determine its decay. Second, there will be a period of approximately exponential decay, governed by a pole responsible for a resonance. Third, oscillations of the decay rate can result from cross terms between the residue at the resonance pole and the contributions from the low-energy part of the spectrum. Fourth, only the low-energy end of the spectrum is important, and the decay rate decreases as some inverse power of the time.

There is no formal obstacle to the observation of these effects, as can be seen by examining any thought experiment of the kind commonly used in discussions of this sort. We must find an operator $R$ such that the required $\Psi(x,0)$ is an eigenfunction of $R$ with eigenvalue $r$. We measure $R$, and know, whenever $r$ results, that we have at that instant prepared $\Psi(x,0)$. After some time $t_1$, we examine the system to see whether it has decayed; it does not matter that this examination disturbs the system. Many repetitions of these operations for each of many times $t_1$ will then yield all required information. There are no uncertainty principle limitations on the observability of all the features that have been discussed, even though we must measure times much smaller than $E/\hbar$.

The experimental difficulties of such measurements are, of course, tremendous. The frequency $E/h$ of the oscillations will usually be so high that only the time average can be observed. Furthermore, the first and second terms in (7) become equal when

$$n^2\pi \exp(T/\tau) = (T/\tau)^4(E/\Delta E)^5,$$

that is, for narrow states, when $T/\tau \simeq 5 \ln(E/\Delta E)$. Most quasi-stationary states that we can examine with any precision are so narrow that the remaining sample at this time is vastly too small.

**ACKNOWLEDGMENTS**

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